EE 703: Digital Message Transmission (Autumn 2022)
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1. Let $p(t)$ be a signal which is nonzero in the interval $[0, T)$ with Fourier transform $P(f)$. Suppose $u(t)$ is a polar NRZ signal given by

$$u(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

where the symbols b_n are independent and identically distributed with probability mass function $\Pr[b_n = A] = \Pr[b_n = -A] = \frac{1}{2}$ for some real value $A > 0$.

Suppose $v(t) = u(t) + u(t - 2T)$. Note that the delay in the second term on the right hand side of the equality is $2T$.

(a) [2½ points] Prove that $v(t)$ is a cyclostationary random process with respect to period T.

(b) [2½ points] Derive the power spectral density of $v(t)$. Simplify your answer such that it does not contain any infinite summations.

2. A course instructor gave a 5-mark question in an assignment and then repeated it in a quiz on a later date. She noticed that the marks scored by the students were inconsistent. Some students who had solved the question in the assignment could not solve it in the quiz. She suspected that these students had not solved the assignment by themselves and had obtained the solution from someone else. To identify the students who had cheated, she formulated a binary hypothesis testing problem.

- She used hypothesis H_1 to represent the situation that the student had solved the question on their own for the assignment and hypothesis H_0 to represent the situation that the student had obtained the question's solution from someone else.
- Let Y_a be the marks scored by a student in the question in the assignment. The instructor assumed that the Y_a was equally likely to take any value in the set $\{4, 5\}$ under both hypotheses.
- Let Y_q be the marks scored by a student in the question in the quiz.
  - Under hypothesis H_0, the instructor assumed that the Y_q was equally likely to take any value in the set $\{0, 1, 2, 3, 4, 5\}$. She chose this to model situation that the student may have understood the solution after they had copied. So their score in the quiz could be high even though they had cheated on the assignment.
  - Under hypothesis H_1, the instructor assumed that the Y_q was equally likely to take any value in the set $\{Y_a - 1, Y_a, \min(Y_a + 1, 5)\}$. She chose this model based on the reasoning that an honest student would have a quiz score close to his assignment score. The min operation was chosen to prevent overflow in case $Y_a = 5$, i.e. a student should not get more than 5 marks in a 5-mark question.
- Both hypotheses were assumed to be equally likely.

(a) [3 points] Derive the optimal decision rule.

(b) [2 points] Find the decision error probability of the optimal decision rule.

3. [5 points] Consider the following binary hypothesis testing problem where the hypotheses are equally likely.

$$\begin{aligned} H_0 &: Y \sim U[a,b] \\ H_1 &: Y \sim U[c,d] \end{aligned}$$

where U denotes the uniform distribution and a, b, c, d are real numbers satisfying $a < c < b < d$.

(a) Derive the optimal decision rule. *Hint: There are two cases.*

(b) Find the decision error probability of the optimal decision rule.

4. Suppose X_1, X_2 are jointly Gaussian random variables each having variance $\sigma^2 > 0$. Suppose $E[X_1] = \mu$ and $E[X_2] = 2\mu$ where $\mu > 0$. We are also given that $\text{cov}(X_1, X_2) = 0$.

Consider the following hypothesis testing problem where the hypotheses are equally likely.

$$\begin{aligned} H_1 &: \mathbf{Y} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \\ H_2 &: \mathbf{Y} = \begin{bmatrix} -X_1 \\ X_2 \end{bmatrix} \\ H_3 &: \mathbf{Y} = \begin{bmatrix} X_1 \\ -X_2 \end{bmatrix} \\ H_4 &: \mathbf{Y} = \begin{bmatrix} -X_1 \\ -X_2 \end{bmatrix} \end{aligned}$$

(a) [2½ points] Derive the optimal decision rule. Show your steps and simplify the rule as much as possible.

(b) [2½ points] Find the decision error probability of the optimal decision rule.