Optimal Receiver for the AWGN Channel

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Additive White Gaussian Noise Channel

$$y(t) = s(t) + n(t)$$

$s(t)$ Transmitted Signal
$y(t)$ Received Signal
$n(t)$ White Gaussian Noise

$$S_n(f) = \frac{N_0}{2} = \sigma^2$$

$$R_n(\tau) = \sigma^2 \delta(\tau)$$

M-ary Signaling in AWGN Channel

- One of M continuous-time signals $s_1(t), \dots, s_M(t)$ is sent
- The received signal is the transmitted signal corrupted by AWGN
- M hypotheses with prior probabilities π_i, $i = 1, \dots, M$

$$\begin{aligned} H_1 &: \quad y(t) = s_1(t) + n(t) \\ H_2 &: \quad y(t) = s_2(t) + n(t) \\ &\vdots \\ H_M &: \quad y(t) = s_M(t) + n(t) \end{aligned}$$

- Random variables are easier to handle than random processes
- We derive an equivalent M-ary hypothesis testing problem involving only random variables

White Gaussian Noise through Correlators

- Consider the output of a correlator with WGN input

$$Z = \int_{-\infty}^{\infty} n(t)u(t)\, dt = \langle n, u \rangle$$

  where $u(t)$ is a deterministic finite-energy signal
- Z is a Gaussian random variable
- The mean of Z is

$$E[Z] = \int_{-\infty}^{\infty} E\left[n(t)\right] u(t)\, dt = 0$$

- The variance of Z is

$$\text{var}[Z] = \sigma^2 \|u\|^2$$

White Gaussian Noise through Correlators

Proposition

Let $u_1(t)$ and $u_2(t)$ be linearly independent finite-energy signals and let $n(t)$ be WGN with PSD $S_n(f) = \sigma^2$. Then $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian with covariance

$$\text{cov}\left(\langle n, u_1 \rangle, \langle n, u_2 \rangle\right) = \sigma^2 \langle u_1, u_2 \rangle.$$

Proof

To prove that $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian, consider a non-trivial linear combination $a\langle n, u_1 \rangle + b\langle n, u_2 \rangle$

$$a\langle n, u_1 \rangle + b\langle n, u_2 \rangle = \int n(t) \left[a u_1(t) + b u_2(t)\right] \, dt$$

White Gaussian Noise through Correlators

Proof (continued)

$$
\begin{aligned}
\text{cov}\left(\langle n, u_1\rangle, \langle n, u_2\rangle\right) &= E\left[\langle n, u_1\rangle\langle n, u_2\rangle\right] \\
&= E\left[\int n(t)u_1(t)\, dt \int n(s)u_2(s)\, ds\right] \\
&= \int\int u_1(t)u_2(s)E\left[n(t)n(s)\right]\, dt\, ds \\
&= \int\int u_1(t)u_2(s)\sigma^2\delta(t-s)\, dt\, ds \\
&= \sigma^2 \int u_1(t)u_2(t)\, dt \\
&= \sigma^2\langle u_1, u_2\rangle
\end{aligned}
$$

If $u_1(t)$ and $u_2(t)$ are orthogonal, $\langle n, u_1\rangle$ and $\langle n, u_2\rangle$ are independent.

Restriction to Signal Space is Optimal

Theorem

For the M-ary hypothesis testing given by

$$\begin{aligned} H_1 &: \quad y(t) = s_1(t) + n(t) \\ \vdots & \qquad\qquad \vdots \\ H_M &: \quad y(t) = s_M(t) + n(t) \end{aligned}$$

there is no loss in detection performance by using the optimal decision rule for the following M-ary hypothesis testing problem

$$\begin{aligned} H_1 &: \quad \mathbf{Y} = \mathbf{s}_1 + \mathbf{N} \\ \vdots & \qquad\quad \vdots \\ H_M &: \quad \mathbf{Y} = \mathbf{s}_M + \mathbf{N} \end{aligned}$$

where $\mathbf{Y}$, $\mathbf{s}_i$ *and* $\mathbf{N}$ *are the projections of* $y(t)$, $s_i(t)$ *and* $n(t)$ *respectively onto the signal space spanned by* $\{s_i(t)\}$.

Projections onto Signal Space

- Consider an orthonormal basis $\{\psi_i | i = 1, \ldots, K\}$ for the space spanned by $\{s_i(t) | i = 1, \ldots, M\}$
- Projection of $s_i(t)$ onto the signal space is

$$\mathbf{s}_i = \begin{bmatrix} \langle s_i, \psi_1 \rangle & \cdots & \langle s_i, \psi_K \rangle \end{bmatrix}^T$$

- Projection of $n(t)$ onto the signal space is

$$\mathbf{N} = \begin{bmatrix} \langle n, \psi_1 \rangle & \cdots & \langle n, \psi_K \rangle \end{bmatrix}^T$$

- Projection of $y(t)$ onto the signal space is

$$\mathbf{Y} = \begin{bmatrix} \langle y, \psi_1 \rangle & \cdots & \langle y, \psi_K \rangle \end{bmatrix}^T$$

- Component of $y(t)$ orthogonal to the signal space is

$$y^{\perp}(t) = y(t) - \sum_{i=1}^{K} \langle y, \psi_i \rangle \psi_i(t)$$

Proof of Theorem

$y(t)$ is equivalent to $(\mathbf{Y}, y^{\perp}(t))$. We will show that $y^{\perp}(t)$ is an irrelevant statistic.

$$
\begin{aligned}
y^{\perp}(t) &= y(t) - \sum_{i=1}^{K} \langle y, \psi_i \rangle \psi_i(t) \\
&= s_i(t) + n(t) - \sum_{j=1}^{K} \langle s_i + n, \psi_j \rangle \psi_j(t) \\
&= n(t) - \sum_{j=1}^{K} \langle n, \psi_j \rangle \psi_j(t) \\
&= n^{\perp}(t)
\end{aligned}
$$

where $n^{\perp}(t)$ is the component of $n(t)$ orthogonal to the signal space.
$n^{\perp}(t)$ is independent of which $s_i(t)$ was transmitted

