EE 703: Digital Message Transmission

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Midsem Exam: **30 points** Date: September 10, 2013

Each question is worth 5 points.

1. Let $u_p(t)$ and $v_p(t)$ be passband signals centered at the same carrier frequency f_c. Let $u(t) = u_c(t) + ju_s(t)$ and $v(t) = v_c(t) + jv_s(t)$ be the complex baseband representations of $u_p(t)$ and $v_p(t)$ respectively. Prove that

$$\langle u_p, v_p \rangle = \langle u_c, v_c \rangle + \langle u_s, v_s \rangle = \text{Re}\left(\langle u, v \rangle\right).$$

2. Suppose the Gram-Schmidt orthogonalization procedure is performed on a set of M signals $s_1(t), \dots, s_M(t)$ to obtain an orthonormal basis $\phi_1(t), \dots, \phi_N(t)$.

   (a) In which situation is N equal to one? Give a condition on the signals $s_i(t), 1 \le i \le M$.

   (b) In which situation is N equal to M? Give a condition on the signals $s_i(t), 1 \le i \le M$.

   (c) In which situation is N equal to $M - 1$? Give a condition on the signals $s_i(t), 1 \le i \le M$.

   (d) If $\sum_{i=1}^{M} s_i(t) = 0$ for all t, what can you say about the orthonormal basis $\phi_1(t), \dots, \phi_N(t)$?

   (e) Can $\sum_{i=1}^{N} \phi_i(t)$ be equal to zero for all t? Explain why or why not.

3. Determine the power spectral density of the following line coding scheme:

$$u(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

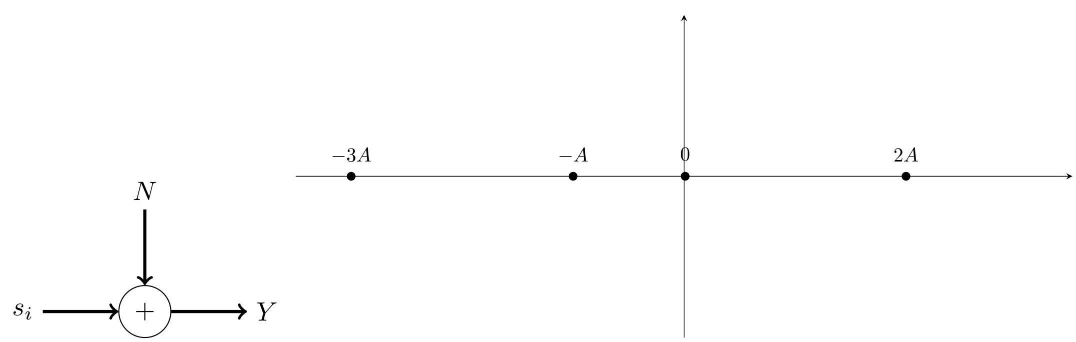
   where $p(t) = I_{[0,T)}(t)$ and the symbol b_n is the obtained by mapping a zero bit to amplitude $-A$ and mapping a one bit to amplitude $2A$. Assume that the bits used to generate b_n are independent and equally likely to be zero or one. Simplify your answer such that it does not contain any infinite summations. The formula for the PSD is as follows.

$$S_u(f) = \frac{|P(f)|^2}{T} \sum_{k=-\infty}^{\infty} R_b[k] e^{-j2\pi kfT}$$

4. The constellation $s_0 = -3A, s_1 = -A, s_2 = 0, s_3 = 2A$ is corrupted by noise N which is a zero mean Gaussian random variable having variance σ^2. Assume all four constellation points are equally likely to be transmitted.

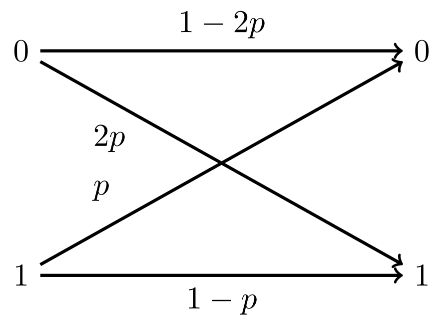
   (a) Find the optimal decision rule based on the observation Y.

   (b) Find the average probability of decision error for the optimal decision rule. Express your final answer in terms of the Q function.



5. Suppose an encoder maps a 0 bit to a binary codeword $\mathbf{v}_0$ of length n and maps a 1 bit to a binary codeword $\mathbf{v}_1$ of length n. The codewords are passed through a binary symmetric channel with crossover probability p. Suppose $\mathbf{r}$ is the received word corresponding to a single transmitted codeword. If $\mathbf{v}_0$ and $\mathbf{v}_1$ share the same prefix[1] of length $k < n$, show that the optimal decoder can ignore the first k bits in the received word $\mathbf{r}$. Assume that the probability of a 0 bit appearing at the input to the encoder is π_0 and the probability of a 1 bit appearing at the input to the encoder is π_1.

6. Suppose the input to the following binary channel is equally likely to be 0 or 1. Assuming $0 \leq p < \frac{1}{2}$, derive the minimum probability of decision error which can be achieved for this channel as a function of p.



[1] For instance, codewords 01011 and 01001 share a prefix of length 3