ML Estimation of Signal Parameters

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ML Estimation Requires Conditional Densities

- ML estimation involves maximizing the conditional density wrt unknown parameters

$$\hat{\theta}_{ML}(y) = \underset{\theta}{\operatorname{argmax}}\, p(y|\theta)$$

- Example: $Y \sim \mathcal{N}(\theta, \sigma^2)$ where θ is unknown and σ^2 is known

$$p\,(y|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\theta)^2}{2\sigma^2}}$$

- Suppose the observation is the realization of a random process

$$y(t) = Ae^{j\theta} s(t-\tau) + n(t)$$

- What is the conditional density of $y(t)$ given A, θ and τ?

Maximizing Likelihood Ratio for ML Estimation

- Consider $Y \sim \mathcal{N}(\theta, \sigma^2)$ where θ is unknown and σ^2 is known

$$p(y|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\theta)^2}{2\sigma^2}}$$

- Let $q(y)$ be the density of a Gaussian with distribution $\mathcal{N}(0, \sigma^2)$

$$q(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$$

- The ML estimate of θ is obtained as

$$\begin{aligned} \hat{\theta}_{ML}(y) &= \underset{\theta}{\operatorname{argmax}}\, p(y|\theta) = \underset{\theta}{\operatorname{argmax}} \frac{p(y|\theta)}{q(y)} \\ &= \underset{\theta}{\operatorname{argmax}}\, L(y|\theta) \end{aligned}$$

where $L(y|\theta)$ is called the likelihood ratio

Likelihood Ratio and Hypothesis Testing

- The likelihood ratio $L(y|\theta)$ is the ML decision statistic for the following binary hypothesis testing problem

$$\begin{aligned} H_1 &: \quad Y \sim \mathcal{N}(\theta, \sigma^2) \\ H_0 &: \quad Y \sim \mathcal{N}(0, \sigma^2) \end{aligned}$$

- H_0 is a dummy hypothesis which does not give any advantage for the case of random vectors
- But it makes calculation of the ML estimator easy for random processes

Likelihood Ratio of a Signal in AWGN

- Let $H_s(\theta)$ be the hypothesis corresponding the following received signal

$$H_s(\theta) \quad : \quad y(t) = s_\theta(t) + n(t)$$

  where θ can be a vector parameter

- Define a noise-only dummy hypothesis H_0

$$H_0 \quad : \quad y(t) = n(t)$$

- Define Z and $y^\perp(t)$ as follows

$$\begin{aligned} Z &= \langle y, s_\theta \rangle \\ y^\perp(t) &= y(t) - \langle y, s_\theta \rangle \frac{s_\theta(t)}{\|s_\theta\|^2} \end{aligned}$$

- Z and $y^\perp(t)$ completely characterize $y(t)$

Likelihood Ratio of a Signal in AWGN

- Under both hypotheses $y^{\perp}(t)$ is equal to $n^{\perp}(t)$ where

$$n^{\perp}(t) = n(t) - \langle n, s_\theta \rangle \frac{s_\theta(t)}{\|s_\theta\|^2}$$

- $n^{\perp}(t)$ has the same distribution under both hypotheses
- $n^{\perp}(t)$ is irrelevant for this binary hypothesis testing problem
- The likelihood ratio of $y(t)$ equals the likelihood ratio of Z under the following hypothesis testing problem

$$\begin{aligned} H_s(\theta) &: \quad Z \sim \mathcal{N}(\|s_\theta\|^2, \sigma^2\|s_\theta\|^2) \\ H_0(\theta) &: \quad Z \sim \mathcal{N}(0, \sigma^2\|s_\theta\|^2) \end{aligned}$$

Likelihood Ratio of Signals in AWGN

- The likelihood ratio of signals in real AWGN is

$$L(y|s_\theta) = \exp\left(\frac{1}{\sigma^2}\left[\langle y, s_\theta\rangle - \frac{\|s_\theta\|^2}{2}\right]\right)$$

- The likelihood ratio of signals in complex AWGN is

$$L(y|s_\theta) = \exp\left(\frac{1}{\sigma^2}\left[\mathrm{Re}(\langle y, s_\theta\rangle) - \frac{\|s_\theta\|^2}{2}\right]\right)$$

- Maximizing these likelihood ratios as functions of θ results in the ML estimator

References

- Section 4.2, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008