Proof of Theorem

To prove $y^{\perp}(t)$ is irrelevant, it is enough to show that $n^{\perp}(t)$ is independent of $\mathbf{Y}$. For a given t and k

$$
\begin{aligned}
\mathrm{cov}(n^{\perp}(t), N_k) &= E[n^{\perp}(t)N_k] \\
&= E\left[\left\{n(t) - \sum_{j=1}^{n} N_j\psi_j(t)\right\} N_k\right] \\
&= E[n(t)N_k] - \sum_{j=1}^{K} E[N_jN_k]\psi_j(t) \\
&= \sigma^2\psi_k(t) - \sigma^2\psi_k(t) = 0
\end{aligned}
$$

M-ary Signaling in AWGN Channel

- M hypotheses with prior probabilities π_i, $i = 1, \ldots, M$

$$
\begin{aligned}
H_1 &: \quad \mathbf{Y} = \mathbf{s}_1 + \mathbf{N} \\
&\vdots \\
H_M &: \quad \mathbf{Y} = \mathbf{s}_M + \mathbf{N}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Y} &= \begin{bmatrix} \langle y, \psi_1 \rangle & \cdots & \langle y, \psi_K \rangle \end{bmatrix}^T \\
\mathbf{s}_i &= \begin{bmatrix} \langle s_i, \psi_1 \rangle & \cdots & \langle s_i, \psi_K \rangle \end{bmatrix}^T \\
\mathbf{N} &= \begin{bmatrix} \langle n, \psi_1 \rangle & \cdots & \langle n, \psi_K \rangle \end{bmatrix}^T
\end{aligned}
$$

- $\mathbf{N} \sim N(\mathbf{m}, \mathbf{C})$ where $\mathbf{m} = \mathbf{0}$ and $\mathbf{C} = \sigma^2 \mathbf{I}$

$$
\operatorname{cov}\left(\langle n, \psi_1 \rangle, \langle n, \psi_2 \rangle\right) = \sigma^2 \langle \psi_1, \psi_2 \rangle.
$$

Optimal Receiver for the AWGN Channel

Theorem (MPE Decision Rule)

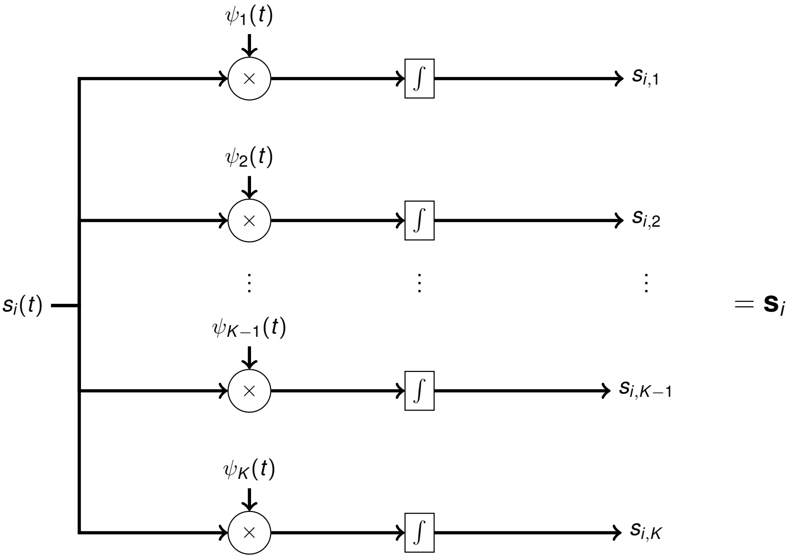
The MPE decision rule for M-ary signaling in AWGN channel is given by

$$\begin{aligned}\delta_{MPE}(\mathbf{y}) &= \arg\min_{1\le i\le M} \|\mathbf{y}-\mathbf{s}_i\|^2 - 2\sigma^2 \log \pi_i \\ &= \arg\max_{1\le i\le M} \langle \mathbf{y}, \mathbf{s}_i\rangle - \frac{\|\mathbf{s}_i\|^2}{2} + \sigma^2 \log \pi_i\end{aligned}$$

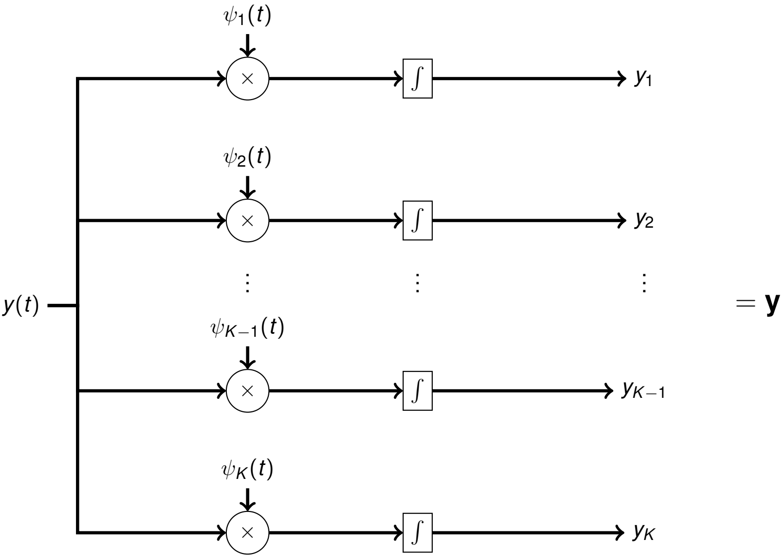
Proof

$$\begin{aligned}\delta_{MPE}(\mathbf{y}) &= \arg\max_{1\le i\le M} \pi_i p_i(\mathbf{y}) \\ &= \arg\max_{1\le i\le M} \pi_i \exp\left(-\frac{\|\mathbf{y}-\mathbf{s}_i\|^2}{2\sigma^2}\right)\end{aligned}$$

