Lecture 2: Palm Probabilities and Rate Conservation Laws

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In Lecture 1 we introduced Palm probabilities as evaluating probabilities conditioning on a point of an underlying point process that is jointly defined on the same space.

In this lecture we will study Palm probabilities in more detail and a Rate Conservation Laws (RCL) that relate calculations under the Palm measure to the underlying reference probability. The RCLs actually allow us to derive very useful formulae in many applications.
Recall from Lecture 1: Palm Probability

Let \((\Omega, F, \mathbb{P})\) be a complete probability space which carries a measurable flow (shift) \(\{\theta\}_t\). Let \(\mathbb{P}\) be stationary w.r.t. \(\{\theta_t\}\) i.e.

\[ P \circ \theta_t^{-1} = P \]

Let \(N\) be a point stationary point process (w.r.t the flow \(\{\theta_t\}\) defined on \((\Omega, F, \mathbb{P})\)

\[ N(\theta_t \omega, C) = N(\omega, C + t) \]

where \(C\) is a Borel set in \(\mathbb{R}\).

Let \(\lambda_N\) denote the average intensity of \(N\) given by:

\[ \lambda_N = E[N(0, 1)] \]

The Palm Probability of \((N, \mathbb{P})\) is defined by:

\[ P^0_N(A) = \frac{1}{\lambda_N \ell(C)} E[\int_C 1_A(\theta_s)N(ds)] \]

where \(\ell(c)\) denotes the Lebesgue measure of \(C\) and the definition does not depend on \(C\).
Properties of $P^0_N$

1. $P^0_N(N\{0\} = 1) = P^0_N[T_0 = 0] = 1$
2. $P^0_N \circ \theta_{T_n} = P^0_N$
3. $E^0_N[T_1] = \lambda_N^{-1}$.

An immediate consequence of the definition is the so-called Mathes-Mecke formula

$$\lambda_N E^0_N \left[ \int_\mathbb{R} v(s)ds \right] = E \left[ \int_\mathbb{R} v(0) \circ \theta_s N(ds) \right]$$

for any $\theta_t$ compatible process $v(t)$

Papangelou’s Formula:

$$\lambda_N E^0_N[X] = E[\lambda(0)X]$$

for all non-negative $\mathcal{F}_{0-}$ measurable r.v. $X$. 
Via Papangelou’s theorem we can define a so-called Rate Conservation Law (RCL) for càdlàg processes that allows us to derive most of the important formulae associated with Palm theory. To do so let us first obtain a representation for all càdlàg processes of bounded variation.

Every càdlàg process \( \{X_t\} \) càdlàg having jump discontinuities can be written as:

\[
X_t = X_0 + X_t^c + X_t^d
\]

where \( X_t^c \) is purely continuous (w.r.t. \( t \)) and \( X_t^d \) is purely discontinuous. If the number of jumps in each compact interval is finite then the process \( \{N_t\} \) defined by:

\[
N_t = \sum_{s \leq t} 1[X_s \neq X_{s-}]
\]

is a locally finite, simple point process.
Let $\Delta X_t = X_t - X_{t-}$ denote the jump of $\{X_t\}$. Then:

$$X_t^d = \sum_{s \leq t} \Delta X_s$$

and by the definition of $\{N_t\}$ $X_t^d = \int_0^t Y_s dN_s$ where

$$Y_t = \sum_{n=0}^{\infty} (\Delta X_{T_n}) 1_{[T_n, T_{n+1})}(t)$$

Finally if $\{X_t\}$ is of bounded variation then

$$X_t^c = \int_0^t X_s^+ ds$$

where $X_s^+$ denotes the right derivative of $\{X_t\}$. Therefore we obtain the following evolution equation:

$$X_t = X_0 + \int_0^t X_s^+ ds + \int_0^t Y_s dN_s$$
We now state the main result.

**Theorem** Let \( \{X_t\} \) be a cadlag process of bounded variation that is stationary w.r.t \( \theta_t \) on \((\Omega, \mathcal{F}, \mathbb{P})\). Then:

\[
\mathbb{E}[X_0^+] + \lambda_N \mathbb{E}^0_N[\Delta X_0] = 0
\]

In particular for any \( f(.) \) that is \( C^1 \) we have:

\[
\mathbb{E}[f'(X_0)X_0^+] + \lambda_N \mathbb{E}^0_N[\Delta f(X_0)] = 0
\]

where \( \Delta f(X_0) = f(X_0) - f(X_0^-) \).
Applications

The first simple application is the level crossing formula due to Brill and Posner.

