EE 703: Digital Message Transmission (Autumn 2022)
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Endsem Exam: 40 points — Date: November 12, 2022

1. [4 points] Suppose X_1, X_2, X_3, X_4 are jointly Gaussian random variables. We are also given that X_1, X_2, X_3, X_4 are independent random variables.

   All four random variables have variance $\sigma^2 > 0$. X_1 and X_2 have mean $\mu > 0$ while X_3 and X_4 have mean $-\mu$.

   Consider the following binary hypothesis testing problem where the hypotheses are equally likely.

$$H_1 \quad : \quad \mathbf{Y} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix},$$

$$H_2 \quad : \quad \mathbf{Y} = \begin{bmatrix} X_2 \\ X_3 \\ X_4 \end{bmatrix}.$$

   (a) Derive the optimal decision rule. Show your steps and simplify the rule as much as possible.

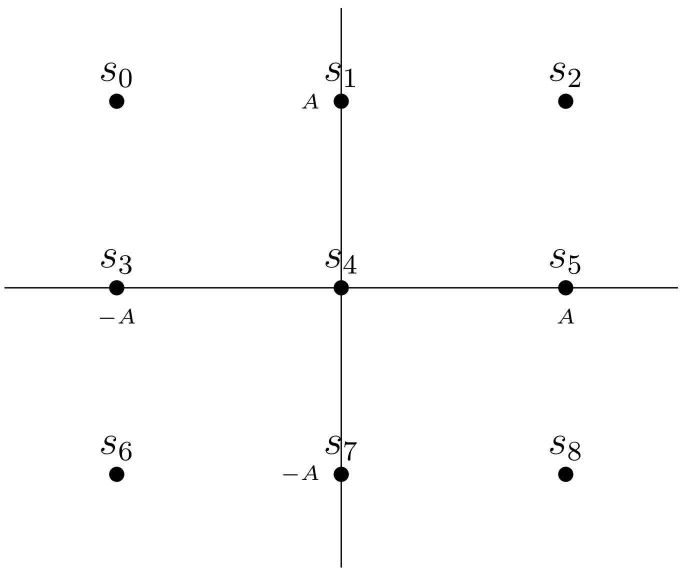
   (b) Find the decision error probability of the optimal decision rule.

2. [6 points] For the below constellation of 9 symbols, assume that the transmitted symbol is corrupted by adding $N = N_c + jN_s$ where N_c and N_s are independent Gaussian random variables with zero mean and variance σ^2. All the constellation points are equally likely to be transmitted. Calculate the following for the optimal decision rule in terms of A and σ.

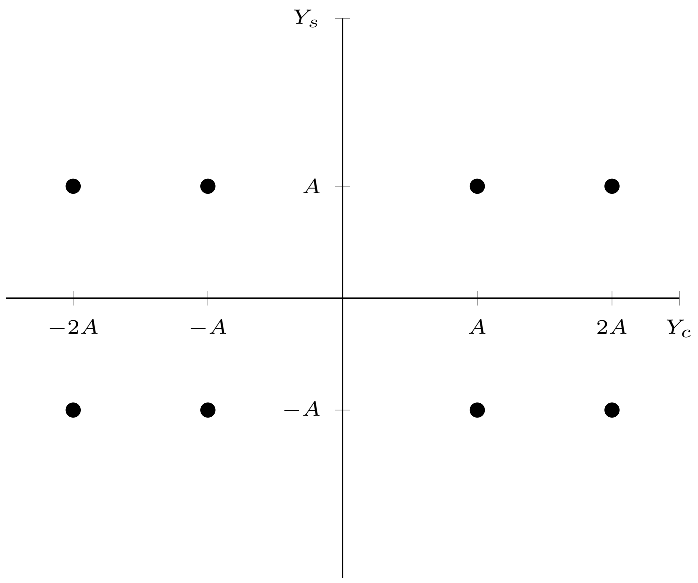
   (a) The union bound on the exact error probability.

