Performance of ML Receiver for M-ary Signaling

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Performance of ML Decision Rule for *M*-ary signaling

ML Decision Rule for *M*-ary Signaling

- M equally likely hypotheses

$$\begin{array}{ccc} H_1 & : & y(t) = s_1(t) + n(t) \\ H_2 & : & y(t) = s_2(t) + n(t) \\ \vdots & & \vdots \\ H_M & : & y(t) = s_M(t) + n(t) \end{array}$$

- The ML decision rule for real AWGN channel is

$$\delta_{ML}(y) = \underset{1 \le i \le M}{\operatorname{argmin}} \|y - s_i\|^2 = \underset{1 \le i \le M}{\operatorname{argmax}} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2}$$

- The ML decision rule for complex AWGN channel is

$$\delta_{ML}(y) = \underset{1 \le i \le M}{\operatorname{argmin}} \|y - s_i\|^2 = \underset{1 \le i \le M}{\operatorname{argmax}} \operatorname{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2}$$

- For $M = 2$, we found that $P_e = Q\left(\frac{\|s_1 - s_2\|}{2\sigma}\right)$
- In general, there is no neat expression for P_e as in the binary case

QPSK

- QPSK signals where $q(t)$ is a real baseband pulse of duration T

$$
\begin{aligned}
s_{1,p}(t) &= \sqrt{2}q(t)\cos\left(2\pi f_c t + \frac{\pi}{4}\right) \\
s_{2,p}(t) &= \sqrt{2}q(t)\cos\left(2\pi f_c t + \frac{3\pi}{4}\right) \\
s_{3,p}(t) &= \sqrt{2}q(t)\cos\left(2\pi f_c t + \frac{5\pi}{4}\right) \\
s_{4,p}(t) &= \sqrt{2}q(t)\cos\left(2\pi f_c t + \frac{7\pi}{4}\right)
\end{aligned}
$$

- Complex envelopes of QPSK Signals

$$
s_1(t) = q(t)e^{j\frac{\pi}{4}},\, s_2(t) = q(t)e^{j\frac{3\pi}{4}},\, s_3(t) = q(t)e^{j\frac{5\pi}{4}},\, s_4(t) = q(t)e^{j\frac{7\pi}{4}}
$$

- Orthonormal basis for the complex envelopes consists of only

$$
\phi(t) = \frac{q(t)}{\sqrt{E_q}}
$$

ML Receiver for QPSK

- $E_b = E_q/2 \implies \sqrt{E_b} = \sqrt{E_q}/\sqrt{2}$
- The vector representation of the QPSK signals is

$$\begin{aligned} s_1 &= \sqrt{E_b} + j\sqrt{E_b} \\ s_2 &= -\sqrt{E_b} + j\sqrt{E_b} \\ s_3 &= -\sqrt{E_b} - j\sqrt{E_b} \\ s_4 &= \sqrt{E_b} - j\sqrt{E_b} \end{aligned}$$

- The hypothesis testing problem is

$$H_i : Y = s_i + N, \quad i = 1, \dots, 4$$

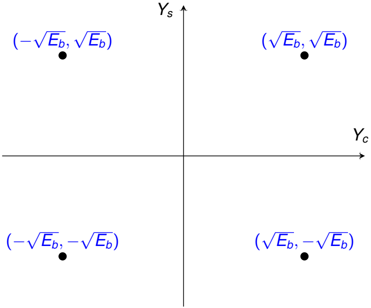
where $N = N_c + jN_s, N_c \sim \mathcal{N}(0, \sigma^2), N_s \sim \mathcal{N}(0, \sigma^2), N_c \perp N_s$

- The ML decision rule is given by

$$\delta_{ML}(y) = \underset{1 \le i \le 4}{\text{argmin}} \|y - s_i\|^2 = \underset{1 \le i \le 4}{\text{argmax}} \left[\text{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2} \right]$$

- The ML decision rule decides s_i was transmitted if y belongs to the ith quadrant