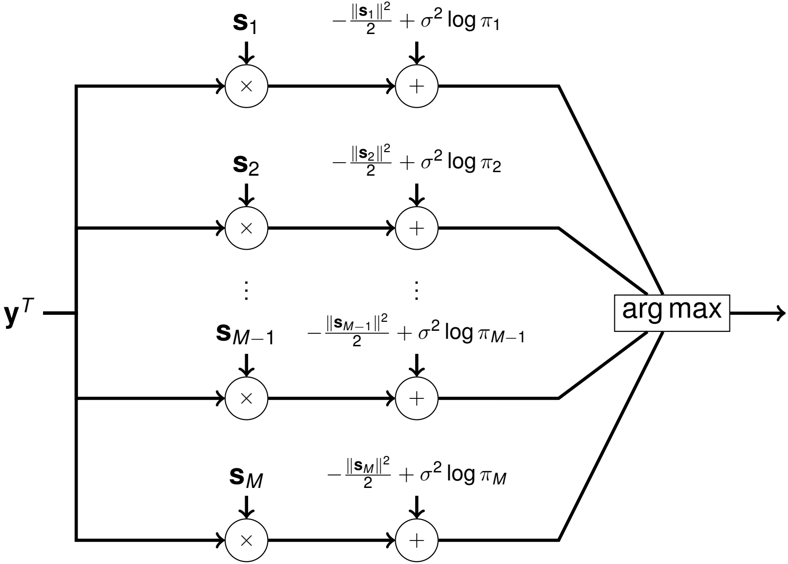
Vector Representation of Real Signal Waveforms



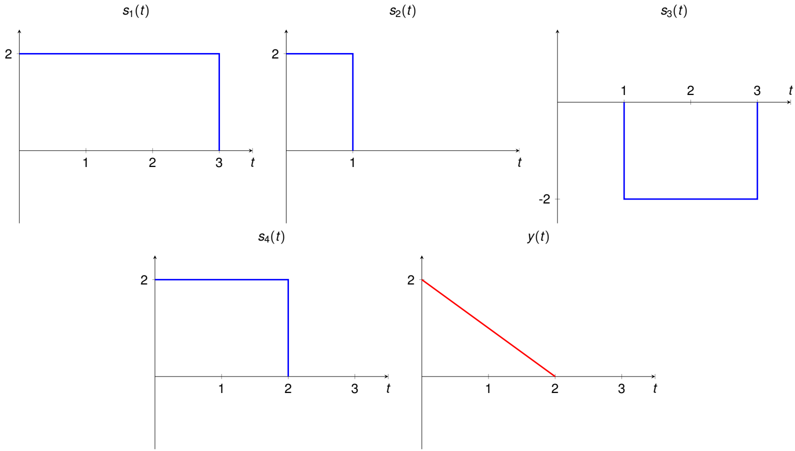
Vector Representation of the Real Received Signal



MPE Decision Rule



MPE Decision Rule Example



$$\text{Let } \pi_1 = \pi_2 = \frac{1}{3}, \pi_3 = \pi_4 = \frac{1}{6}, \sigma^2 = 1, \text{ and } \log 2 = 0.69.$$

ML Receiver for the AWGN Channel

Theorem (ML Decision Rule)

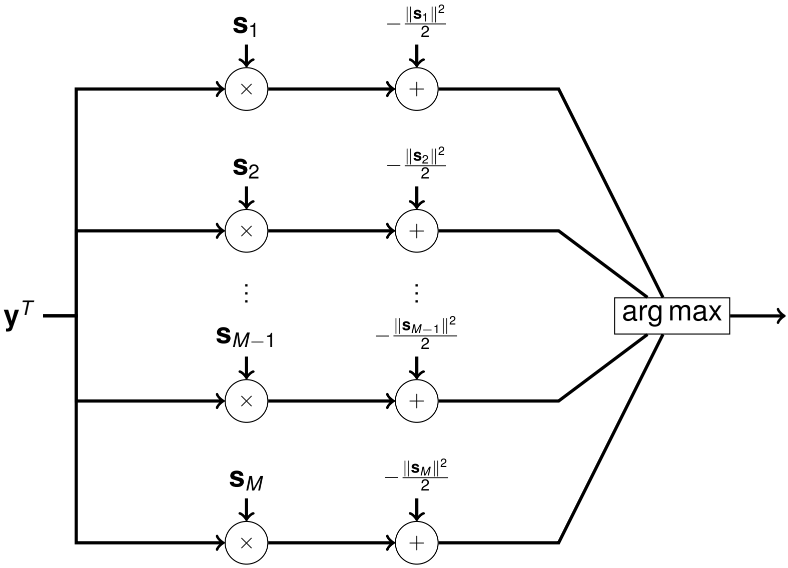
The ML decision rule for M-ary signaling in AWGN channel is given by

$$\begin{aligned}\delta_{ML}(\mathbf{y}) &= \arg\min_{1\le i\le M} \|\mathbf{y}-\mathbf{s}_i\|^2 \\ &= \arg\max_{1\le i\le M} \langle \mathbf{y}, \mathbf{s}_i\rangle - \frac{\|\mathbf{s}_i\|^2}{2}\end{aligned}$$

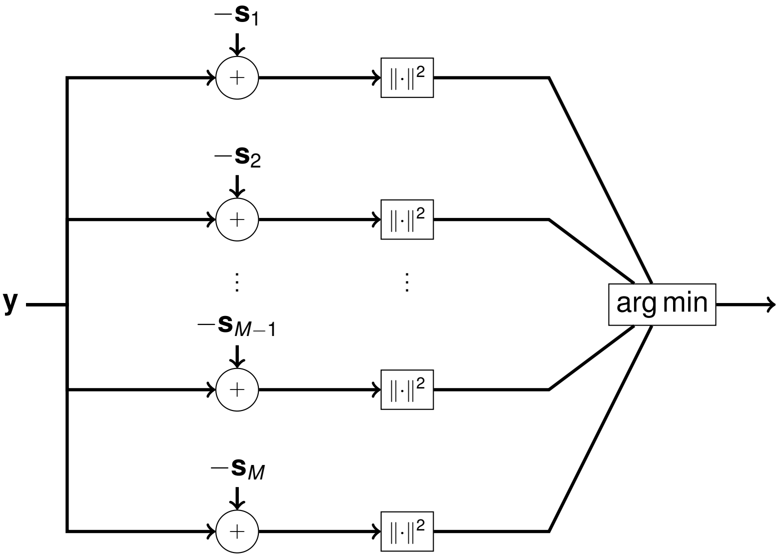
Proof

$$\begin{aligned}\delta_{ML}(\mathbf{y}) &= \arg\max_{1\le i\le M} p_i(\mathbf{y}) \\ &= \arg\max_{1\le i\le M} \exp\left(-\frac{\|\mathbf{y}-\mathbf{s}_i\|^2}{2\sigma^2}\right)\end{aligned}$$

