Phase and Timing Synchronization

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The System Model

- Consider the following complex baseband signal $s(t)$

$$s(t) = \sum_{i=0}^{K-1} b_i p(t - iT)$$

where b_i's are complex symbols

- Suppose the LO frequency at the transmitter is f_c

$$s_p(t) = \text{Re}\left[\sqrt{2} s(t) e^{j2\pi f_c t}\right].$$

- Suppose that the LO frequency at the receiver is $f_c - \Delta f$
- The received passband signal is

$$y_p(t) = A s_p(t - \tau) + n_p(t)$$

- The complex baseband representation of the received signal is then

$$y(t) = A e^{j(2\pi \Delta f t + \theta)} s(t - \tau) + n(t)$$

The System Model

$$y(t) = Ae^{j(2\pi\Delta ft+\theta)}\sum_{i=0}^{K-1} b_i p(t - iT - \tau) + n(t)$$

- The unknown parameters are A, τ, θ and Δf

  Timing Synchronization Estimation of τ
  Carrier Synchronization Estimation of θ and Δf

- The preamble of a packet contains known symbols called the training sequence
- The b_i's are known during the preamble

Carrier Phase Estimation

- The change in phase due to the carrier offset Δf is $2\pi\Delta fT$ in a symbol interval T
- The phase can be assumed to be constant over multiple symbol intervals
- Assume that the phase θ is the only unknown parameter
- Assume that $s(t)$ is a known signal in the following

$$y(t) = s(t)e^{j\theta} + n(t)$$

- The likelihood function for this scenario is given by

$$L(y|\theta) = \exp\left(\frac{1}{\sigma^2}\left[\mathrm{Re}(\langle y, se^{j\theta}\rangle) - \frac{\|se^{j\theta}\|^2}{2}\right]\right)$$

- Let $\langle y, s\rangle = Z = |Z|e^{j\phi} = Z_c + jZ_s$

$$\begin{aligned}
\langle y, se^{j\theta}\rangle &= e^{-j\theta}Z = |Z|e^{j(\phi-\theta)} \\
\mathrm{Re}(\langle y, se^{j\theta}\rangle) &= |Z|\cos(\phi-\theta) \\
\|se^{j\theta}\|^2 &= \|s\|^2
\end{aligned}$$

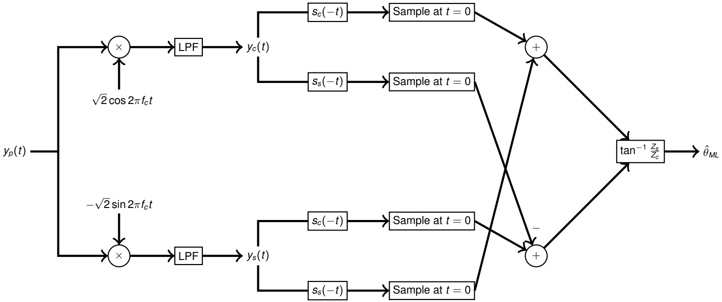
Carrier Phase Estimation

- The likelihood function for this scenario is given by

$$L(y|s_\theta) = \exp\left(\frac{1}{\sigma^2}\left[|Z|\cos(\phi-\theta) - \frac{\|s\|^2}{2}\right]\right)$$

- The ML estimate of θ is given by

$$\hat{\theta}_{ML} = \phi = \arg(\langle y, s\rangle) = \tan^{-1}\frac{Z_s}{Z_c}$$



Phase Locked Loop

- The carrier offset will cause the phase to change slowly
- A tracking mechanism is required to track the changes in phase
- For simplicity, consider an unmodulated carrier

$$y_p(t) = \sqrt{2}\cos(2\pi f_c t + \theta(t)) + n_p(t)$$

- The complex baseband representation is

$$y(t) = e^{j\theta(t)} + n(t)$$

- For an observation interval T_o, the log likelihood function is given by

$$\ln L(y|\theta) = \frac{1}{\sigma^2}\left[\mathrm{Re}\left(\langle y, e^{j\theta(t)}\rangle\right) - \frac{T_o}{2}\right]$$