   (b) The intelligent union bound on the exact error probability.

   (c) The nearest neighbor approximation of the exact error probability.



3. [6 points] For the below constellation of 8 symbols, assume that the transmitted symbol is corrupted by adding $N = N_c + jN_s$ where N_c and N_s are independent Gaussian random variables with zero mean and variance $\frac{N_0}{2}$. All the constellation points are equally likely to be transmitted. Calculate the BER performance of the ML receiver under a Gray mapping in terms of E_b and N_0.



4. [6 points] Let $p(t) = \frac{1}{\sqrt{T}} I_{[0,T)}(t)$ be the unit energy rectangular pulse of duration T. Suppose we are given a set of 16 signals $\{\pm p(t), \pm 3p(t), \pm 5p(t), \pm 7p(t), \pm 9p(t), \pm 11p(t), \pm 13p(t), \pm 15p(t)\}$. Due to some engineering constraints, a **transmitter can transmit signals only from this set**. The transmitted signal will be corrupted by real additive white Gaussian noise with PSD σ^2.

   (a) Suppose the transmitter wants to convey a **single bit** in the duration T. To minimize the decision error probability of the ML decision rule, which signals from the set should the transmitter use? Explain your answer.

   (b) Suppose the transmitter wants to convey **two bits** in the duration T. The transmitter chooses four signals $a_1 p(t), a_2 p(t), a_3 p(t), a_4 p(t)$ from the set where $a_1 < a_2 < a_3 < a_4$ and $a_i \in \{\pm 1, \pm 3, \ldots, \pm 15\}$ for $i = 1, 2, 3, 4$. Derive the decision error probability of the ML decision rule as a function of a_1, a_2, a_3, a_4 and σ.

   (c) For the situation described in part (b), the transmitter wants to choose a_1, a_2, a_3, a_4 such that the decision error probability of the ML decision rule **is minimized**. What values of a_1, a_2, a_3, a_4 should the transmitter use? Explain your answer.

5. [6 points] Suppose we observe $Y_i,\ i = 1, 2, \ldots, M$ such that

$$Y_i \sim U[-\theta, \theta]$$

where Y_i's are independent and θ is unknown. Assume $\theta > 0$.

(a) Derive the ML estimator of θ.

(b) For any $\epsilon > 0$, show that $\Pr\left[\left|\hat{\theta}_{ML}(\mathbf{Y}) - \theta\right| > \epsilon\right]$ tends to zero as the number of observations M goes to infinity. This will imply that the ML estimator approaches the true value of θ as the number of observations becomes large.

6. [6 points] Suppose we have two biased coins C_1 and C_2. Let the probability that C_2 shows Heads when tossed be equal to the probability C_1 shows Tails when it is tossed. Each coin is tossed M times. Let the observations be given by the following, where X_i is the random variable representing the ith toss of C_1 and Y_i is the random variable representing the ith toss of C_2. For both X_i and Y_i, the value 1 corresponds to Heads and the value 0 corresponds to Tails.

$$X_i \sim \text{Bernoulli}(p), \quad i = 1, 2, \ldots, M,$$
$$Y_i \sim \text{Bernoulli}(1-p), \quad i = 1, 2, \ldots, M.$$

The parameter p is the probability that C_1 shows Heads when tossed. Also assume that the X_i's and Y_i's are pairwise independent, and that the X_i's are independent of the Y_i's.

Find the ML estimator of the parameter p. Show your steps.

7. [6 points] Suppose X and Y are jointly Gaussian random variables. Let their joint pdf be given by

$$p_{XY}(x, y) = \frac{1}{2\pi|\mathbf{C}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{s}-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\mathbf{s}-\boldsymbol{\mu})\right)$$

where $\mathbf{s} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\boldsymbol{\mu} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$

Suppose Y is observed and we want to estimate X. Show that the MMSE estimator of X is given by

$$\hat{X}_{MMSE}(y) = \mu_x + \frac{\sigma_x}{\sigma_y}\rho(y - \mu_y).$$