Performance of ML Receiver for Binary Signaling

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Real AWGN Channel

M-ary Signaling in AWGN Channel

- One of M continuous-time signals $s_1(t), \ldots, s_M(t)$ is transmitted
- The received signal is the transmitted signal corrupted by real AWGN
- M hypotheses with prior probabilities $\pi_i, i = 1, \ldots, M$

$$
\begin{array}{ccc}
H_1 & : & y(t) = s_1(t) + n(t) \\
H_2 & : & y(t) = s_2(t) + n(t) \\
\vdots & & \vdots \\
H_M & : & y(t) = s_M(t) + n(t)
\end{array}
$$

- If the prior probabilities are equal, ML decision rule is optimal
- The ML decision rule is

$$
\delta_{ML}(y) = \underset{1 \le i \le M}{\text{argmin}} \|y - s_i\|^2 = \underset{1 \le i \le M}{\text{argmax}} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2}
$$

- We want to study the performance of the ML decision rule

ML Decision Rule for Binary Signaling

- Consider the special case of binary signaling

$$\begin{aligned} H_0 &: \quad y(t) = s_0(t) + n(t) \\ H_1 &: \quad y(t) = s_1(t) + n(t) \end{aligned}$$

- The ML decision rule decides H_0 is true if

$$\langle y, s_0 \rangle - \frac{\|s_0\|^2}{2} > \langle y, s_1 \rangle - \frac{\|s_1\|^2}{2}$$

- The ML decision rule decides H_1 is true if

$$\langle y, s_0 \rangle - \frac{\|s_0\|^2}{2} \leq \langle y, s_1 \rangle - \frac{\|s_1\|^2}{2}$$

- The ML decision rule

$$\langle y, s_0 - s_1 \rangle \underset{H_1}{\overset{H_0}{\gtrless}} \frac{\|s_0\|^2}{2} - \frac{\|s_1\|^2}{2}$$

- The distribution of $\langle y, s_0 - s_1 \rangle$ is required to evaluate decision rule performance

Performance of ML Decision Rule for Binary Signaling

- Let $Z = \langle y, s_0 - s_1 \rangle$
- Z is a Gaussian random variable

$$Z = \langle y, s_0 - s_1 \rangle = \langle s_i, s_0 - s_1 \rangle + \langle n, s_0 - s_1 \rangle$$

- The mean and variance of Z under H_0 are

$$\begin{aligned} E[Z|H_0] &= \|s_0\|^2 - \langle s_0, s_1 \rangle \\ \text{var}[Z|H_0] &= \sigma^2 \|s_0 - s_1\|^2 \end{aligned}$$

  where σ^2 is the PSD of $n(t)$
- Probability of error under H_0 is

$$P_{e|0} = \Pr\left[Z \le \frac{\|s_0\|^2 - \|s_1\|^2}{2} \middle| H_0 \right] = Q\left(\frac{\|s_0 - s_1\|}{2\sigma} \right)$$

Performance of ML Decision Rule for Binary Signaling

- The mean and variance of Z under H_1 are

$$\begin{aligned} E[Z|H_1] &= \langle s_1, s_0 \rangle - \|s_1\|^2 \\ \text{var}[Z|H_1] &= \sigma^2 \|s_0 - s_1\|^2 \end{aligned}$$

- Probability of error under H_1 is

$$P_{e|1} = \Pr\left[Z > \frac{\|s_0\|^2 - \|s_1\|^2}{2} \middle| H_1 \right] = Q\left(\frac{\|s_0 - s_1\|}{2\sigma} \right)$$

- The average probability of error is

$$P_e = \frac{P_{e|0} + P_{e|1}}{2} = Q\left(\frac{\|s_0 - s_1\|}{2\sigma} \right)$$

Different Types of Binary Signaling

- Let $E_b = \frac{1}{2}\left(\|s_0\|^2 + \|s_1\|^2\right)$
- For antipodal signaling, $s_1(t) = -s_0(t)$
  $E_b = \|s_0\|^2 = \|s_1\|^2$ and $\|s_0 - s_1\| = 2\|s_0\| = 2\|s_1\| = 2\sqrt{E_b}$

$$P_e = Q\left(\frac{\sqrt{E_b}}{\sigma}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

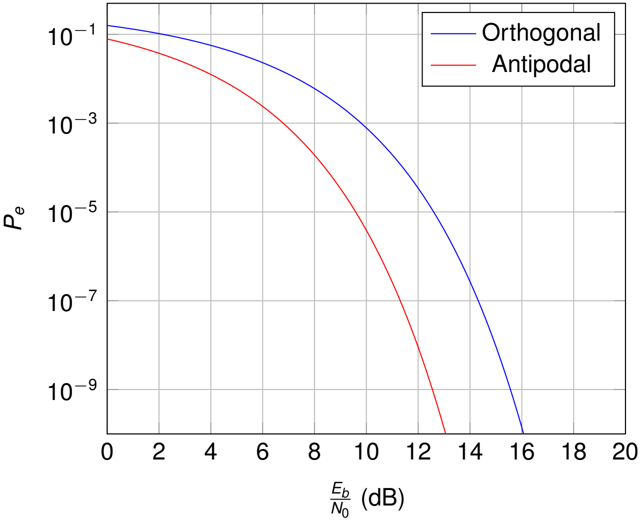
  where $\sigma^2 = \frac{N_0}{2}$
- For on-off keying, $s_1(t) = s(t)$ and $s_0(t) = 0$ and

