Optimal Receiver using Complex Baseband Representation

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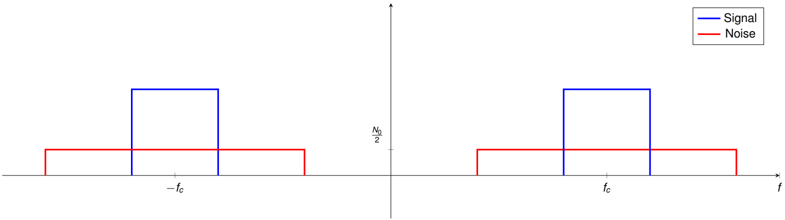
Passband Signals in Passband Noise

Consider M-ary passband signaling over a channel with passband Gaussian noise

$$H_i : y_p(t) = s_{i,p}(t) + n_p(t), \quad i = 1, \ldots, M$$

where

- $y_p(t)$ Real passband received signal
- $s_{i,p}(t)$ Real passband signals
- $n_p(t)$ Real passband GN with PSD $\frac{N_0}{2}$



Note: A WSS random process is passband if its autocorrelation function is a passband signal

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The equivalent problem in complex baseband is

$$H_i : y(t) = s_i(t) + n(t), \quad i = 1, \dots, M$$

where

- $y(t)$ Complex envelope of $y_p(t)$
- $s_i(t)$ Complex envelope of $s_{i,p}(t)$
- $n(t)$ Complex envelope of $n_p(t)$

What is the optimal receiver in terms of the complex baseband signals?

Complex Envelope of Passband Gaussian Noise

- The complex baseband representation of $n_p(t)$ is given by

$$n(t) = n_c(t) + jn_s(t) = \frac{1}{\sqrt{2}} \left[n_p(t) + j\hat{n}_p(t) \right] e^{-j2\pi f_c t}$$

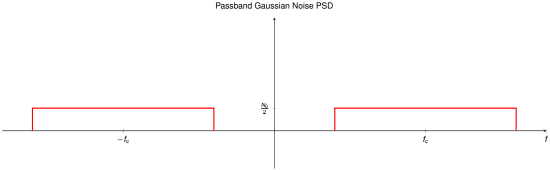
  where $\hat{n}_p(t)$ is the Hilbert transform of $n_p(t)$
- The inphase and quadrature components of $n(t)$ are given by

$$\begin{aligned} n_c(t) &= \frac{1}{\sqrt{2}} \left[n_p(t) \cos 2\pi f_c t + \hat{n}_p(t) \sin 2\pi f_c t \right] \\ n_s(t) &= \frac{1}{\sqrt{2}} \left[\hat{n}_p(t) \cos 2\pi f_c t - n_p(t) \sin 2\pi f_c t \right] \end{aligned}$$

- $n_c(t)$ and $n_s(t)$ are jointly Gaussian and independent random processes (Proof in Proakis Section 2.9)

In-Phase and Quadrature Component PSDs

$$S_{n_p}(f) = \begin{cases} \frac{N_0}{2} & |f - f_c| < W \\ 0 & \text{otherwise} \end{cases}$$



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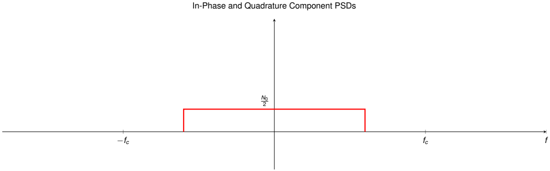
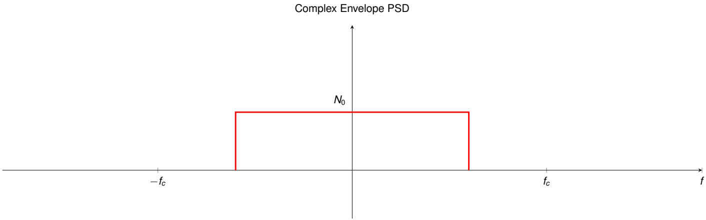


Complex Envelope PSD

- By the independence of $n_c(t)$ and $n_s(t)$, we have

$$R_n(\tau) = E\left[n(t+\tau)n^*(t)\right] = R_{n_c}(\tau) + R_{n_s}(\tau)$$

- $S_n(f) = S_{n_c}(f) + S_{n_s}(f)$



- If $n_c(t)$ and $n_s(t)$ are approximated by white Gaussian noise, $n(t)$ is said to be complex white Gaussian noise

Complex White Gaussian Noise

Definition (Complex Gaussian Random Process)

A complex random process $Z(t) = X(t) + jY(t)$ is a Gaussian random process if $X(t)$ and $Y(t)$ are jointly Gaussian random processes.

Definition (Complex White Gaussian Noise)

A complex Gaussian random process $Z(t) = X(t) + jY(t)$ is complex white Gaussian noise with PSD N_0 if $X(t)$ and $Y(t)$ are independent white Gaussian noise processes with PSD $\frac{N_0}{2}$.