ML Decision Rule for QPSK



$$
\begin{aligned}
P_{e|1} &= \Pr\left[\left(\underset{1\le i\le 4}{\operatorname{argmin}} \|Y - s_i\|^2\right) \neq 1 \,\middle|\, \left(\sqrt{E_b}, \sqrt{E_b}\right) \text{ was sent}\right] \\
&= \Pr\left[\left(\underset{1\le i\le 4}{\operatorname{argmax}} \operatorname{Re}\left(\langle Y, s_i\rangle\right) - \frac{\|s_i\|^2}{2}\right) \neq 1 \,\middle|\, \left(\sqrt{E_b}, \sqrt{E_b}\right) \text{ was sent}\right] \\
&= \Pr\left[Y_c < 0 \text{ or } Y_s < 0 \,\middle|\, \left(\sqrt{E_b}, \sqrt{E_b}\right) \text{ was sent}\right]
\end{aligned}
$$

ML Decision Rule for QPSK

- Probability of error when s_1 is transmitted is

$$\begin{aligned} P_{e|1} &= \Pr\left[Y_c < 0 \text{ or } Y_s < 0 \,\middle|\, (\sqrt{E_b}, \sqrt{E_b}) \text{ was sent} \right] \\ &= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \end{aligned}$$

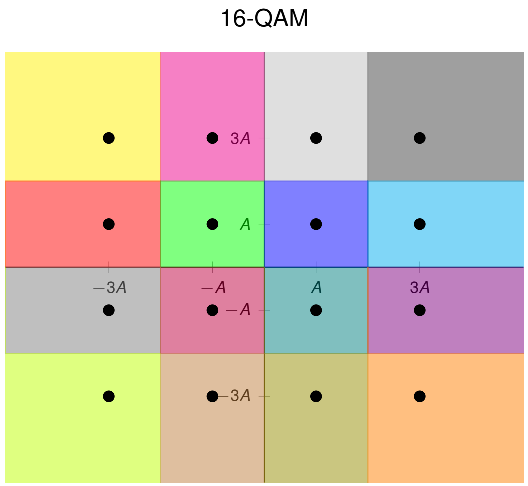
- By symmetry,

$$P_{e|1} = P_{e|2} = P_{e|3} = P_{e|4}$$

- The average probability of error is

$$P_e = \frac{1}{4}\sum_{i=1}^{4} P_{e|i} = P_{e|1} = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

ML Decision Rule for 16-QAM



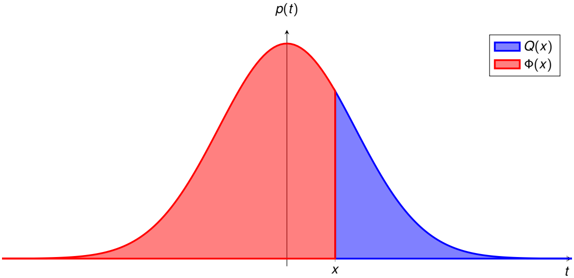
Exact analysis is tedious. Approximate analysis is sufficient.