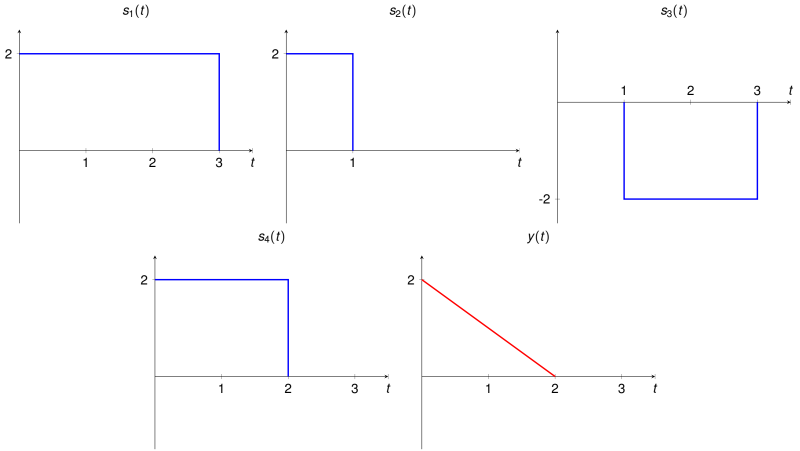
ML Decision Rule



ML Decision Rule



ML Decision Rule Example



Continuous-Time Versions of Optimal Decision Rules

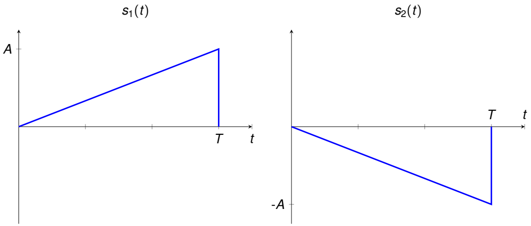
- Discrete-time decision rules

$$\begin{aligned}
\delta_{MPE}(\mathbf{y}) &= \arg\max_{1\le i\le M} \langle \mathbf{y}, \mathbf{s}_i \rangle - \frac{\|\mathbf{s}_i\|^2}{2} + \sigma^2 \log \pi_i \\
\delta_{ML}(\mathbf{y}) &= \arg\max_{1\le i\le M} \langle \mathbf{y}, \mathbf{s}_i \rangle - \frac{\|\mathbf{s}_i\|^2}{2}
\end{aligned}$$

- Continuous-time decision rules

$$\begin{aligned}
\delta_{MPE}(y) &= \arg\max_{1\le i\le M} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2} + \sigma^2 \log \pi_i \\
\delta_{ML}(y) &= \arg\max_{1\le i\le M} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2}
\end{aligned}$$

ML Decision Rule for Antipodal Signaling



$$\delta_{ML}(y) = \arg\max_{1 \le i \le 2} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2} = \arg\max_{1 \le i \le 2} \langle y, s_i \rangle$$

$$\delta_{ML}(y) = 1 \iff \langle y, s_1 \rangle \ge \langle y, s_2 \rangle \iff \langle y, s_1 \rangle \ge 0$$

$$\langle y, s_1 \rangle = \int_0^T y(\tau) s_1(\tau) \, d\tau = y \star s_{MF}(T)$$

where $s_{MF}(t) = s_1(T - t)$ is the matched filter.

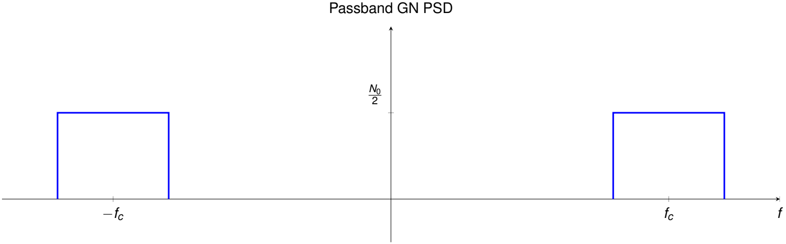
Optimal Receiver for Passband Signals

Consider M-ary passband signaling over the AWGN channel

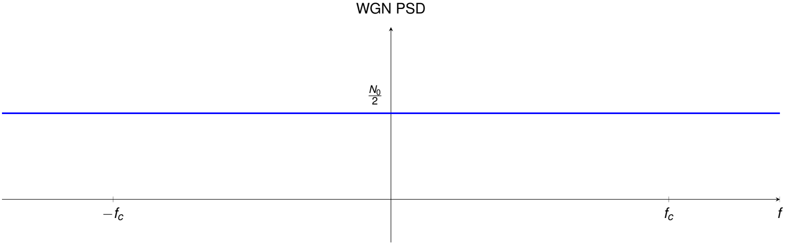
$$H_i : y_p(t) = s_{i,p}(t) + n_p(t), \quad i = 1, \dots, M$$

where

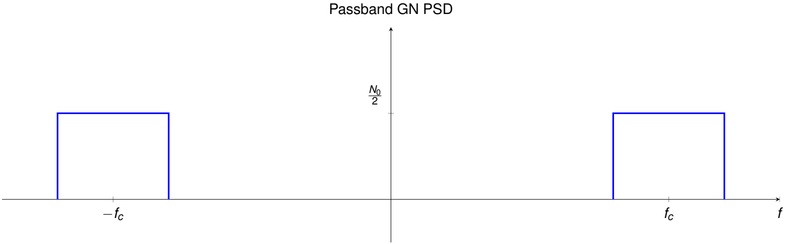
$y_p(t)$	Real passband received signal
$s_{i,p}(t)$	Real passband signals
$n_p(t)$	Real passband GN with PSD $\frac{N_0}{2}$