**Theorem** Let \( \{X_t\} \) be a stationary cadlag process that possesses a density. Then:

\[
p(x)E[X_0^+/X_0 = x] = \lambda_N E_N^0 \left[ 1_{X_0^- > x} - 1_{X_0 > x} \right]
\]

Noting that:

\[
1_{X_0^- > x} - 1_{X_0 > x} = 1_{X_0^- > x} 1_{X_0 \leq x} - 1_{X_0^- \leq x} 1_{X_0 > x}
\]

We can re-write the result as:

\[
p(x)E[X_0^+/X_0^- = x] = \lambda_N E_N^0 \left[ 1_{X_0^- > x} 1_{X_0 \leq x} - 1_{X_0^- \leq x} 1_{X_0 > x} \right]
\]
The proof follows by directly applying the RCL to the process: \( Y_t = 1_{[X_t > x]} \). Formally (since the indicator function is not differentiable), \( Y_t^+ = \delta(X_t - x)X_t^+ \) and hence, \( \mathbb{E}[Y_0^+] = p(x)\mathbb{E}[X_0^+|X_0 = x] \). Substituting in the RCL we obtain:

\[
p(x)\mathbb{E}[X_0^+|X_0 = x] = \lambda_N \mathbb{E}_N[1_{[X_0 > x]} - 1_{[X_0 > x]}]
\]

Noting that:

\[
1_{[X_0 > x]} - 1_{[X_0 > x]} = 1_{[X_0 > x]}1_{[X_0 \leq x]} - 1_{[X_0 \leq x]}1_{[X_0 > x]}
\]
Now we see two very fundamental formulae arising in Palm theory namely: 1) The Palm inversion formula and 2) Neveu’s cycle formula.

For any $X_t$ that is compatible with $\theta_t$ (stationary):

$$E[X_0] = \lambda_N \mathbb{E}_N^0 \left[ \int_0^{T_1} X_s ds \right]$$

**Proof:** Let $T_+(t)$ be the first point of $N_t$ after $t$.

Define $Y_t = \int_t^{T_+(t)} X_s ds$ Then $Y_t^+ = -X_t$ and $Y_0^- = 0$ since $T_+(0-) = T_0$ and $T_0 = 0$ under $\mathbb{P}_0^N$ Hence

$$\mathbb{E}[Y_0^+] = -\mathbb{E}[X_0] = -\lambda_N \mathbb{E}_N^0 [\Delta Y_0] = -\lambda_N \mathbb{E}_N^0 \left[ \int_0^{T_1} X_s ds \right]$$
Neveu’s Exchange Formula

Let $N$ and $N'$ be two stationary point processes defined on $(\Omega, \mathcal{F}, P)$ compatible with $\theta_t$.

Then:

$$\lambda_N \mathbb{E}^N_0[f(0)] = \lambda_{N'} \mathbb{E}^0_{N'} \left[ \int_0^{T'_1} (f \circ \theta_t) dN_t \right]$$

Proof; We give a proof with a stochastic intensity. Define $g(t) = \int_t^{T'_1(t)} f_s \lambda_s ds$

Then it is easy to see $g^+t = -f_t \lambda_t$ and $\Delta g(0) = \int_0^{T'_1} f_s dN_s$. Then applying RCL w.r.t. $N'$ and using Papangelou’s formula we have:

$$\lambda_N \mathbb{E}^0_N[f(0)] = \lambda_{N'} \mathbb{E}^0_{N'} \left[ \int_0^{T'_1} f_s dN_s \right]$$

Taking $f = 1$ we see $\lambda_{N'} \mathbb{E}^0_{N'}[N[0, T'_1]] = \lambda_N$ we have

$$\mathbb{E}_N[f(0)] = \frac{\mathbb{E}^0_{N'} \left[ \int_0^{T'_1} (f \circ \theta_t) dN_t \right]}{\lambda_N \mathbb{E}^0_{N'}[N[0, T'_1]]}$$

which gives the cycle representation (the Palm distribution can be obtained as an average over a selection of points of the original point process).
Recall: Let $R_t = T_+ (t) - t$ where by definition $T_+ (t) = T_{N_t + 1}$ where $\{T_n\}$ are the points of a stationary point process $N_t$. Define: $X_t = (R_t - x)^+$. Then $X_t^+ = -\mathbb{I}_{[R_t > x]}$ and the jumps of $X_t$ are given by:

