Circularly Symmetric Gaussian Random Vectors

Saravanan Vijayakumaran
sarva@ee.iitb.ac.in

Department of Electrical Engineering
Indian Institute of Technology Bombay

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Definition (Complex Gaussian Random Variable)

If X and Y are jointly Gaussian random variables, $Z = X + jY$ is a complex Gaussian random variable.

Definition (Complex Gaussian Random Vector)

If $\mathbf{X}$ and $\mathbf{Y}$ are jointly Gaussian random vectors, $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ is a complex Gaussian random vector.

Definition (Circularly Symmetric Gaussian RV)

A complex Gaussian random vector $\mathbf{Z}$ is circularly symmetric if $e^{j\phi}\mathbf{Z}$ has the same distribution as $\mathbf{Z}$ for all real ϕ.

If $\mathbf{Z}$ is circularly symmetric, then

- $E[\mathbf{Z}] = E[e^{j\phi}\mathbf{Z}] = e^{j\phi}E[\mathbf{Z}] \implies E[\mathbf{Z}] = 0$.
- $\text{cov}(e^{j\phi}\mathbf{Z}) = E[e^{j\phi}\mathbf{Z}e^{-j\phi}\mathbf{Z}^H] = \text{cov}(\mathbf{Z})$. (See footnote)
- Define pseudocovariance of $\mathbf{Z}$ as $E[\mathbf{Z}\mathbf{Z}^T]$.

$$E[\mathbf{Z}\mathbf{Z}^T] = E[e^{j\phi}\mathbf{Z}e^{j\phi}\mathbf{Z}^T] = e^{2j\phi}E[\mathbf{Z}\mathbf{Z}^T] \implies E[\mathbf{Z}\mathbf{Z}^T] = \mathbf{0}$$

$\text{cov}(e^{j\phi}\mathbf{Z}) = \text{cov}(\mathbf{Z})$ holds for any zero mean $\mathbf{Z}$ (even non-circulary symmetric $\mathbf{Z}$)

Circular Symmetry for Random Variables

- Consider a circularly symmetric complex Gaussian RV $Z = X + jY$
- $E[Z] = 0 \implies E[X] = 0$ and $E[Y] = 0$
- Pseudocovariance zero $\implies E[ZZ^T] = E[Z^2] = 0 \implies$ $\text{var}(X) = \text{var}(Y)$, $\text{cov}(X, Y) = 0$
- If Z is a circularly symmetric complex Gaussian random variable, its real and imaginary parts are independent and have equal variance

Complex Gaussian Random Vector PDF

- The pdf of a complex random vector $\mathbf{Z}$ is the joint pdf of its real and imaginary parts i.e. the pdf of

$$\tilde{\mathbf{Z}} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}$$

- For a complex Gaussian random vector, the pdf is given by

$$p(\mathbf{z}) = p(\tilde{\mathbf{z}}) = \frac{1}{(2\pi)^n (\det(\mathbf{C}_{\tilde{\mathbf{Z}}}))^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\mathbf{m}})^T \mathbf{C}_{\tilde{\mathbf{Z}}}^{-1}(\tilde{\mathbf{z}} - \tilde{\mathbf{m}})\right)$$

where $\tilde{\mathbf{m}} = E[\tilde{\mathbf{Z}}]$ and

$$\mathbf{C}_{\tilde{\mathbf{Z}}} = \begin{bmatrix} \mathbf{C}_{\mathbf{X}} & \mathbf{C}_{\mathbf{XY}} \\ \mathbf{C}_{\mathbf{YX}} & \mathbf{C}_{\mathbf{Y}} \end{bmatrix}$$

$$\begin{aligned} \mathbf{C}_{\mathbf{X}} &= E\left[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T\right] \\ \mathbf{C}_{\mathbf{Y}} &= E\left[(\mathbf{Y} - E[\mathbf{Y}])(\mathbf{Y} - E[\mathbf{Y}])^T\right] \\ \mathbf{C}_{\mathbf{XY}} &= E\left[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^T\right] \\ \mathbf{C}_{\mathbf{YX}} &= E\left[(\mathbf{Y} - E[\mathbf{Y}])(\mathbf{X} - E[\mathbf{X}])^T\right] \end{aligned}$$