White Gaussian Noise is an Idealization



Infinite Power! Ideal model of passband Gaussian noise



Detection using Complex Baseband Representation

- M-ary passband signaling over the AWGN channel

$$H_i : y_p(t) = s_{i,p}(t) + n_p(t), \quad i = 1, \dots, M$$

where

$y_p(t)$ Real passband received signal
$s_{i,p}(t)$ Real passband signals
$n_p(t)$ Real passband GN with PSD $\frac{N_0}{2}$

- The equivalent problem in complex baseband is

$$H_i : y(t) = s_i(t) + n(t), \quad i = 1, \dots, M$$

where

$y(t)$ Complex envelope of $y_p(t)$
$s_i(t)$ Complex envelope of $s_{i,p}(t)$
$n(t)$ Complex envelope of $n_p(t)$

Complex Envelope of Passband Signals (Recap)

- Frequency Domain Representation

$$S(f) = \sqrt{2}S_p^+(f + f_c) = \sqrt{2}S_p(f + f_c)u(f + f_c)$$

- Time Domain Representation of Positive Spectrum

$$s_p^+(t) = s_p(t) \star \left[\frac{1}{2}\delta(t) + \frac{j}{2\pi t}\right] = \frac{1}{2}\left[s_p(t) + j\hat{s}_p(t)\right]$$

where $\hat{s}_p(t) = s_p(t) \star \frac{1}{\pi t}$ is the Hilbert transform of $s_p(t)$

- Time Domain Representation of Complex Envelope

$$\sqrt{2}S_p(f + f_c)u(f + f_c) \rightleftharpoons \frac{1}{\sqrt{2}}\left[s_p(t) + j\hat{s}_p(t)\right]e^{-j2\pi f_c t}$$

$$s(t) = \frac{1}{\sqrt{2}}\left[s_p(t) + j\hat{s}_p(t)\right]e^{-j2\pi f_c t}$$

Complex Envelope of Passband Signals (Recap)

- Complex Envelope

$$s(t) = s_c(t) + js_s(t)$$

$s_c(t)$ In-phase component
$s_s(t)$ Quadrature component

- Time domain relationship between $s(t)$ and $s_p(t)$

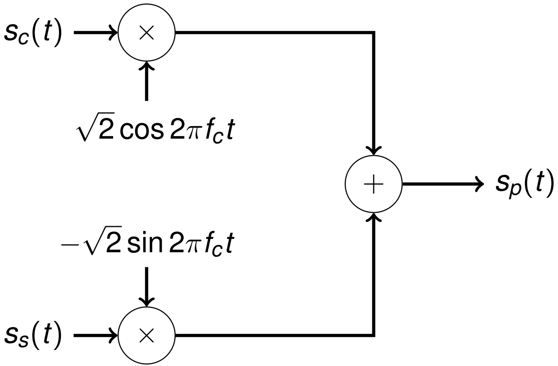
$$\begin{aligned} s_p(t) &= \text{Re}\left[\sqrt{2}s(t)e^{j2\pi f_c t}\right] \\ &= \sqrt{2}s_c(t)\cos 2\pi f_c t - \sqrt{2}s_s(t)\sin 2\pi f_c t \end{aligned}$$

- Frequency domain relationship between $s(t)$ and $s_p(t)$

$$S_p(f) = \frac{S(f - f_c) + S^*(-f - f_c)}{\sqrt{2}}$$

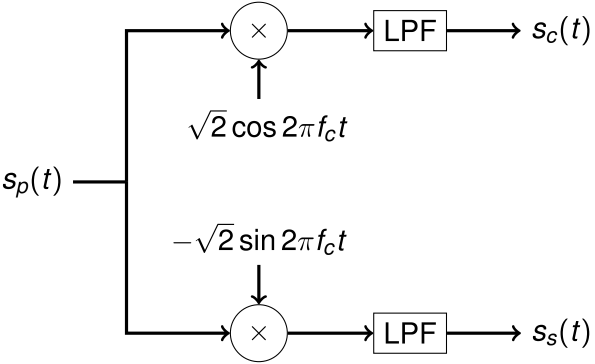
Upconversion (Recap)

$$s_p(t) = \sqrt{2} s_c(t) \cos 2\pi f_c t - \sqrt{2} s_s(t) \sin 2\pi f_c t$$



Downconversion (Recap)

$$\begin{aligned}
&\sqrt{2}s_p(t)\cos 2\pi f_c t \\
&= 2s_c(t)\cos^2 2\pi f_c t - 2s_s(t)\sin 2\pi f_c t \cos 2\pi f_c t \\
&= s_c(t) + s_c(t)\cos 4\pi f_c t - s_s(t)\sin 4\pi f_c t
\end{aligned}$$



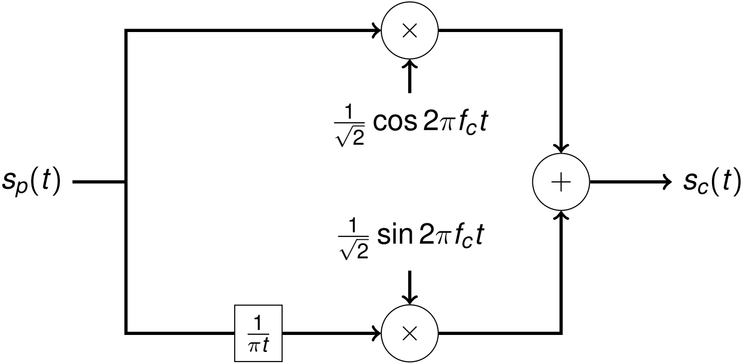
Downconversion (Alternative)

$$
\begin{aligned}
s(t) &= \frac{1}{\sqrt{2}} \left[s_p(t) + j\hat{s}_p(t) \right] e^{-j2\pi f_c t} \\
s_c(t) + js_s(t) &= \frac{1}{\sqrt{2}} \left[s_p(t) + j\hat{s}_p(t) \right] e^{-j2\pi f_c t} \\
s_c(t) &= \frac{1}{\sqrt{2}} \left[s_p(t) \cos 2\pi f_c t + \hat{s}_p(t) \sin 2\pi f_c t \right] \\
s_s(t) &= \frac{1}{\sqrt{2}} \left[\hat{s}_p(t) \cos 2\pi f_c t - s_p(t) \sin 2\pi f_c t \right]
\end{aligned}
$$

