EE 703: Digital Message Transmission (Autumn 2016)
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Quiz 2 : 15 points October 6, 2016

1. (5 points) Consider the M-ary hypothesis testing problem in AWGN where $s_i(t) = A_i p(t)$ for $A_i \in \mathbb{R}$ such that $A_1 < A_2 < \cdots < A_M$ and unit energy pulse $p(t)$ which is nonzero for $0 \le t \le T$.

$$\begin{aligned} H_1 &: \quad y(t) = s_1(t) + n(t) \\ H_2 &: \quad y(t) = s_2(t) + n(t) \\ \vdots & \qquad\qquad \vdots \\ H_M &: \quad y(t) = s_M(t) + n(t) \end{aligned}$$

If all the hypotheses are equally likely, show that the optimal receiver compares the output of a matched filter to a set of thresholds.

2. (5 points) Let $\phi_1(t)$ and $\phi_2(t)$ be real unit energy signals which are orthogonal, i.e. $\langle \phi_1, \phi_2 \rangle = 0$. Consider the following hypothesis testing problem in AWGN where the hypotheses are equally likely, $s_1(t) = 2\phi_1(t) + 4\phi_2(t)$, and $s_2(t) = -\phi_1(t) + 3\phi_2(t)$.

$$\begin{aligned} H_1 &: \quad y(t) = s_1(t) + n(t) \\ H_2 &: \quad y(t) = s_2(t) + n(t) \end{aligned}$$

Let the observed signal be given by $y(t) = \frac{3}{2}\phi_1(t) + \phi_2(t)$. Find the output of the optimal decision rule. You have to show the calculations which lead to your answer.

3. (5 points) Consider the following binary hypothesis testing problem where the hypotheses are equally likely and $\lambda_1 > \lambda_2$.

$$\begin{aligned} H_1 &: \quad X_i \sim \text{Poisson}(\lambda_1), \quad i = 1, 2, \ldots, N \\ H_2 &: \quad X_i \sim \text{Poisson}(\lambda_2), \quad i = 1, 2, \ldots, N \end{aligned}$$

Assume that X_i and X_j are independent for $i \neq j$. The probability mass function of a Poisson random variable Z with parameter γ, i.e. $Z \sim \text{Poisson}(\gamma)$, is given by $P(Z = n) = \frac{\gamma^n}{n!} e^{-\gamma}$ for $n = 0, 1, 2, 3, \ldots$.

   (a) Find the optimal decision rule which minimizes the decision error probability. Simplify it as much as possible.

   (b) Let $F(x; \gamma)$ be the cumulative distribution function of a Poisson random variable with parameter γ. Find the decision error probability of the optimal decision rule in terms of F, λ_1, λ_2, and N. *Hint: Sum of two independent Poisson random variables with parameters γ_1 and γ_2 is a Poisson random variable with parameter $\gamma_1 + \gamma_2$.*