Fourier Transform

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Definition

- Fourier transform of a signal $s(t)$

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} \, dt$$

- Inverse Fourier transform

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} \, df$$

- Notation

$$s(t) \leftrightarrow S(f)$$

Properties of Fourier Transform

- Linearity

$$as_1(t) + bs_2(t) \leftrightarrow aS_1(f) + bS_2(f)$$

- Duality

$$S(t) \leftrightarrow s(-f)$$

- Conjugation in time corresponds to conjugation and reflection in frequency, and vice versa

$$s^*(t) \leftrightarrow S^*(-f)$$

$$s^*(-t) \leftrightarrow S^*(f)$$

- Real-valued signals have conjugate symmetric Fourier transforms

$$s(t) = s^*(t) \implies S(f) = S^*(-f)$$

Properties of Fourier Transform

- Time scaling

$$s(at) \leftrightarrow \frac{1}{|a|} S\left(\frac{f}{a}\right)$$

- Time shift

$$s(t - t_0) \leftrightarrow S(f) e^{-j2\pi f t_0}$$

- Modulation

$$s(t) e^{j2\pi f_0 t} \leftrightarrow S(f - f_0)$$

- Convolution

$$s_1(t) * s_2(t) \leftrightarrow S_1(f) S_2(f)$$

- Multiplication

$$s_1(t) s_2(t) \leftrightarrow S_1(f) * S_2(f)$$

Fourier Transforms using Dirac Function

- DC Signal

$$1 \leftrightarrow \delta(f)$$

- Complex Exponential

$$e^{j2\pi f_c t} \leftrightarrow \delta(f - f_c)$$

- Sinusoidal Functions

$$\cos(2\pi f_c t) \leftrightarrow \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

$$\sin(2\pi f_c t) \leftrightarrow \frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$$

Properties of Fourier Transform

- Parseval's identity

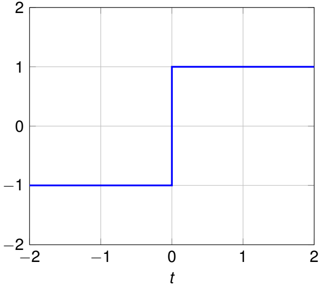
$$\int_{-\infty}^{\infty} s_1(t)s_2^*(t)\,dt = \int_{-\infty}^{\infty} S_1(f)S_2^*(f)\,df$$

- Energy is independent of representation

$$E_s = \|s\|^2 = \int_{-\infty}^{\infty} |s(t)|^2\,dt = \int_{-\infty}^{\infty} |S(f)|^2\,df$$

Signum Function

$$\operatorname{sgn}(t) = \begin{cases} +1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

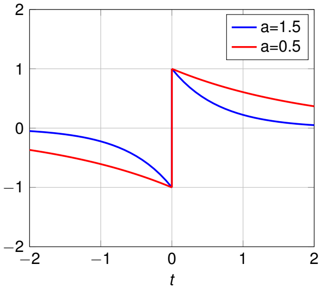


Fourier Transform

$$\operatorname{sgn}(t) \leftrightarrow \frac{1}{j\pi f}$$

Signum Function

$$g(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t = 0 \\ -e^{at}, & t < 0 \end{cases}$$

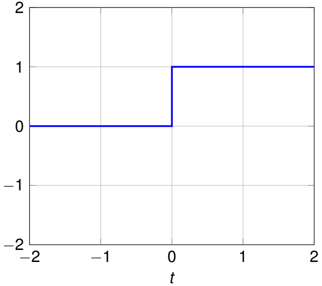


$$\mathrm{sgn}(t) = \lim_{a \to 0^+} g(t)$$

$$G(f) = \frac{-j4\pi f}{a^2 + (2\pi f)^2}$$

Unit Step Function

$$u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$$



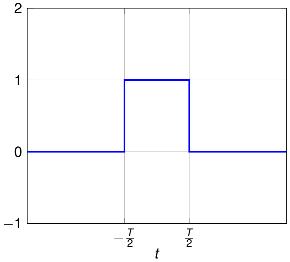
Fourier Transform

$$u(t) = \frac{1}{2}[\mathrm{sgn}(t) + 1]$$

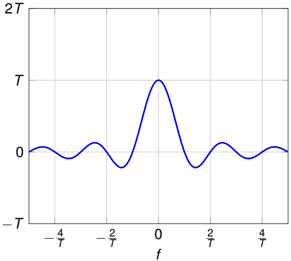
$$u(t) \leftrightarrow \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

Rectangular Pulse

$$I_{[-\frac{T}{2},\frac{T}{2}]}(t) = \begin{cases} 1, & |t| \leq \frac{T}{2} \\ 0, & |t| > \frac{T}{2} \end{cases}$$



$$I_{[-\frac{T}{2},\frac{T}{2}]}(t) \leftrightarrow T\mathrm{sinc}(fT)$$



References

- pp 13 — 14, Section 2.1, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008