Quiz 2 : 20 points March 23, 2018

1. Let $G = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$ be a group with multiplication as the operation performed using the rules
$$\sigma^3 = e, \quad \tau^2 = e, \quad \tau\sigma = \sigma^2\tau,$$
where e is the identity of the group G.
   (a) (1 point) Simplify $\tau\sigma^2$ to one of the six elements in G.
   (b) (1 point) Simplify $\tau(\sigma\tau)$ to one of the six elements in G.
   (c) (1 point) Simplify $(\sigma\tau)(\sigma\tau)$ to one of the six elements in G.
   (d) (1 point) Simplify $(\sigma\tau)(\sigma^2\tau)$ to one of the six elements in G.
   (e) (1 point) Give an example to show that G is **not** an abelian group.

2. (a) (2½ points) Prove that every subgroup of a cyclic group is cyclic.
   (b) (2½ points) Prove that a cyclic group of order n has $\phi(n)$ generators.

   Note: $\phi(1) = 1$. For $n > 1$, the value of $\phi(n)$ is the number of integers in $\{1, 2, \ldots, n-1\}$ which are relatively prime to n, i.e. which satisfy $\gcd(i, n) = 1$.

3. Let G and H be groups. A function $\phi : G \mapsto H$ is called a **group homomorphism** if it satisfies
$$\phi(g_1 \star g_2) = \phi(g_1) \circ \phi(g_2), \text{ for all } g_1, g_2 \in G.$$
Here $\star$ is the group operation in G and $\circ$ is the group operation in H.
   (a) (2½ points) Let e_G be the identity of G and let e_H be the identity of H. Prove that $\phi(e_G) = e_H$.
   (b) (2½ points) For all $g \in G$, prove that $\phi(g^{-1}) = [\phi(g)]^{-1}$.

4. (5 points) Let $m_1, m_2, \ldots, m_k$ be positive integers greater than 1 and let $m = m_1 m_2 \cdots m_k$ be their product. Assume that $m_1, m_2, \ldots, m_k$ are **pairwise** relatively prime, i.e. $\gcd(m_i, m_j) = 1$ for $i \neq j$. Prove that the function $f : \mathbb{Z}_m \mapsto \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$ given by
$$f(a) = (a \bmod m_1, a \bmod m_2, \ldots, a \bmod m_k)$$
is a one-to-one function. That is, for all $a_1, a_2 \in \mathbb{Z}_m$ with $a_1 \neq a_2$ you have to show that $f(a_1) \neq f(a_2)$.