Revisiting the Q function

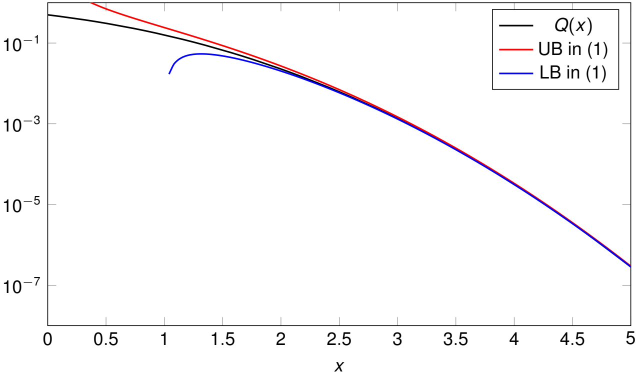
Revisiting the Q function

$X \sim N(0, 1)$

$$Q(x) = P[X > x] = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$

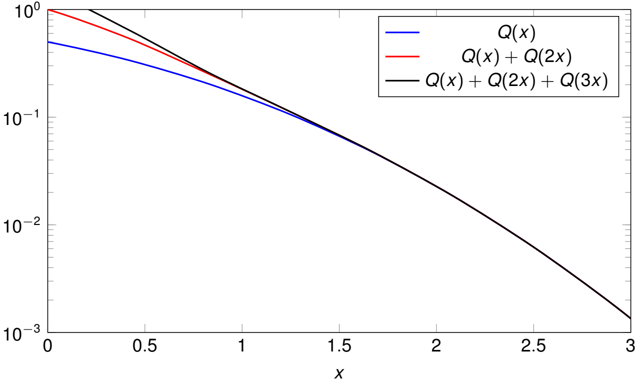


Bounds on $Q(x)$ for Large Arguments



$$\left(1 - \frac{1}{x^2}\right) \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} \leq Q(x) \leq \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} \qquad (1)$$

Q Functions with Smallest Arguments Dominate



- P_e in AWGN channels can be bounded by a sum of Q functions
- The Q function with the smallest argument is used to approximate P_e

Union Bound

Union Bound for M-ary Signaling in AWGN

- Let Z_i be $\langle y, s_i \rangle - \frac{\|s_i\|^2}{2}$ or $\text{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2}$
- The conditional error probability given H_i is true is

$$P_{e|i} = \Pr\left[\bigcup_{j \neq i} \{Z_i < Z_j\} \,\middle|\, H_i\right]$$

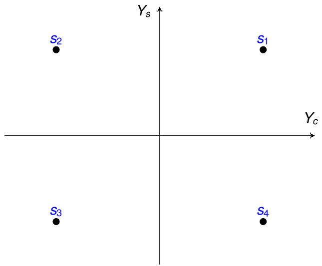
- Since $P(A \cup B) \leq P(A) + P(B)$, we have

$$P_{e|i} \leq \sum_{j \neq i} \Pr\left[Z_i < Z_j \,\middle|\, H_i\right] = \sum_{j \neq i} Q\left(\frac{\|s_j - s_i\|}{2\sigma}\right)$$

- The error probability is given by

$$P_e = \frac{1}{M}\sum_{i=1}^{M} P_{e|i} \leq \frac{1}{M}\sum_{i=1}^{M}\sum_{j \neq i} Q\left(\frac{\|s_j - s_i\|}{2\sigma}\right)$$

Union Bound for QPSK



$$
\begin{aligned}
P_{e|1} &= \Pr\left[\cup_{j\neq 1}\{Z_1 < Z_j\}\middle| H_1\right] \leq \sum_{j\neq 1} \Pr\left[Z_1 < Z_j \middle| H_1\right] \\
P_{e|1} &\leq Q\left(\frac{\|s_2 - s_1\|}{2\sigma}\right) + Q\left(\frac{\|s_3 - s_1\|}{2\sigma}\right) + Q\left(\frac{\|s_4 - s_1\|}{2\sigma}\right) \\
&= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{N_0}}\right)
\end{aligned}
$$

Union Bound for QPSK

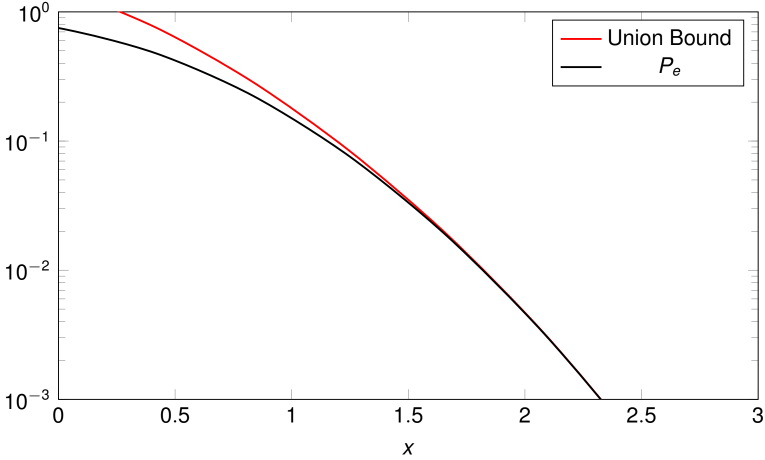
- Union bound on error probability of ML rule