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- For orthogonal signaling, $s_0(t)$ and $s_1(t)$ are orthogonal ($\langle s_0, s_1 \rangle = 0$)

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Performance Comparison of Antipodal and Orthogonal Signaling



Optimal Choice of Signal Pair

- For any $s_0(t)$ and $s_1(t)$, the probability of error of the ML decision rule is

$$P_e = Q\left(\frac{\|s_0 - s_1\|}{2\sigma}\right)$$

- How to chose $s_0(t)$ and $s_1(t)$ to minimize P_e?
- If E_b is not fixed, the problem is ill-defined
- For a given E_b, we have

$$P_e = Q\left(\sqrt{\frac{\|s_0 - s_1\|^2}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

where

$$\rho = \frac{\langle s_0, s_1 \rangle}{E_b}, \; -1 \leq \rho \leq 1$$

- $\rho = -1$ for antipodal signaling, $s_0(t) = -s_1(t)$
- Any pair of antipodal signals is the optimal choice

Complex AWGN Channel

ML Rule for Complex Baseband Binary Signaling

- Consider binary signaling in the complex AWGN channel

$$\begin{aligned} H_0 &: \quad y(t) = s_0(t) + n(t) \\ H_1 &: \quad y(t) = s_1(t) + n(t) \end{aligned}$$

  where

$y(t)$	Complex envelope of received signal
$s_i(t)$	Complex envelope of transmitted signal under H_i
$n(t)$	Complex white Gaussian noise with PSD $N_0 = 2\sigma^2$

- $n(t) = n_c(t) + jn_s(t)$ where $n_c(t)$ and $n_s(t)$ are independent WGN with PSD σ^2
- The ML decision rule is

$$\text{Re}\left(\langle y, s_0 \rangle\right) - \frac{\|s_0\|^2}{2} \underset{H_1}{\overset{H_0}{\gtrless}} \text{Re}\left(\langle y, s_1 \rangle\right) - \frac{\|s_1\|^2}{2}$$

$$\text{Re}\left(\langle y, s_0 - s_1 \rangle\right) \underset{H_1}{\overset{H_0}{\gtrless}} \frac{\|s_0\|^2 - \|s_1\|^2}{2}$$

- The distribution of $\text{Re}\left(\langle y, s_0 - s_1 \rangle\right)$ is required to evaluate decision rule performance

Performance of ML Rule for Complex Baseband Binary Signaling

- Let $Z = \text{Re}\left(\langle y, s_0 - s_1 \rangle\right)$
- Z is a Gaussian random variable

$$
\begin{aligned}
Z &= \text{Re}\left(\langle y, s_0 - s_1 \rangle\right) = \langle y_c, s_{0,c} - s_{1,c} \rangle + \langle y_s, s_{0,s} - s_{1,s} \rangle \\
&= \langle s_{i,c} + n_c, s_{0,c} - s_{1,c} \rangle + \langle s_{i,s} + n_s, s_{0,s} - s_{1,s} \rangle \\
&= \langle s_{i,c}, s_{0,c} - s_{1,c} \rangle + \langle n_c, s_{0,c} - s_{1,c} \rangle \\
&\quad + \langle s_{i,s}, s_{0,s} - s_{1,s} \rangle + \langle n_s, s_{0,s} - s_{1,s} \rangle
\end{aligned}
$$

- The mean and variance of Z under H_0 are

$$
\begin{aligned}
E[Z|H_0] &= \|s_{0,c}\|^2 + \|s_{0,s}\|^2 - \langle s_{0,c}, s_{1,c} \rangle - \langle s_{0,s}, s_{1,s} \rangle \\
&= \|s_0\|^2 - \text{Re}\left(\langle s_0, s_1 \rangle\right) \\
\text{var}[Z|H_0] &= \sigma^2 \|s_{0,c} - s_{1,c}\|^2 + \sigma^2 \|s_{0,s} - s_{1,s}\|^2 = \sigma^2 \|s_0 - s_1\|^2
\end{aligned}
$$

- Probability of error under H_0 is

$$
P_{e|0} = \Pr\left[Z \le \frac{\|s_0\|^2 - \|s_1\|^2}{2} \,\middle|\, H_0 \right] = Q\left(\frac{\|s_0 - s_1\|}{2\sigma} \right)
$$

Performance of ML Rule for Complex Baseband Binary Signaling

- The mean and variance of Z under H_1 are

$$\begin{aligned} E[Z|H_1] &= \langle s_{1,c}, s_{0,c}\rangle + \langle s_{1,s}, s_{0,s}\rangle - \|s_{1,c}\|^2 - \|s_{1,s}\|^2 \\ &= \mathrm{Re}\left(\langle s_1, s_0\rangle\right) - \|s_1\|^2 \\ \mathrm{var}[Z|H_1] &= \sigma^2\|s_{0,c} - s_{1,c}\|^2 + \sigma^2\|s_{0,s} - s_{1,s}\|^2 = \sigma^2\|s_0 - s_1\|^2 \end{aligned}$$

- Probability of error under H_1 is

$$P_{e|1} = \Pr\left[Z > \frac{\|s_0\|^2 - \|s_1\|^2}{2} \middle| H_1\right] = Q\left(\frac{\|s_0 - s_1\|}{2\sigma}\right)$$

- The average probability of error is

$$P_e = \frac{P_{e|0} + P_{e|1}}{2} = Q\left(\frac{\|s_0 - s_1\|}{2\sigma}\right)$$

References

- Section 3.5.1, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008