Circularly Symmetry and Pseudocovariance

- Covariance of $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$

$$\mathbf{C_Z} = E\left[(\mathbf{Z} - E[\mathbf{Z}])(\mathbf{Z} - E[\mathbf{Z}])^H\right] = \mathbf{C_X} + \mathbf{C_Y} + j\left(\mathbf{C_{YX}} - \mathbf{C_{XY}}\right)$$

- Pseudocovariance of $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ is

$$E\left[(\mathbf{Z} - E[\mathbf{Z}])(\mathbf{Z} - E[\mathbf{Z}])^T\right] = \mathbf{C_X} - \mathbf{C_Y} + j\left(\mathbf{C_{XY}} + \mathbf{C_{YX}}\right)$$

- Pseudocovariance is zero $\implies$
  - $\mathbf{C_X} = \mathbf{C_Y}, \mathbf{C_{XY}} = -\mathbf{C_{YX}}$
  - $\mathbf{C_Z} = 2\mathbf{C_X} + 2j\mathbf{C_{YX}}$
- The covariance of $\tilde{\mathbf{Z}} = \begin{bmatrix}\mathbf{X} & \mathbf{Y}\end{bmatrix}^T$ for zero pseudocovariance is

$$\mathbf{C_{\tilde{Z}}} = \begin{bmatrix}\mathbf{C_X} & \mathbf{C_{XY}} \\ \mathbf{C_{YX}} & \mathbf{C_Y}\end{bmatrix} = \begin{bmatrix}\mathbf{C_X} & -\mathbf{C_{YX}} \\ \mathbf{C_{YX}} & \mathbf{C_X}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\,\mathrm{Re}(\mathbf{C_Z}) & -\frac{1}{2}\,\mathrm{Im}(\mathbf{C_Z}) \\ \frac{1}{2}\,\mathrm{Im}(\mathbf{C_Z}) & \frac{1}{2}\,\mathrm{Re}(\mathbf{C_Z})\end{bmatrix}$$

Circularly Symmetry and Pseudocovariance

Theorem

A complex Gaussian vector is circularly symmetric if and only if its mean and pseudocovariance are zero.

Proof.

- The forward direction was shown in the first slide.
- For reverse direction, assume $\mathbf{Z}$ is a complex Gaussian vector with zero mean and zero pseudocovariance.
- The pdf of $\mathbf{Z}$ is the the pdf of $\tilde{\mathbf{Z}} = \begin{bmatrix} \mathbf{X} & \mathbf{Y} \end{bmatrix}^T$

$$p(\mathbf{z}) = p(\tilde{\mathbf{z}}) = \frac{1}{(2\pi)^n (\det(\mathbf{C}_{\tilde{\mathbf{Z}}}))^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\tilde{\mathbf{z}}^T \mathbf{C}_{\tilde{\mathbf{Z}}}^{-1} \tilde{\mathbf{z}}\right)$$

  where

$$\mathbf{C}_{\tilde{\mathbf{Z}}} = \begin{bmatrix} \frac{1}{2}\operatorname{Re}(\mathbf{C}_{\mathbf{Z}}) & -\frac{1}{2}\operatorname{Im}(\mathbf{C}_{\mathbf{Z}}) \\ \frac{1}{2}\operatorname{Im}(\mathbf{C}_{\mathbf{Z}}) & \frac{1}{2}\operatorname{Re}(\mathbf{C}_{\mathbf{Z}}) \end{bmatrix}$$

- We want to show that $e^{j\phi}\mathbf{Z}$ has the same pdf as $\mathbf{Z}$

Circularly Symmetry and Pseudocovariance

Proof Continued.