- We get $\hat{\theta}_{ML}$ by maximizing

$$J[\theta(t)] = \mathrm{Re}\left(\langle y, e^{j\theta(t)}\rangle\right) = \int_0^{T_o} [y_c(t)\cos\theta(t) + y_s(t)\sin\theta(t)]\ dt$$

Phase Locked Loop

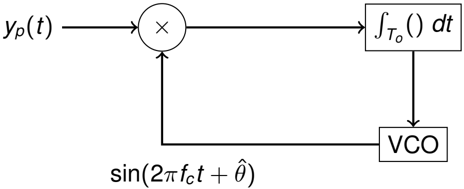
- A necessary condition for a maximum at $\hat{\theta}_{ML}$ is

$$\frac{\partial}{\partial\theta} J[\theta(t)]\Big|_{\hat{\theta}_{ML}} = 0 \implies \int_0^{T_o} \left[-y_c(t)\sin\hat{\theta}_{ML} + y_s(t)\cos\hat{\theta}_{ML}\right] dt = 0$$

$$\implies \mathrm{Re}\left(\langle y, je^{j\hat{\theta}_{ML}}\rangle\right) = 0$$

$$\implies \langle y_p, -\sin(2\pi f_c t + \hat{\theta}_{ML})\rangle = 0$$

$$\implies -\int_{T_o} y_p(t)\sin(2\pi f_c t + \hat{\theta}_{ML})\, dt = 0$$



Symbol Timing Estimation

- Consider the complex baseband received signal

$$y(t) = As(t-\tau)e^{j\theta} + n(t)$$

  where A, τ and θ are unknown and $s(t)$ is known
- For $\gamma = [\tau, \theta, A]$ and $s_\gamma(t) = As(t-\tau)e^{j\theta}$, the likelihood function is

$$L(y|\gamma) = \exp\left(\frac{1}{\sigma^2}\left[\mathrm{Re}\left(\langle y, s_\gamma\rangle\right) - \frac{\|s_\gamma\|^2}{2}\right]\right)$$

- For a large enough observation interval, the signal energy does not depend on τ and $\|s_\gamma\|^2 = A^2\|s\|^2$
- For $s_{MF}(t) = s^*(-t)$ we have

$$\begin{aligned}
\langle y, s_\gamma\rangle &= Ae^{-j\theta}\int y(t)s^*(t-\tau)\,dt \\
&= Ae^{-j\theta}\int y(t)s_{MF}(\tau-t)\,dt \\
&= Ae^{-j\theta}(y \star s_{MF})(\tau)
\end{aligned}$$

Symbol Timing Estimation

- Maximizing the likelihood function is equivalent to maximizing the following cost function

$$J(\tau, A, \theta) = \text{Re}\left(Ae^{-j\theta}(y \star s_{MF})(\tau)\right) - \frac{A^2\|s\|^2}{2}$$

- For $(y \star s_{MF})(\tau) = Z(\tau) = |Z(\tau)|e^{j\phi(\tau)}$ we have

$$\text{Re}\left(Ae^{-j\theta}(y \star s_{MF})(\tau)\right) = A|Z(\tau)|\cos(\phi(\tau) - \theta)$$

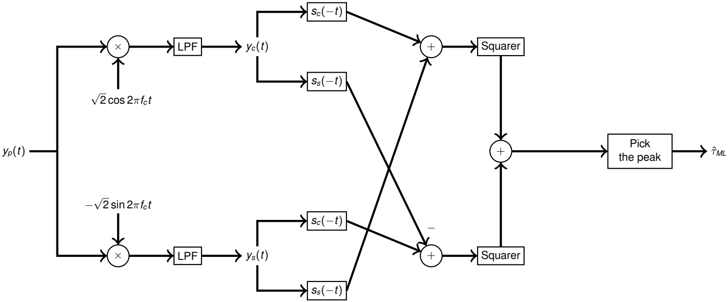
- The maximizing value of θ is equal to $\phi(\tau)$
- Substituting this value of θ gives us the following cost function

$$J(\tau, A) = \underset{\theta}{\text{argmax}}\, J(\tau, A, \theta) = A|(y \star s_{MF})(\tau)| - \frac{A^2\|s\|^2}{2}$$

