Gaussian Random Vectors and Processes

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Gaussian Random Vectors

Jointly Gaussian Random Variables

Definition (Jointly Gaussian RVs)

Random variables $X_1, X_2, \ldots, X_n$ are jointly Gaussian if any non-trivial linear combination is a Gaussian random variable.

$$a_1 X_1 + \cdots + a_n X_n \text{ is Gaussian for all } (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \mathbf{0}$$

Example (Not Jointly Gaussian)

$X \sim N(0, 1)$

$$Y = \begin{cases} X, & \text{if } |X| > 1 \\ -X, & \text{if } |X| \leq 1 \end{cases}$$

$Y \sim N(0, 1)$ and $X + Y$ is not Gaussian.

Gaussian Random Vector

Definition (Gaussian Random Vector)

A random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ whose components are jointly Gaussian.

Notation

$\mathbf{X} \sim N(\mathbf{m}, \mathbf{C})$ where

$$\begin{aligned} \mathbf{m} &= E[\mathbf{X}] \text{ is the } n \times 1 \text{ mean vector} \\ \mathbf{C} &= E\left[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T\right] \text{ is the } n \times n \text{ covariance matrix} \end{aligned}$$

$m_i = E[X_i]$, $C_{ij} = E[(X_i - m_i)(X_j - m_j)] = \text{cov}(X_i, X_j)$

Definition (Joint Gaussian Density)

For a Gaussian random vector, $\mathbf{C}$ is invertible and the joint density is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m})\right)$$

For derivation, see Problems 3.31(f) and 3.32 in Madhow's book.

Uncorrelated Jointly Gaussian RVs are Independent

If $X_1, \ldots, X_n$ are jointly Gaussian and pairwise uncorrelated, then they are independent. For pairwise uncorrelated random variables,

$$C_{ij} = E[(X_i - m_i)(X_j - m_j)] = \begin{cases} 0 & \text{if } i \neq j \\ \sigma_i^2 & \text{otherwise.} \end{cases}$$

The joint probability density function is given by

$$\begin{aligned} p(\mathbf{x}) &= \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x_i - m_i)^2}{2\sigma_i^2}\right) \end{aligned}$$

where $m_i = E[X_i]$ and $\sigma_i^2 = \text{var}(X_i)$.

Uncorrelated Gaussian RVs may not be Independent

Example

- $X \sim N(0, 1)$
- W is equally likely to be +1 or -1
- W is independent of X
- $Y = WX$
- $Y \sim N(0, 1)$
- X and Y are uncorrelated
- X and Y are not independent

Gaussian Random Processes

Gaussian Random Process

Definition

A random process $X(t)$ is Gaussian if its samples $X(t_1), \ldots, X(t_n)$ are jointly Gaussian for any $n \in \mathbb{N}$ and distinct sample locations $t_1, t_2, \ldots, t_n$.

Let $\mathbf{X} = \begin{bmatrix} X(t_1) & \cdots & X(t_n) \end{bmatrix}^T$ be the vector of samples. The joint density is given by

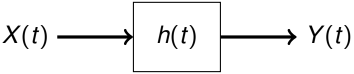
$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2} (\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m}) \right)$$

where

$$\mathbf{m} = E[\mathbf{X}], \;\; \mathbf{C} = E\left[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T \right]$$

Properties of Gaussian Random Process

- The mean and autocorrelation functions completely characterize a Gaussian random process.
- Wide-sense stationary Gaussian processes are strictly stationary.
- If the input to a stable linear filter is a Gaussian random process, the output is also a Gaussian random process.



White Gaussian Noise

Definition

A zero mean WSS Gaussian random process with power spectral density

$$S_n(f) = \frac{N_0}{2}.$$

$\frac{N_0}{2}$ is termed the two-sided PSD and has units Watts per Hertz.

Remarks

- Autocorrelation function $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$
- Infinite Power! Ideal model of Gaussian noise occupying more bandwidth than the signals of interest.

White Gaussian Noise through Correlators

- Consider the output of a correlator with WGN input

$$Z = \int_{-\infty}^{\infty} n(t)u(t)\,dt = \langle n, u\rangle$$

  where $u(t)$ is a deterministic finite-energy signal
- Z is a Gaussian random variable
- The mean of Z is

$$E[Z] = \int_{-\infty}^{\infty} E\left[n(t)\right] u(t)\,dt = 0$$

- The variance of Z is

$$\begin{aligned} \mathrm{var}(Z) &= E\left[(\langle n, u\rangle)^2\right] = E\left[\int n(t)u(t)\,dt \int n(s)u(s)\,ds\right] \\ &= \int\int u(t)u(s)E\left[n(t)n(s)\right]\,dt\,ds \\ &= \int\int u(t)u(s)\frac{N_0}{2}\delta(t-s)\,dt\,ds \\ &= \frac{N_0}{2}\int u^2(t)\,dt = \frac{N_0}{2}\|u\|^2 \end{aligned}$$

White Gaussian Noise through Correlators

Proposition

Let $u_1(t)$ and $u_2(t)$ be linearly independent finite-energy signals and let $n(t)$ be WGN with PSD $S_n(f) = \frac{N_0}{2}$. Then $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian with covariance

$$\text{cov}\left(\langle n, u_1 \rangle, \langle n, u_2 \rangle\right) = \frac{N_0}{2} \langle u_1, u_2 \rangle.$$

Proof

To prove that $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian, consider a non-trivial linear combination $a\langle n, u_1 \rangle + b\langle n, u_2 \rangle$

$$a\langle n, u_1 \rangle + b\langle n, u_2 \rangle = \int n(t) \left[a u_1(t) + b u_2(t)\right] \, dt.$$

This is the result of passing $n(t)$ through a correlator. So it is a Gaussian random variable.

White Gaussian Noise through Correlators

Proof (continued)

$$
\begin{aligned}
\operatorname{cov}(\langle n, u_1 \rangle, \langle n, u_2 \rangle) &= E\left[\langle n, u_1 \rangle \langle n, u_2 \rangle\right] \\
&= E\left[\int n(t) u_1(t)\, dt \int n(s) u_2(s)\, ds\right] \\
&= \int \int u_1(t) u_2(s) E\left[n(t) n(s)\right]\, dt\, ds \\
&= \int \int u_1(t) u_2(s) \frac{N_0}{2} \delta(t - s)\, dt\, ds \\
&= \frac{N_0}{2} \int u_1(t) u_2(t)\, dt \\
&= \frac{N_0}{2} \langle u_1, u_2 \rangle
\end{aligned}
$$

If $u_1(t)$ and $u_2(t)$ are orthogonal, $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are independent.

Thanks for your attention