- $e^{j\phi}\mathbf{Z}$ has zero mean and zero pseudocovariance
- $\text{cov}(e^{j\phi}\mathbf{Z}) = \text{cov}(\mathbf{Z})$
- If $\mathbf{Z}$ is a complex Gaussian vector, $e^{j\phi}\mathbf{Z}$ is a complex Gaussian vector (Assignment 4)
- Let $\mathbf{U} = e^{j\phi}\mathbf{Z}$ and $\tilde{\mathbf{U}} = \begin{bmatrix}\text{Re}(\mathbf{U}) & \text{Im}(\mathbf{U})\end{bmatrix}^T$
- The pdf of $\mathbf{U}$ is the the pdf of $\tilde{\mathbf{U}}$

$$p(\mathbf{u}) = p(\tilde{\mathbf{u}}) = \frac{1}{(2\pi)^n(\det(\mathbf{C}_{\tilde{\mathbf{U}}}))^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\tilde{\mathbf{u}}^T\mathbf{C}_{\tilde{\mathbf{U}}}^{-1}\tilde{\mathbf{u}}\right)$$

where

$$\mathbf{C}_{\tilde{\mathbf{U}}} = \begin{bmatrix}\frac{1}{2}\text{Re}(\mathbf{C}_\mathbf{U}) & -\frac{1}{2}\text{Im}(\mathbf{C}_\mathbf{U}) \\ \frac{1}{2}\text{Im}(\mathbf{C}_\mathbf{U}) & \frac{1}{2}\text{Re}(\mathbf{C}_\mathbf{U})\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\text{Re}(\mathbf{C}_\mathbf{Z}) & -\frac{1}{2}\text{Im}(\mathbf{C}_\mathbf{Z}) \\ \frac{1}{2}\text{Im}(\mathbf{C}_\mathbf{Z}) & \frac{1}{2}\text{Re}(\mathbf{C}_\mathbf{Z})\end{bmatrix}$$

- $\mathbf{U}$ has the same pdf as $\mathbf{Z}$

□

PDF of Circularly Symmetric Gaussian Vectors

- The pdf of a complex Gaussian vector $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ is the pdf of $\tilde{\mathbf{Z}} = \begin{bmatrix}\mathbf{X} & \mathbf{Y}\end{bmatrix}^T$

$$p(\mathbf{z}) = p(\tilde{\mathbf{z}}) = \frac{1}{(2\pi)^n(\det(\mathbf{C}_{\tilde{\mathbf{Z}}}))^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\mathbf{m}})^T \mathbf{C}_{\tilde{\mathbf{Z}}}^{-1}(\tilde{\mathbf{z}} - \tilde{\mathbf{m}})\right)$$

- If $\mathbf{Z}$ is circularly symmetric, the pdf is given by

$$p(\mathbf{z}) = \frac{1}{\pi^n \det(\mathbf{C}_{\mathbf{Z}})} \exp\left(-\mathbf{z}^H \mathbf{C}_{\mathbf{Z}}^{-1}\mathbf{z}\right)$$

We write $\mathbf{Z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{C}_{\mathbf{Z}})$

Projection of Complex WGN is Circularly Symmetric

- Let $\psi_1(t), \psi_2(t), \ldots, \psi_K(t)$ be a complex orthonormal basis
- Let $n(t)$ be complex white Gaussian noise with PSD $2\sigma^2$
- Consider the projection of $n(t)$ onto the orthonormal basis

$$\mathbf{N} = \begin{bmatrix} \langle n, \psi_1 \rangle \\ \vdots \\ \langle n, \psi_K \rangle \end{bmatrix}$$

- $\mathbf{N}$ is circularly symmetric and its pdf is

$$p(\mathbf{n}) = \frac{1}{\pi^K \det(\mathbf{C_N})} \exp\left(-\mathbf{n}^H \mathbf{C_N}^{-1} \mathbf{n}\right) = \frac{1}{(2\pi\sigma^2)^K} \exp\left(-\frac{\|\mathbf{n}\|^2}{2\sigma^2}\right)$$

- If $\mathbf{Y} = \mathbf{s}_i + \mathbf{N}$, then the pdf of $\mathbf{Y}$ is

$$p(\mathbf{y}) = \frac{1}{(2\pi\sigma^2)^K} \exp\left(-\frac{\|\mathbf{y} - \mathbf{s}_i\|^2}{2\sigma^2}\right)$$

Reference

Circularly Symmetric Gaussian Random Vectors
Robert G. Gallager
`http://www.rle.mit.edu/rgallager/documents/CircSymGauss.pdf`

Thanks for your attention