Probability Theory

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Probability Theory

- Branch of mathematics which pertains to random phenomena
- Used to model uncertainty in the real world
- Applications
    - Communications
    - Signal Processing
    - Statistical Inference
    - Finance
    - Gambling

What is Probability?

- Classical definition: Ratio of outcomes favorable to an event to the total number of outcomes provided all outcomes are equally likely.

$$P(A) = \frac{N_A}{N}$$

- Relative frequency definition:

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N}$$

- Axiomatic definition: A countably additive function defined on the set of events with range in the interval $[0, 1]$.

Sample Space

Definition

The set of all possible outcomes of an experiment is called the sample space and is denoted by Ω.

Examples

- Coin toss: $\Omega = \{\text{Heads}, \text{Tails}\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Tossing of two coins: $\Omega = \{(H, H), (T, H), (H, T), (T, T)\}$
- Coin is tossed until heads appear. What is Ω?
- Life expectancy of a random person. $\Omega = [0, 120]$ years

Events

- An event is a subset of the sample space

Examples

- Coin toss: $\Omega = \{\text{Heads}, \text{Tails}\}$.
  $E = \{\text{Heads}\}$ is the event that a head appears on the flip of a coin.
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
  $E = \{2, 4, 6\}$ is the event that an even number appears.
- Life expectancy. $\Omega = [0, 120]$.
  $E = [50, 120]$ is the event that a random person lives beyond 50 years.

Definition (Mutually Exclusive Events)

Events E and F are said to be mutually exclusive if $E \cap F = \phi$.

Probability Measure

Definition

A mapping P on the event space which satisfies

1. $0 \leq P(E) \leq 1$
2. $P(\Omega) = 1$
3. For any sequence of events $E_1, E_2, \ldots$ that are pairwise mutually exclusive, i.e. $E_n \cap E_m = \phi$ for $n \neq m$,

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

Example (Coin Toss)

$S = \{\text{Heads}, \text{Tails}\}$, $P(\{\text{Heads}\}) = P(\{\text{Tails}\}) = \frac{1}{2}$

Some Properties of the Probability Measure

- $P(A^c) = 1 - P(A)$
- If $A \subseteq B$, then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots A_n)$$
- $$P\left(\bigcap_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cup A_j) + \sum_{i<j<k} P(A_i \cup A_j \cup A_k) - \cdots + (-1)^{n+1} P(A_1 \cup A_2 \cup \cdots A_n)$$

Conditional Probability

Definition

If $P(B) > 0$ then the conditional probability that A occurs given that B occurs is defined to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Examples

- Two fair dice are thrown. Given that the first shows 3, what is the probability that the total exceeds 6?
- A family has two children. What is the probability that both are boys, given that at least one is a boy?
- A family has two children. What is the probability that both are boys, given that the younger is a boy?
- A box has three white balls w_1, w_2, and w_3 and two red balls r_1 and r_2. Two random balls are removed in succession. What is the probability that the first removed ball is white and the second is red?

Law of Total Probability

Theorem

For any events A and B such that $0 < P(B) < 1$*,*

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c).$$

More generally, let $B_1, B_2, \ldots, B_n$ *be a partition of* Ω *such that* $P(B_i) > 0$ *for all* i*. Then*

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

Examples

- Box 1 contains 3 white and 2 black balls. Box 2 contains 4 white and 6 black balls. If a box is selected at random and a ball is chosen at random from it, what is the probability that it is white?
- We have two coins; the first is fair and the second has heads on both sides. A coin is picked at random and tossed twice. What is the probability of heads showing up in both tosses?

Bayes' Theorem

Theorem

For any events A and B such that $P(A) > 0, P(B) > 0$*,*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

If $A_1, \ldots, A_n$ *is a partition of* Ω *such that* $P(A_i) > 0$ *and* $P(B) > 0$*, then*

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}.$$

Examples

- We have two coins; the first is fair and the second has heads on both sides. A coin is picked at random and tossed twice. If heads showed up in both tosses, what is the probability that the coin is fair?

Independence

Definition

Events A and B are called independent if

$$P(A \cap B) = P(A)P(B).$$

More generally, a family $\{A_i : i \in I\}$ is called independent if

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i)$$

for all finite subsets J of I.

Examples

- A fair coin is tossed twice. The first toss is independent of the second toss.
- Two fair dice are rolled. Is the the sum of the faces independent of the number shown by the first die?

Conditional Independence

Definition

Let C be an event with $P(C) > 0$. Two events A and B are called conditionally independent given C if

$$P(A \cap B|C) = P(A|C)P(B|C).$$

Example

- We have two coins; the first is fair and the second has heads on both sides. A coin is picked at random and tossed twice. Are the results of the two tosses independent? Are they independent if we know which coin was picked?

Questions?