Downconversion (Alternative)

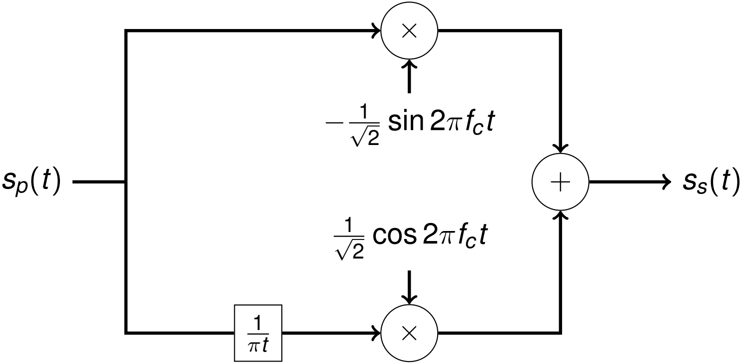
$$s_c(t) = \frac{1}{\sqrt{2}}\left[s_p(t)\cos 2\pi f_c t + \hat{s}_p(t)\sin 2\pi f_c t\right]$$

$$s_s(t) = \frac{1}{\sqrt{2}}\left[\hat{s}_p(t)\cos 2\pi f_c t - s_p(t)\sin 2\pi f_c t\right]$$

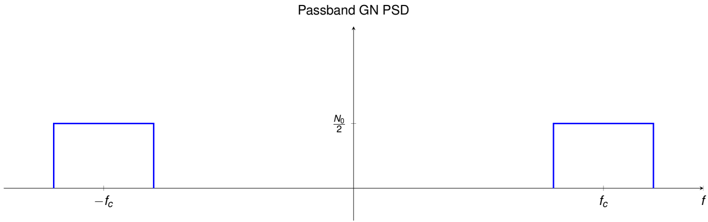


Downconversion (Alternative)

$$
\begin{aligned}
s_c(t) &= \frac{1}{\sqrt{2}}\left[s_p(t)\cos 2\pi f_c t + \hat{s}_p(t)\sin 2\pi f_c t\right] \\
s_s(t) &= \frac{1}{\sqrt{2}}\left[\hat{s}_p(t)\cos 2\pi f_c t - s_p(t)\sin 2\pi f_c t\right]
\end{aligned}
$$



What is the Complex Envelope of Passband GN?



How to characterize $n_c(t)$ and $n_s(t)$ where

$$n_c(t) + jn_s(t) = \frac{1}{\sqrt{2}} \left[n_p(t) + j\hat{n}_p(t) \right] e^{-j2\pi f_c t}$$

Complex Envelope PSD for Passband Random Processes

- Let $S_p(f)$ be the PSD of a passband random process and let $S(f)$ be the PSD of its complex envelope
- $S_p(f)$ in terms of $S(f)$

$$S_p(f) = \frac{S(f - f_c) + S(-f - f_c)}{2}$$

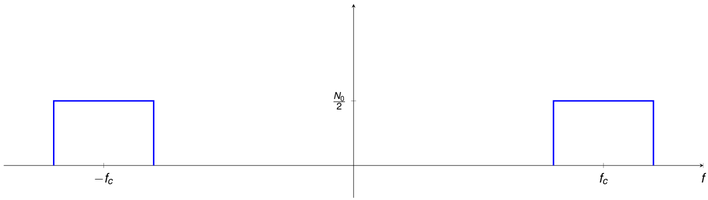
- $S(f)$ in terms of $S_p(f)$

$$S(f) = 2S_p(f + f_c)u(f + f_c)$$

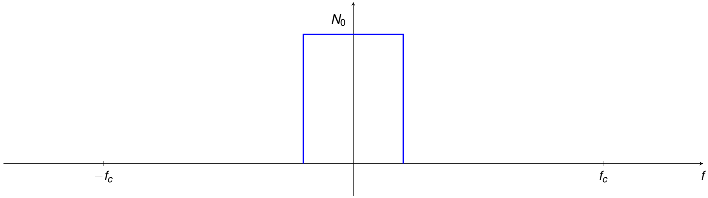
- See explanation in Section 2.3.1 of Madhow's textbook

PSD of the Complex Envelope of Passband GN

Passband GN PSD



GN Complex Envelope PSD



But we need to characterize $n_c(t)$ and $n_s(t)$ where $n(t) = n_c(t) + jn_s(t)$ is the complex envelope of passband GN.

Characterizing the Complex Envelope of a Passband Random Process

- Passband Random Process: A real, zero-mean, WSS random process whose autocorrelation function is passband
- The in-phase and quadrature components of a passband random process $X_p(t)$ are given by

$$
\begin{aligned}
X_c(t) &= \frac{1}{\sqrt{2}}\left[X_p(t)\cos 2\pi f_c t + \hat{X}_p(t)\sin 2\pi f_c t\right] \\
X_s(t) &= \frac{1}{\sqrt{2}}\left[\hat{X}_p(t)\cos 2\pi f_c t - X_p(t)\sin 2\pi f_c t\right]
\end{aligned}
$$

- The complex envelope of $X_p(t)$ is given by

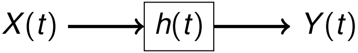
$$
X(t) = X_c(t) + jX_s(t)
$$

Characterizing the In-phase Component

$$
\begin{aligned}
R_{X_c}(t+\tau, t) &= E\left[X_c(t+\tau)X_c(t)\right] \\
&= \frac{1}{2}R_{X_p}(\tau)\cos 2\pi f_c(t+\tau)\cos 2\pi f_c t + \\
&\quad \frac{1}{2}R_{\hat{X}_p}(\tau)\sin 2\pi f_c(t+\tau)\sin 2\pi f_c t + \\
&\quad \frac{1}{2}R_{X_p\hat{X}_p}(\tau)\cos 2\pi f_c(t+\tau)\sin 2\pi f_c t + \\
&\quad \frac{1}{2}R_{\hat{X}_p X_p}(\tau)\sin 2\pi f_c(t+\tau)\cos 2\pi f_c t
\end{aligned}
$$

