# Random Variables

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# Random Variable

## Definition

A real-valued function defined on a sample space.

$$X : \Omega \to \mathbb{R}$$

## Example (Coin Toss)

$\Omega = \{\text{Heads}, \text{Tails}\}$
$X = 1$ if outcome is Heads and $X = 0$ if outcome is Tails.

## Example (Rolling Two Dice)

$\Omega = \{(i, j) : 1 \leq i, j \leq 6\}$, $X = i + j$.

# Cumulative Distribution Function

## Definition

The cdf $F$ of a random variable $X$ is defined for any real number $a$ by

$$F(a) = P(X \leq a).$$

## Properties

- $F(a)$ is a nondecreasing function of $a$
- $F(\infty) = 1$
- $F(-\infty) = 0$

# Discrete Random Variables

## Definition

A random variable is called discrete if it takes values only in some countable subset $\{x_1, x_2, x_3, \ldots\}$ of $\mathbb{R}$.

## Definition

A discrete random variable $X$ has a probability mass function $f : \mathbb{R} \to [0, 1]$ given by $f(x) = P[X = x]$

## Example

- Bernoulli random variable

  $\Omega = \{0, 1\}$

$$f(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

  where $0 \le p \le 1$

# Independent Discrete Random Variables

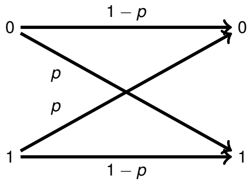
- Discrete random variables $X$ and $Y$ are independent if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all $x$ and $y$
- A family of discrete random variables $\{X_i : i \in I\}$ is an independent family if

$$P\left(\bigcap_{i \in J} \{X_i = x_i\}\right) = \prod_{i \in J} P(X_i = x_i)$$

for all sets $\{x_i : i \in I\}$ and for all finite subsets $J \in I$

## Example

Binary symmetric channel with crossover probability $p$




If the input is equally likely to be 0 or 1, are the input and output independent?

# Consequences of Independence

- If $X$ and $Y$ are independent, then the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for any subsets $A$ and $B$ of $\mathbb{R}$
- If $X$ and $Y$ are independent, then for any functions $g, h : \mathbb{R} \to \mathbb{R}$ the random variables $g(X)$ and $h(Y)$ are independent
- Let $X$ and $Y$ be discrete random variables with probability mass functions $f_X(x)$ and $f_Y(y)$ respectively

  Let $f_{X,Y}(x, y) = P(\{X = x\} \cap \{Y = y\})$ be the joint probability mass function of $X$ and $Y$

  $X$ and $Y$ are independent if and only if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \qquad \text{for all } x, y \in \mathbb{R}$$

# Continuous Random Variable

## Definition

A random variable is called continuous if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^{x} f(u)\, du \ \text{ for all } \ x \in \mathbb{R}$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the probability density function of $X$.

## Example (Uniform Random Variable)

A continuous random variable $X$ on the interval $[a, b]$ with cdf

$$F(x) = \begin{cases} 0, & \text{if } x < a \\ \frac{x-a}{b-a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } x > b \end{cases}$$

The pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

# Probability Density Function Properties

- $F(a) = \int_{-\infty}^{a} f(x)\, dx$
- $P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- The numerical value $f(x)$ is not a probability. It can be larger than 1.
- $f(x)dx$ can be intepreted as the probability $P(x < X \leq x + dx)$ since

$$P(x < X \leq x + dx) = F(x + dx) - F(x) \approx f(x)\, dx$$

# Independent Continuous Random Variables

- Continuous random variables $X$ and $Y$ are independent if the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for all $x$ and $y$ in $\mathbb{R}$
- If $X$ and $Y$ are independent, then the random variables $g(X)$ and $h(Y)$ are independent
- Let the joint probability distribution function of $X$ and $Y$ be $F(x, y) = P(X \leq x, Y \leq y)$.

  Then $X$ and $Y$ are said to be jointly continuous random variables with joint pdf $f_{X,Y}(x, y)$ if

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)\, du\, dv$$

  for all $x, y$ in $\mathbb{R}$
- $X$ and $Y$ are independent if and only if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \qquad \text{for all } x, y \in \mathbb{R}$$

# Expectation

- The expectation of a discrete random variable $X$ with probability mass function $f$ is defined to be

$$E(X) = \sum_{x:f(x)>0} xf(x)$$

- The expectation of a continuous random variable with density function $f$ is given by

$$E(X) = \int_{-\infty}^{\infty} xf(x)\, dx$$

- If $a, b \in \mathbb{R}$, then $E(aX + bY) = aE(X) + bE(Y)$
- If $X$ and $Y$ are independent, $E(XY) = E(X)E(Y)$
- $X$ and $Y$ are said to be uncorrelated if $E(XY) = E(X)E(Y)$
- Independent random variables are uncorrelated but uncorrelated random variables need not be independent

## Example

$Y$ and $Z$ are independent random variables such that $Z$ is equally likely to be 1 or $-1$ and $Y$ is equally likely to be 1 or 2. Let $X = YZ$.

Then $X$ and $Y$ are uncorrelated but not independent.

# Variance

- $\text{var}(X) = E\left[(X - E[X])^2\right] = E(X^2) - [E(X)]^2$
- For $a \in \mathbb{R}$, $\text{var}(aX) = a^2\,\text{var}(X)$
- $\text{var}(X + Y) = \text{var}(X) + \text{var}(Y)$ if $X$ and $Y$ are uncorrelated

# Complex Random Variable

## Definition

A complex random variable $Z = X + jY$ is a pair of real random variables $X$ and $Y$.

## Remarks

- The cdf of a complex RV is the joint cdf of its real and imaginary parts.
- $E[Z] = E[X] + jE[Y]$
- $\text{var}[Z] = E[|Z|^2] - |E[Z]|^2 = \text{var}[X] + \text{var}[Y]$

Thanks for your attention