Symbol Timing Estimation

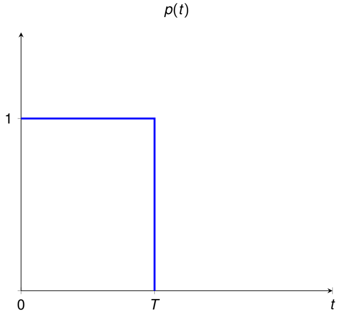
- The ML estimator of the delay picks the peak of the matched filter output

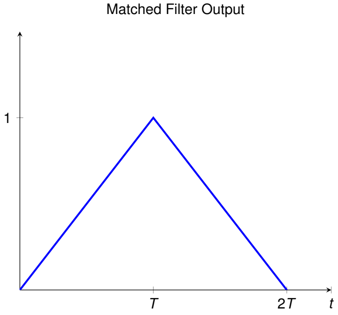
$$\hat{\tau}_{ML} = \underset{\tau}{\text{argmax}} \left| (y \star s_{MF})(\tau) \right|$$



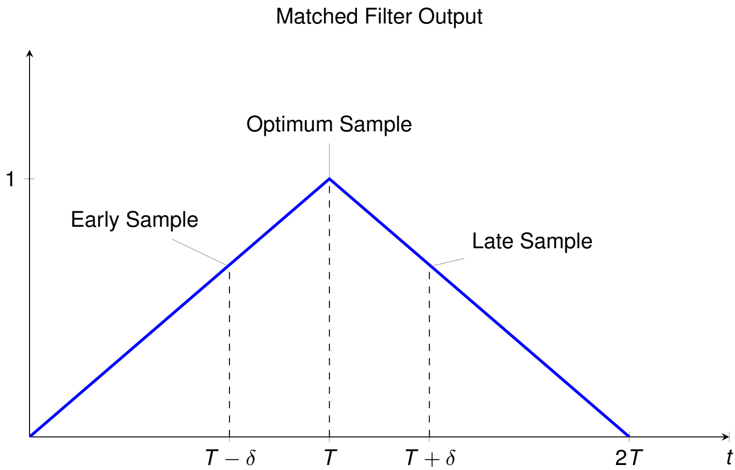
Early-Late Gate Synchronizer

- Timing tracker which exploits symmetry in matched filter output



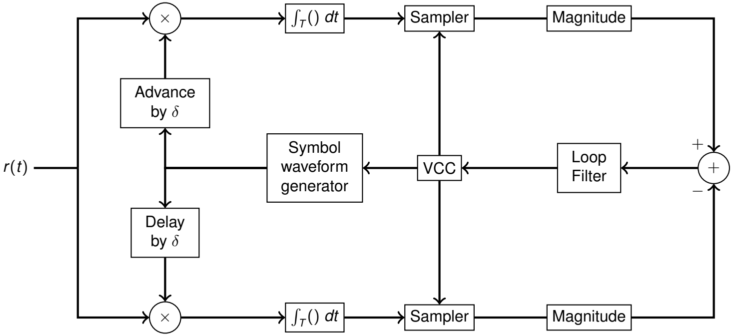


Early-Late Gate Synchronizer



- The values of the early and late samples are equal

Early-Late Gate Synchronizer



- The motivation for this structure can be seen from the following approximation

$$\frac{dJ(\tau)}{d\tau} \approx \frac{J(\tau+\delta) - J(\tau-\delta)}{2\delta}$$

References

- Section 4.3, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008