LTI Filtering of a WSS Process (Cheatsheet)

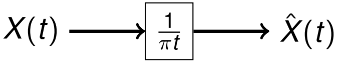
$X(t)$ is a WSS process and $h(t)$ is the impulse reponse of an LTI system



$X(t)$ and $Y(t)$ are jointly WSS and the following relations hold

$$
\begin{aligned}
m_Y &= m_X \int_{-\infty}^{\infty} h(t)\, dt \\
R_{XY}(\tau) &= R_X(\tau) \star h^*(-\tau) \\
R_Y(\tau) &= R_X(\tau) \star h(\tau) \star h^*(-\tau) \\
S_{XY}(f) &= S_X(f) H^*(f) \\
S_Y(f) &= S_X(f) |H(f)|^2
\end{aligned}
$$

A Zero Mean WSS Process and its Hilbert Transform



$X(t)$ and $\hat{X}(t)$ are jointly WSS and the following relations hold

$$
\begin{aligned}
m_{\hat{X}} &= m_X \int_{-\infty}^{\infty} h(t)\, dt = 0 \\
R_{X\hat{X}}(\tau) &= R_X(\tau) \star h^*(-\tau) = -\hat{R}_X(\tau) \\
R_{\hat{X}}(\tau) &= R_X(\tau) \star h(\tau) \star h^*(-\tau) = R_X(\tau) \\
R_{\hat{X}X}(\tau) &= R_{\hat{X}}(\tau) \star [-h^*(-\tau)] = \hat{R}_X(\tau)
\end{aligned}
$$

Back to Characterizing the In-phase Component

$$
\begin{aligned}
R_{X_p \hat{X}_p}(\tau) &= -\hat{R}_{X_p}(\tau) \\
R_{\hat{X}_p}(\tau) &= R_{X_p}(\tau) \\
R_{\hat{X}_p X_p}(\tau) &= \hat{R}_{X_p}(\tau)
\end{aligned}
$$

$$
\begin{aligned}
R_{X_c}(t+\tau, t) &= \frac{1}{2} R_{X_p}(\tau) \cos 2\pi f_c (t+\tau) \cos 2\pi f_c t + \\
&\quad \frac{1}{2} R_{\hat{X}_p}(\tau) \sin 2\pi f_c (t+\tau) \sin 2\pi f_c t + \\
&\quad \frac{1}{2} R_{X_p \hat{X}_p}(\tau) \cos 2\pi f_c (t+\tau) \sin 2\pi f_c t + \\
&\quad \frac{1}{2} R_{\hat{X}_p X_p}(\tau) \sin 2\pi f_c (t+\tau) \cos 2\pi f_c t \\
&= \frac{1}{2} \left[R_{X_p}(\tau) \cos 2\pi f_c \tau + \hat{R}_{X_p}(\tau) \sin 2\pi f_c \tau \right]
\end{aligned}
$$

Characterizing both the Components

Autocorrelations and Crosscorrelations

$$
\begin{aligned}
R_{X_c}(\tau) &= \frac{1}{2}\left[R_{X_p}(\tau)\cos 2\pi f_c\tau + \hat{R}_{X_p}(\tau)\sin 2\pi f_c\tau\right] \\
R_{X_s}(\tau) &= R_{X_c}(\tau) \\
R_{X_cX_s}(\tau) &= \frac{1}{2}\left[R_{X_p}(\tau)\sin 2\pi f_c\tau - \hat{R}_{X_p}(\tau)\cos 2\pi f_c\tau\right] \\
R_{X_cX_s}(\tau) &= -R_{X_sX_c}(\tau)
\end{aligned}
$$

To derive the PSDs we will use the following

$$
\begin{aligned}
R_{X_p}(\tau) &\rightleftharpoons S_{X_p}(f) \\
\hat{R}_{X_p}(\tau) &\rightleftharpoons -j\mathrm{sgn}(f)S_{X_p}(f)
\end{aligned}
$$

Characterizing both the Components

- In-phase PSD

$$S_{X_c}(f) = \begin{cases} \frac{1}{2}\left[S_{X_p}(f - f_c) + S_{X_p}(f + f_c)\right] & |f| < f_c \\ 0 & \text{otherwise} \end{cases}$$

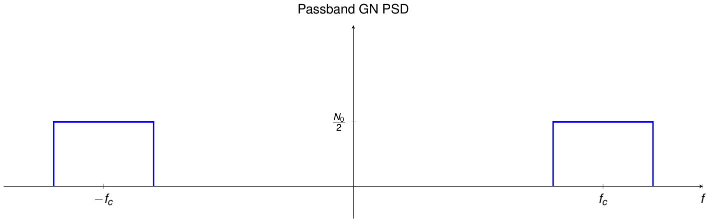
- Quadrature PSD: $S_{X_s}(f) = S_{X_c}(f)$
- Fourier transform of crosscorrelation functions

$$S_{X_c X_s}(f) = \begin{cases} \frac{j}{2}\left[S_{X_p}(f - f_c) - S_{X_p}(f + f_c)\right] & |f| < f_c \\ 0 & \text{otherwise} \end{cases}$$

$$S_{X_s X_c}(f) = -S_{X_c X_s}(f)$$

- If $S_{X_p}(f - f_c) = S_{X_p}(f + f_c)$ for $|f| < f_c$, $R_{X_c X_s}(\tau) = 0$
- Passband RPs with PSDs satisfying above condition have uncorrelated in-phase and quadrature components

Back to the Complex Envelope of Passband GN



- In-phase component PSD

$$S_{n_c}(f) = \begin{cases} \frac{N_0}{2} & |f| < W < f_c \\ 0 & \text{otherwise} \end{cases}$$

- Quadrature component PSD: $S_{n_s}(f) = S_{n_c}(f)$
- Since $S_{n_p}(f - f_c) = S_{n_p}(f + f_c)$ for $|f| < f_c$, the components are uncorrelated
- By joint Gaussianity, the components are independent random processes

Back to Optimal Detection in Complex Baseband

- The continuous time hypothesis testing problem in complex baseband

$$H_i : y(t) = s_i(t) + n(t), \quad i = 1, \dots, M$$

where

$$\begin{aligned} y(t) &\ \text{Complex envelope of } y_p(t) \\ s_i(t) &\ \text{Complex envelope of } s_{i,p}(t) \\ n(t) &\ \text{Complex envelope of } n_p(t) \end{aligned}$$

- The equivalent problem in terms of complex random vectors

$$H_i : \mathbf{Y} = \mathbf{s}_i + \mathbf{N}, \quad i = 1, \dots, M$$

where $\mathbf{Y}$, $\mathbf{s}_i$ and $\mathbf{N}$ are the projections of $y(t)$, $s_i(t)$ and $n(t)$ respectively onto the signal space spanned by $\{s_i(t)\}$.