$$\Delta X_{T_n} = (R_{T_n} - x)^+ - (R_{T_n-} - x)^+.$$ By definition $R_{T_n} = T_{n+1} - T_n$ and $R_{T_n-} = 0$ and therefore for any $x \geq 0$ we have

$$\Delta X_{T_n} = (T_{n+1} - T_n - x)^+$$

Applying the RCL we obtain:

$$\mathbb{E} [\mathbb{I}_{[R_0 > x]}] = \lambda N \mathbb{E}_N [(T_1 - x)^+] = \lambda N \int_x^\infty (y - x) dF(y)$$

where $F(x)$ is the distribution of the inter-arrival time $S_0 = T_1 - T_0 = T_1$ and under $\mathbb{P}_N$ we have $T_0 = 0$. 
Let us define $\bar{F}_e(x) = \lambda_N \mathbb{E}_N[(T_1 - x)^+]$, then it is easy to see that $\bar{F}_e(0) = 1$, $\lim_{x \to \infty} \bar{F}_e(x) = 0$, and thus $F_e(x) = 1 - \bar{F}_e(x)$ defines the distribution of a r.v. that we call $R_e$. $F_e(x)$ is called the equilibrium distribution of $R_0$ with density $\lambda_N(1 - F(x))$ where $F(x)$ is the distribution of $T_1$ under $\mathbb{P}_N$. By direct calculation $\mathbb{E}[R_e] = \int_0^\infty \bar{F}_e(x)dx = \frac{1}{2} \mathbb{E}_N[T_1^2]$. 
A similar argument can be carried out for the backward recurrence time denoted by $B_0$ and the stationary distribution is also equal to $F_e(x)$.

Let $S_e = R_e + B_e$ where $R_e$ is the stationary forward recurrence time and $B_e$ is the stationary backward recurrence time. $S_e$ is called the equilibrium spread or lifetime of the point process.

By definition the spread is defined as $S_t = T_{N_t+1} - T_{N_t}$.

Therefore:

$$\frac{1}{t} \int_0^t \mathbb{I}_{[S_u > x]} du = \frac{1}{t} \sum_{k=1}^{N_t} S_k \mathbb{I}_{[S_k > x]}$$

i.e, we count all the interarrival times that are larger than $x$.

Hence noting that $\lim_{t \to \infty} \frac{N_t}{t} = \lambda_N$ we obtain that

$$P(S_e > x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{I}_{[S_s > s]} ds$$

satisfies:

$$P(S_e > x) = \lambda_N \mathbb{E}_N[T_1 \mathbb{I}_{[T_1 > x]}]$$

from which it readily follows that:

$$\mathbb{E}[S_e] = \mathbb{E}[R_e] + \mathbb{E}[B_e] = \frac{\mathbb{E}_N[T_1^2]}{\mathbb{E}[T_1]}$$
Proposition

Let $S_e = R_e + B_e$, and $F_S(x)$ denote the equilibrium distribution of a point process whose Palm distribution is given by $F(x) = \mathbb{P}_N(T_1 \leq x)$. Then $S_e$ is stochastically larger under $\mathbb{P}$ than $T_1 - T_0$ under $\mathbb{P}_N$.

Proof Let $F_S(x) = \mathbb{P}(S_e \leq x)$. To show this result we need to show that:

$$1 - F_S(x) = \mathbb{P}(S_e > x) \geq 1 - F(x) \quad x \geq 0$$

Now:

$$\mathbb{P}(S_e > x) = \lambda_N E[N[T_1 \mathbb{I}_{[T_1 > x]}] = \lambda_N x(1 - F(x)) + (1 - F_e(x))$$

Noting that: $\lambda_N = \frac{1}{\int_0^{\infty} (1 - F(y))dy}$. We therefore need to show

$$x(1 - F(x)) + \frac{1}{\lambda_N}(1 - F_e(x)) \geq \frac{1}{\lambda_N}(1 - F(x)) .$$

This readily follows from

$$(\lambda_N)^{-1}(1 - F(x)) = (1 - F(x)) \int_0^x (1 - F(y))dy + (1 - F(x)) \int_x^{\infty} (1 - F$$

$$\leq x(1 - F(x)) + \lambda_N^{-1}(1 - F_e(x))$$
Queue from customer and workload viewpoints

![Diagram](image-url)