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{N_0}}\right)$$

- Exact error probability of ML rule

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

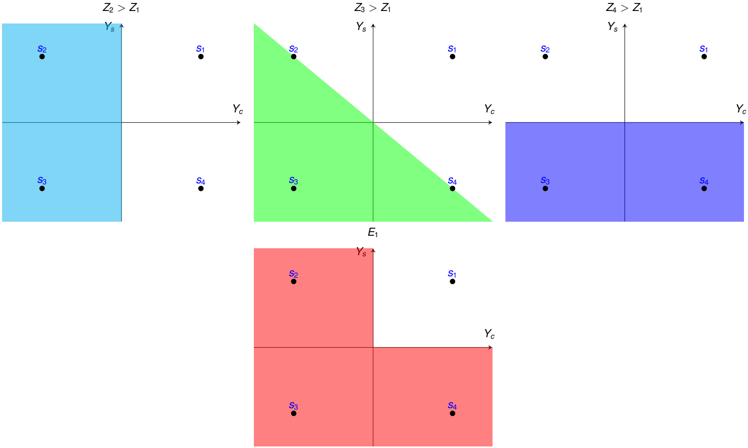
Union Bound and Exact Error Probability for QPSK



Intelligent Union Bound

QPSK Error Events

$$E_1 = [Z_2 > Z_1] \cup [Z_3 > Z_1] \cup [Z_4 > Z_1] = [Z_2 > Z_1] \cup [Z_4 > Z_1]$$



Intelligent Union Bound for QPSK

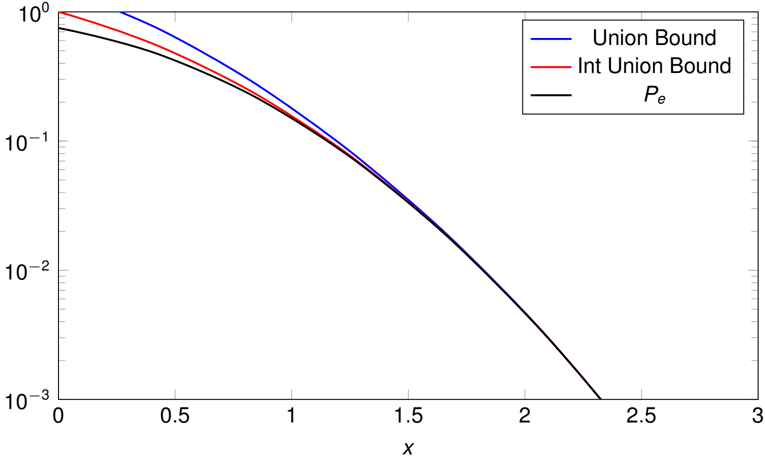
- Intelligent union bound on $P_{e|1}$

$$
\begin{aligned}
P_{e|1} &= \Pr\left[(Z_2 > Z_1) \cup (Z_4 > Z_1) \middle| H_1\right] \\
&\leq \Pr\left[Z_2 > Z_1 \middle| H_1\right] + \Pr\left[Z_4 > Z_1 \middle| H_1\right] \\
&= Q\left(\frac{\|s_2 - s_1\|}{2\sigma}\right) + Q\left(\frac{\|s_4 - s_1\|}{2\sigma}\right) \\
&= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)
\end{aligned}
$$

- By symmetry $P_{e|1} = P_{e|2} = P_{e|3} = P_{e|4}$ and

$$
P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)
$$

Intelligent Union Bound and Exact Error Probability for QPSK



General Strategy for Intelligent Union Bound

- Let $N_{ML}(i)$ be the smallest set of neighbors of s_i which define the decision region Γ_i

$$\Gamma_i = \left\{ y \middle| \delta_{ML}(y) = i \right\} = \left\{ y \middle| Z_i \geq Z_j \text{ for all } j \in N_{ML}(i) \right\}$$