- $\mathbf{N} \sim CN(\mathbf{m}, \mathbf{C_N})$ where $\mathbf{m} = \mathbf{0}$ and $\mathbf{C_N} = 2\sigma^2\mathbf{I}$

$$\text{cov}\left(\langle n, \psi_1 \rangle, \langle n, \psi_2 \rangle\right) = 2\sigma^2 \langle \psi_2, \psi_1 \rangle.$$

Autocorrelation of Complex White Gaussian Noise

$$
\begin{aligned}
E\left[n(t)n^*(s)\right] &= E\left[\left(n_c(t) + jn_s(t)\right)\left(n_c(s) - jn_s(s)\right)\right] \\
&= E\left[n_c(t)n_c(s) + n_s(t)n_s(s)\right. \\
&\quad \left. + j\left(n_s(t)n_c(s) - n_c(t)n_s(s)\right)\right] \\
&= E\left[n_c(t)n_c(s)\right] + E\left[n_s(t)n_s(s)\right] \\
&\quad + j\left(E\left[n_s(t)n_c(s)\right] - E\left[n_c(t)n_s(s)\right]\right) \\
&= E\left[n_c(t)n_c(s)\right] + E\left[n_s(t)n_s(s)\right] \\
&\quad + j\left(E\left[n_s(t)\right]E\left[n_c(s)\right] - E\left[n_c(t)\right]E\left[n_s(s)\right]\right) \\
&= 2\sigma^2\delta(t-s)
\end{aligned}
$$

Complex White Gaussian Noise through Correlators

$$
\begin{aligned}
\operatorname{cov}(\langle n, \psi_1\rangle, \langle n, \psi_2\rangle) &= E\left[\langle n, \psi_1\rangle (\langle n, \psi_2\rangle)^*\right] \\
&= E\left[\int n(t)\psi_1^*(t)\, dt \int n^*(s)\psi_2(s)\, ds\right] \\
&= \int\int \psi_2(t)\psi_2^*(s) E\left[n(t)n^*(s)\right]\, dt\, ds \\
&= \int\int \psi_2(t)\psi_1^*(s) 2\sigma^2 \delta(t-s)\, dt\, ds \\
&= 2\sigma^2 \int \psi_2(t)\psi_1^*(t)\, dt \\
&= 2\sigma^2 \langle \psi_2, \psi_1\rangle
\end{aligned}
$$

If $u_1(t)$ and $u_2(t)$ are orthogonal, $\langle n, u_1\rangle$ and $\langle n, u_2\rangle$ are independent.

MPE and ML Rules in Complex Baseband

- The pdf of the observation under H_i

$$
\begin{aligned}
p_i(\mathbf{y}) &= \frac{1}{\pi^K \det(\mathbf{C_N})} \exp\left(-(\mathbf{y}-\mathbf{s_i})^H \mathbf{C_N}^{-1} (\mathbf{y}-\mathbf{s_i})\right) \\
&= \frac{1}{(2\pi\sigma^2)^K} \exp\left(-\frac{\|\mathbf{y}-\mathbf{s}_i\|^2}{2\sigma^2}\right)
\end{aligned}
$$

- The MPE rule is given by

$$
\begin{aligned}
\delta_{MPE}(\mathbf{y}) &= \arg\max_{1\le i\le M} \mathrm{Re}\left(\langle \mathbf{y}, \mathbf{s}_i \rangle\right) - \frac{\|\mathbf{s}_i\|^2}{2} + \sigma^2 \log \pi_i \\
&= \arg\max_{1\le i\le M} \mathrm{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2} + \sigma^2 \log \pi_i
\end{aligned}
$$

- The ML rule is given by

$$
\delta_{ML}(\mathbf{y}) = \arg\max_{1\le i\le M} \mathrm{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2}
$$

ML Receiver for QPSK in Passband Gaussian Noise

QPSK signals where $p(t)$ is a real baseband pulse, A is a real number and $1 \leq m \leq 4$

$$
\begin{aligned}
s_m^p(t) &= \sqrt{2}Ap(t)\cos\left(2\pi f_c t + \frac{\pi(2m-1)}{4}\right) \\
&= \mathrm{Re}\left[\sqrt{2}Ap(t)e^{j\left(2\pi f_c t + \frac{\pi(2m-1)}{4}\right)}\right]
\end{aligned}
$$

Complex Envelope of QPSK Signals

$$
s_m(t) = Ap(t)e^{j\frac{\pi(2m-1)}{4}}, \quad 1 \leq m \leq 4
$$

Orthonormal basis for the complex envelope consists of only

$$
\phi(t) = \frac{p(t)}{\sqrt{E_p}}
$$

ML Receiver for QPSK in Passband Gaussian Noise

Let $\sqrt{E_b} = \frac{A\sqrt{E_p}}{\sqrt{2}}$. The vector representation of the QPSK signals is

$$\begin{aligned}
\mathbf{s_1} &= \sqrt{E_b} + j\sqrt{E_b} \\
\mathbf{s_2} &= -\sqrt{E_b} + j\sqrt{E_b} \\
\mathbf{s_3} &= -\sqrt{E_b} - j\sqrt{E_b} \\
\mathbf{s_4} &= \sqrt{E_b} - j\sqrt{E_b}
\end{aligned}$$

The hypothesis testing problem in terms of vectors is

$$H_i : \mathbf{Y} = \mathbf{s}_i + \mathbf{N}, \quad i = 1, \ldots, M$$

where

$$\mathbf{N} \sim CN\left(0, 2\sigma^2\right)$$

Thanks for your attention