**Figure:** Typical sample paths of congestion and workload viewpoints. The diagram shows the buffer occupancy sample path and the workload sample path with busy and idle periods labeled. The sample paths illustrate the dynamic changes in buffer occupancy and workload over time.
**Evolution Equations**

Let $Q_t$ denote the number of packets in the queue at time $t$ or the congestion process at time $t$. Then:

$$Q_t = Q_0 + A(0, t] - D(0, t] \quad t \geq 0$$

Let $\{W_t\}$ denote the workload at time $t$. It denotes the amount of work to be processed at time $t$ in the queue. Clearly the sample-path of $W_t$ is cadlag and the jumps correspond to $\sigma_n$. The evolution of the workload can be written as:

$$W_t = W_0 + X(0, t] - c \int_0^t \mathbb{I}_{[W_s > 0]} ds, \quad t \geq 0$$

where $X(0, t]$ denotes the amount of work arriving in $(0, t]$ and is given by $X(0, t] = \sum_{0 < T_n \leq t} \sigma T_n = \int_0^t \sigma_s dA_s$ where $\sigma_t = \sigma_{T_n} \mathbb{I}_{[T_n, T_{n+1})}(t)$. 
Skorokhod Problem

Note we can re-write equation for $W_t$ as:

$$W_t = W_0 + X(0, t) - ct + c \int_0^t \mathbb{I}_{[W_s=0]} ds = W_0 + J(0, t) + I(0, t)$$

where $J(0, t) = X(0, t) - ct$ is called the netput and $I(0, t) = c \int_0^t \mathbb{I}_{[W_s=0]} ds$ is called the idle time. This form of writing the workload process has a very nice interpretation. It is called a Skorohod reflection problem stated as follows.

Given a cadlag process $x_t$ find a non-negative process $z_t$ such that

$$y_t = x_t + z_t \geq 0 \quad t \geq 0$$

Once again denoting $J(0, t)$ and $I(0, t)$ by $J_t$ and $I_t$ respectively, we note the following.

1. $I_t$ is an increasing process in $t$ with $I_0 = 0$.
2. $\int_0^t W_s dI_s = 0$, i.e., $I_t$ can only increase when $W_t$ is 0.
3. Furthermore, $I_t = \sup_{0 < s \leq t} (-J_s - W_0)^+$ where $(x)^+ = \max\{x, 0\}$
Proposition

Let $Q_t$ be the congestion process resulting from a stationary arrival process $A_t$ and a corresponding departure process $D_t$. If $Q_t$ is stationary, then the stationary distribution seen just before an arrival when the buffer is in state $n$ is equal to the stationary distribution just after a departure in state $n + 1$ i.e.

$$\pi_A(n) = \pi_{D+}(n), \quad n = 0, 1, 2, \ldots$$

where $\pi_A(\cdot)$ denotes the distribution just before an arrival, and $\pi_{D+}(\cdot)$ denotes the distribution just after the departure.
First order equivalence

In the general case, it is of interest to compute quantities under the stationary distribution. For stationary (and ergodic) single server queues there is in very simple relationship that can be found in terms of so-called conditional intensities that we develop below.

Define:

\[
\lambda_i = \lim_{t \to \infty} \frac{\int_0^t \mathbb{I}[Q_s = i] dA_s}{\int_0^t \mathbb{I}[Q_s = i] ds} = \lim_{t \to \infty} \frac{A_t \frac{1}{A_t} \int_0^t \mathbb{I}[Q_s = i] dA_1}{\frac{1}{t} \int_0^t \mathbb{I}[Q_s = i] ds}
\]

Note the first term in the numerator above \( \frac{A_t}{t} \to \lambda_A \), the second term in the numerator goes to \( \pi_A(i) \) while the denominator goes to \( \pi(i) \) as \( t \to \infty \).

Similarly, define

\[
\mu_i = \lim_{t \to \infty} \frac{\int_0^t \mathbb{I}[Q_s = i] dD_s}{\int_0^t \mathbb{I}[Q_s = i] ds}
\]
Therefore, by the definition of $\lambda_i$ and $\mu_i$ we obtain: $\lambda_i \pi_i = \lambda_A \pi_A(i)$ and $\mu_i \pi(i) = \lambda_D \pi_D(i)$, then from above we obtain the following relationship:

$$\lambda_i \pi(i) = \mu_{i+1} \pi(i + 1), \quad i = 0, 1, 2, \ldots$$

and hence:

$$\pi(n) = \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i} \pi(0), \quad n = 1, 2, \ldots$$
Stationary Distributions of $GI/M/1$ queues

The way we analyze this queue is by considering the embedded process at arrival times, say $\{T_n\}$.

