EE 703: Digital Message Transmission (Autumn 2020)
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Assignment 4: 20 points — Date: October 20, 2020

1. [10 points] A complex random vector $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ is said to be a complex Gaussian vector if $\mathbf{X}$ and $\mathbf{Y}$ are jointly Gaussian vectors.

   Let $\psi_1(t), \psi_2(t), \dots, \psi_K(t)$ be a complex orthonormal basis. Let $n(t) = n_c(t) + jn_s(t)$ be complex white Gaussian noise with PSD $2\sigma^2$. Then $n_c(t)$ and $n_s(t)$ are independent real WGN processes with PSD σ^2. Consider the projection of $n(t)$ onto the orthonormal basis

$$\mathbf{N} = \begin{bmatrix} \langle n, \psi_1 \rangle \\ \vdots \\ \langle n, \psi_K \rangle \end{bmatrix}.$$

   Show that $\mathbf{N}$ is a complex Gaussian vector i.e. the components of the following vector $\tilde{\mathbf{N}}$ are jointly Gaussian. Here $\Re(z)$ denotes the real part of a complex number z and $\Im(z)$ denotes the imaginary part of a complex number z.

$$\tilde{\mathbf{N}} = \begin{bmatrix} \Re(\langle n, \psi_1 \rangle) \\ \vdots \\ \Re(\langle n, \psi_K \rangle) \\ \Im(\langle n, \psi_1 \rangle) \\ \vdots \\ \Im(\langle n, \psi_K \rangle) \end{bmatrix}.$$

   Hint: Independent Gaussian random variables are jointly Gaussian.

2. [10 points] Consider binary signaling in the complex AWGN channel

$$\begin{aligned} H_0 &: y(t) = s_0(t) + n(t) \\ H_1 &: y(t) = s_1(t) + n(t) \end{aligned}$$

   where $y(t)$ is the complex envelope of received signal, $s_i(t)$ is the complex envelope of transmitted signal under H_i, and $n(t)$ is complex white Gaussian noise with PSD $2\sigma^2$.

   The ML receiver decides that H_0 is true if $Z = \Re\left(\langle y, s_0 - s_1 \rangle\right)$ is greater than $\frac{\|s_0\|^2 - \|s_1\|^2}{2}$. Otherwise, it decides that H_1 is true.

   (a) Calculate the mean of Z under both H_0 and H_1.

   (b) Show that the variance of Z under both H_0 and H_1 is $\sigma^2 \|s_0 - s_1\|^2$.

   (c) Show that the conditional probability of error of the ML receiver under both H_0 and H_1 is $Q\left(\frac{\|s_0 - s_1\|}{2\sigma}\right)$.