Optimal Detection in Complex Baseband

- The continuous time hypothesis testing problem in complex baseband

$$H_i : y(t) = s_i(t) + n(t), \quad i = 1, \dots, M$$

where

$y(t)$	Complex envelope of $y_p(t)$
$s_i(t)$	Complex envelope of $s_{i,p}(t)$
$n(t)$	Complex white Gaussian noise with PSD $N_0 = 2\sigma^2$

- The equivalent problem in terms of complex random vectors

$$H_i : \mathbf{Y} = \mathbf{s}_i + \mathbf{N}, \quad i = 1, \dots, M$$

where $\mathbf{Y}$, $\mathbf{s}_i$ and $\mathbf{N}$ are the projections of $y(t)$, $s_i(t)$ and $n(t)$ respectively onto the signal space spanned by $\{s_i(t)\}$.

- $\mathbf{m} = E[\mathbf{N}] = \mathbf{0}$, $\mathbf{C_N} = 2\sigma^2\mathbf{I}$

Complex White Gaussian Noise through Correlators

$$
\begin{aligned}
\mathrm{cov}\left(\langle n, \psi_1\rangle, \langle n, \psi_2\rangle\right) &= E\left[\langle n, \psi_1\rangle \left(\langle n, \psi_2\rangle\right)^*\right] \\
&= E\left[\int n(t)\psi_1^*(t)\, dt \int n^*(s)\psi_2(s)\, ds\right] \\
&= \int\int \psi_2(t)\psi_2^*(s) E\left[n(t)n^*(s)\right]\, dt\, ds \\
&= \int\int \psi_2(t)\psi_1^*(s) 2\sigma^2 \delta(t-s)\, dt\, ds \\
&= 2\sigma^2 \int \psi_2(t)\psi_1^*(t)\, dt \\
&= 2\sigma^2 \langle \psi_2, \psi_1\rangle
\end{aligned}
$$

MPE and ML Rules in Complex Baseband

- $\mathbf{N}$ is a circularly symmetric Gaussian vector and the pdf of $\mathbf{Y}$ under H_i is

$$
\begin{aligned}
p_i(\mathbf{y}) &= \frac{1}{\pi^K \det(\mathbf{C_N})} \exp\left(-(\mathbf{y}-\mathbf{s_i})^H \mathbf{C_N}^{-1}(\mathbf{y}-\mathbf{s_i})\right) \\
&= \frac{1}{(2\pi\sigma^2)^K} \exp\left(-\frac{\|\mathbf{y}-\mathbf{s}_i\|^2}{2\sigma^2}\right)
\end{aligned}
$$

- The MPE rule is given by

$$
\begin{aligned}
\delta_{MPE}(\mathbf{y}) &= \operatorname*{argmax}_{1 \le i \le M} \operatorname{Re}\left(\langle \mathbf{y}, \mathbf{s}_i \rangle\right) - \frac{\|\mathbf{s}_i\|^2}{2} + \sigma^2 \log \pi_i \\
&= \operatorname*{argmax}_{1 \le i \le M} \operatorname{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2} + \sigma^2 \log \pi_i
\end{aligned}
$$

- The ML rule is given by

$$
\begin{aligned}
\delta_{ML}(\mathbf{y}) &= \operatorname*{argmin}_{1 \le i \le M} \|\mathbf{y}-\mathbf{s}_i\|^2 = \operatorname*{argmax}_{1 \le i \le M} \operatorname{Re}\left(\langle \mathbf{y}, \mathbf{s}_i \rangle\right) - \frac{\|\mathbf{s}_i\|^2}{2} \\
&= \operatorname*{argmin}_{1 \le i \le M} \|y-s_i\|^2 = \operatorname*{argmax}_{1 \le i \le M} \operatorname{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2}
\end{aligned}
$$

ML Receiver for QPSK

QPSK signals where $p(t)$ is a real baseband pulse, A is a real number and $1 \leq m \leq 4$

$$\begin{aligned} s_m^p(t) &= \sqrt{2}Ap(t)\cos\left(2\pi f_c t + \frac{\pi(2m-1)}{4}\right) \\ &= \text{Re}\left[\sqrt{2}Ap(t)e^{j\left(2\pi f_c t + \frac{\pi(2m-1)}{4}\right)}\right] \end{aligned}$$

Complex Envelope of QPSK Signals

$$s_m(t) = Ap(t)e^{j\frac{\pi(2m-1)}{4}}, \quad 1 \leq m \leq 4$$

Orthonormal basis for the complex envelope consists of only

$$\phi(t) = \frac{p(t)}{\sqrt{E_p}}$$

ML Receiver for QPSK

Let $\sqrt{E_b} = \frac{A\sqrt{E_p}}{\sqrt{2}}$. The vector representation of the QPSK signals is

$$\begin{aligned} s_1 &= \sqrt{E_b} + j\sqrt{E_b} \\ s_2 &= -\sqrt{E_b} + j\sqrt{E_b} \\ s_3 &= -\sqrt{E_b} - j\sqrt{E_b} \\ s_4 &= \sqrt{E_b} - j\sqrt{E_b} \end{aligned}$$

The hypothesis testing problem in terms of vectors is

$$H_i : Y = s_i + N, \quad i = 1, \ldots, 4$$

where $N \sim \mathcal{CN}(0, 2\sigma^2)$
The ML rule is given by

$$\delta_{ML}(y) = \underset{1 \le i \le 4}{\text{argmin}} \|y - s_i\|^2$$

Thanks for your attention