EE 703: Digital Message Transmission (Autumn 2021)
Instructor: Saravanan Vijayakumaran
Indian Institute of Technology Bombay

Endsem Exam 40 points — Date: November 8, 2021

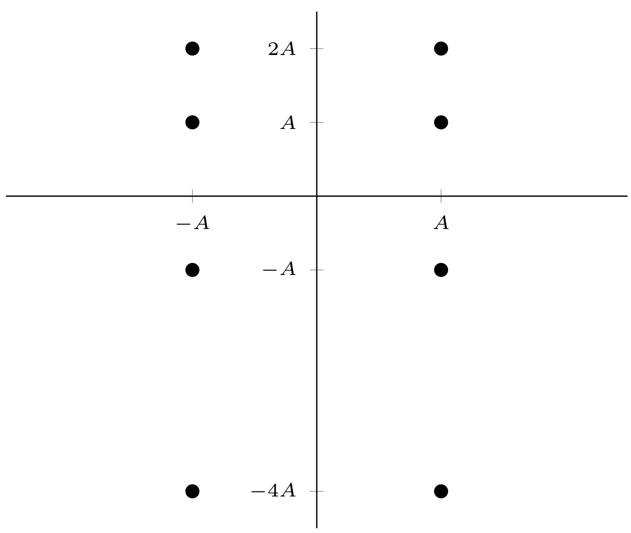
1. [4 points] State whether the following statements are **True** or **False** with a short justification (half a page or less).
   (a) Suppose that $\phi_1(t), \phi_2(t), \ldots, \phi_M(t)$ is an orthonormal basis for the set of signals $s_1(t), s_2(t), \ldots, s_N(t)$. Then $M \geq N$.
   (b) A sum of Gaussian random variables is always a Gaussian random variable.
   (c) In a binary hypothesis testing situation with equally likely hypotheses, the probability of decision error of the optimal decision rule is always less than or equal to $\frac{1}{2}$.
   (d) Suppose we have to choose two signals from a set of three distinct signals $\{s_1(t), s_2(t), s_3(t)\}$ to transmit a single bit over an AWGN channel. We should pick the pair of signals $s_i(t)$ and $s_j(t)$ such that $\|s_i - s_j\|$ is maximum where $i, j \in \{1, 2, 3\}$ and $i \neq j$.

2. [6 points] Consider the following binary hypothesis testing problem where the hypotheses are equally likely.

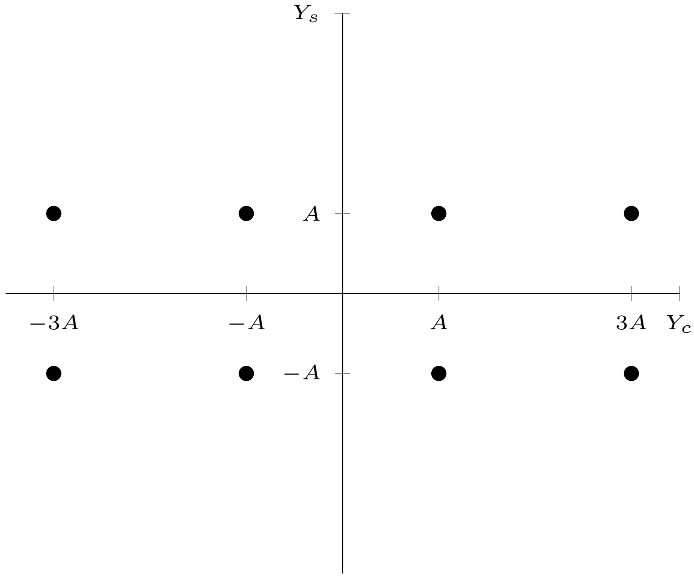
$$\begin{aligned} H_0 &: Y \sim U[a, b] \\ H_1 &: Y \sim U[c, d] \end{aligned}$$

   where U denotes the uniform distribution and a, b, c, d are real numbers satisfying $a < c < d < b$.
   (a) Derive the optimal decision rule.
   (b) Find the decision error probability of the optimal decision rule.

3. [6 points] For the below constellation of 8 symbols, assume that the transmitted symbol is corrupted by adding $N = N_c + jN_s$ where N_c and N_s are independent Gaussian random variables with zero mean and variance $\frac{N_0}{2}$. All the constellation points are equally likely to be transmitted. Calculate the following for the optimal decision rule in terms of E_b and N_0.
   (a) The intelligent union bound on the exact error probability.
   (b) The nearest neighbor approximation of the exact error probability.



4. [6 points] For the below constellation of 8 symbols, assume that the transmitted symbol is corrupted by adding $N = N_c + jN_s$ where N_c and N_s are independent Gaussian random variables with zero mean and variance $\frac{N_0}{2}$. All the constellation points are equally likely to be transmitted. Calculate the BER performance of the ML receiver under a Gray mapping in terms of E_b and N_0.



5. (a) [3 points] Let $b \geq 1$ be an integer. For $M = 2^b$, suppose M **orthogonal** real signals $s_i(t)$, $i = 1, \ldots, M$ are used for transmitting b bits over a real AWGN channel with PSD $\frac{N_0}{2}$. If all the signals have the same energy E and are equally likely to be transmitted, derive the following as a function of E, N_0, b or M when the optimal receiver is used.
   i. The union bound on the symbol error probability
   ii. The nearest neighbor approximation of the symbol error probability

   (b) [3 points] Suppose we use the M signals in the previous part to form a set of $2M$ real signals

$$\{s_1(t), s_2(t), \ldots, s_M(t), -s_1(t), -s_2(t), \ldots, -s_M(t)\}.$$

   So the set contains M signals and their negative versions. These $2M$ signals are used for transmitting $b + 1$ bits over a real AWGN channel with PSD $\frac{N_0}{2}$. If all the $2M$ signals are equally likely to be transmitted, derive the following as a function of E, N_0, b or M when the optimal receiver is used.
   i. The union bound on the symbol error probability
   ii. The nearest neighbor approximation of the symbol error probability

6. [6 points] Suppose N_1, N_2 are independent Gaussian random variables each having mean 0 and variance $\sigma^2 > 0$. The variance σ^2 is assumed to be known. We observe two observations Y_1, Y_2 given by

$$Y_1 = \lambda + N_1 - N_2,$$
$$Y_2 = 2\lambda + N_1 + N_2.$$

(a) Find the ML estimator of the parameter λ.

(b) Find the mean and variance of the ML estimator.

7. [6 points] Suppose we observe $Y_i, \ i = 1, 2, \ldots, M$ such that

$$Y_i \sim \text{Uniform}\left[-\frac{\theta}{2}, 2\theta\right]$$

where Y_i's are independent and θ is unknown. Assume $\theta \geq 0$. Derive the ML estimator of θ.