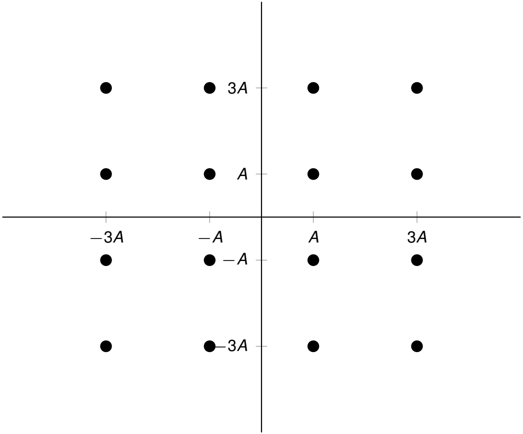
- Probability of error when s_i is transmitted is

$$\begin{aligned} P_{e|i} &= \Pr\left[Y \notin \Gamma_i | H_i\right] = \Pr\left[Z_i < Z_j \text{ for some } j \in N_{ML}(i) \middle| H_i \right] \\ &\leq \sum_{j \in N_{ML}(i)} Q\left(\frac{\|s_j - s_i\|}{2\sigma} \right) \end{aligned}$$

- Average probability of error is bounded by

$$P_e \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \in N_{ML}(i)} Q\left(\frac{\|s_j - s_i\|}{2\sigma} \right)$$

Intelligent Union Bound for 16-QAM



Assignment!

Nearest Neighbors Approximation

Nearest Neighbors Approximation

- Let d_{min} be the minimum distance between constellation points

$$d_{min} = \min_{i \neq j} \|s_i - s_j\|$$

- Let $N_{d_{min}}(i)$ denote the number of nearest neighbors of s_i

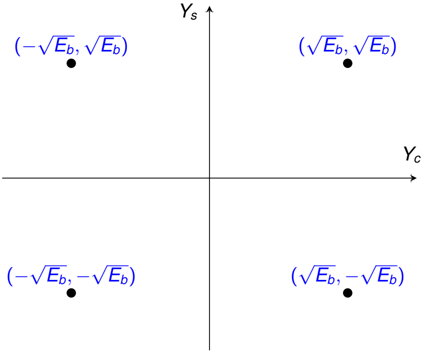
$$P_{e|i} \approx N_{d_{min}}(i) Q\left(\frac{d_{min}}{2\sigma}\right)$$

- Averaging over i we get

$$P_e \approx \bar{N}_{d_{min}} Q\left(\frac{d_{min}}{2\sigma}\right)$$

where $\bar{N}_{d_{min}}$ denotes the average number of nearest neighbors

Nearest Neighbors Approximation for QPSK



$$d_{min} = 2\sqrt{E_b}, \qquad N_{d_{min}}(1) = 2 = \bar{N}_{d_{min}}$$

$$P_e \approx \bar{N}_{d_{min}} Q\left(\frac{d_{min}}{2\sigma}\right) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Summary of results for QPSK

- Exact error probability of ML rule

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Union bound on error probability of ML rule

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{N_0}}\right)$$

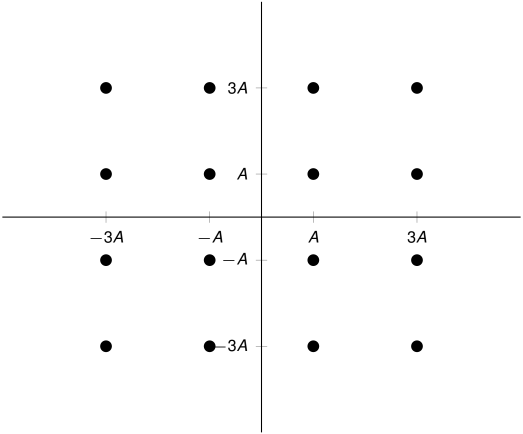
- Intelligent union bound on error probability of ML rule

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Nearest neighbors approximation of error probability of ML rule

$$P_e \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Nearest Neighbors Approximation for 16-QAM



Assignment!

References

- Section 3.5.2, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008