Let $Q_n = Q_{T_n-}$. Then the evolution of the queue is as follows:

$$Q_{n+1} = Q_n + 1 - D(T_n, T_{n+1}]$$

where $D(T_n, T_{n+1}]$ denotes the number of departures in $(T_n, T_{n+1}]$.

Now $D(T_n, T_{n+1}) \leq Q_n + 1$ since the number of departures between 2 arrivals cannot exceed the total number in the queue. Let $H(t) = P_A(T_1 \leq t)$ denote the inter-arrival time distribution and under $P_A$ the inter-arrival times are i.i.d. The departure distribution can be computed explicitly from:

$$p_k = P(D(T_n, T_{n+1}] = k) = \int_0^\infty \frac{(\mu t)^k}{k!} e^{-\mu t} dH(t)$$

since the event corresponds to $k$-departures in one inter-arrival time and the departures are Poisson since the service times are exponential.
Define the moment generating function $P(z)$ of $\{p_k\}$:

$$P(z) = \sum_{k=0}^{\infty} p_k z^k$$

$$= \sum_{k=0}^{\infty} z^k \int_0^\infty \frac{(\mu t)^k}{k!} e^{-\mu t} dH(t)$$

$$= \phi_A(\mu(1 - z)),$$

where $\phi_A(z) = \mathbb{E}_A[e^{-zT_1}]$ (noting that $P_A(T_0 = 0) = 1$ by definition of the Palm probability).
Consider a GI/M/1 queue in equilibrium with $\lambda A E_A[\sigma_0] = \rho < 1$. Then there exists a unique solution in $(0, 1)$ to the fixed point equation:

$$\xi = P(\xi)$$

where $P(z)$ as defined above. Moreover, the arrival stationary distribution of the queue is given by:

$$\pi_A(i) = \xi^i(1 - \xi), \quad i = 0, 1, 2, \ldots$$
Relation between the arrival stationary and (time) stationary distribution

This follows from equations first order equivalence noting the fact that the service times are exponentially distributed. Indeed from the fact that the arrivals are i.i.d. we have:

\[ \lambda_A \pi_A(i) = \mu \pi_D(i + 1) = \mu \pi(i + 1), \quad i = 0, 1, \ldots \]

since \( \pi_D(i + 1) = \pi(i + 1) \) from the fact that the rate associated with exponential distributions is constant (PASTA). Therefore noting that \( \rho = \frac{\lambda_A}{\mu} \) we have:

\[ \pi(i + 1) = \rho \pi_A(i), \quad i = 0, 1, 2 \ldots \]

and from the normalization condition \( \sum_{i=0}^{\infty} \pi(i) = 1 \) we finally obtain the stationary distribution as:

\[ \pi(i) = \rho(1 - \xi)\xi^{i-1}, \quad i = 1, 2, \ldots \]
\[ \pi(0) = 1 - \rho \]
Palm theory for general stationary increasing measures

Because of the high speed of modern networks we actually need to extend Palm theory to general stationary increasing processes.

Let $A(0, t)$ be a continuous increasing process with stationary increments. Let $\theta_t$ be a measurable flow and we assume $A_t = A(0, t)$ is compatible with it. We can define a (fluid) Palm measure readily as follows:
For any Borel set $C$

$$P^0_A(C) = \frac{1}{\lambda_A} \mathbb{E} \left[ \int_{[0,1]} \mathbb{1}_C(\theta_s)A(ds) \right]$$

and in particular:
For all $\mathcal{F}$-measurable processes $(Z(t), t \geq 0)$,

$$E \int_{\mathbb{R}} Z(s) \circ \theta_s A(ds) = \lambda_A E_A \int_{\mathbb{R}} Z(s) ds.$$
For $c > 0$, define $(Q(t), t \geq 0)$ as:

$$Q(t) = Q(0) + A(t) - ct + Z(t),$$

where $(Z(t), t \geq 0)$ is an increasing process, null at 0, which satisfies

- For all $t \geq 0$, $Q(0) + A(t) - ct + Z(t) \geq 0$,
- The support of $Z(dt)$ is included in the set $\{s \geq 0, Q(s) = 0\}$.
Define $\rho_A = \frac{\lambda_A}{c}$. Then, under the condition $\rho_A < 1$, it can be shown that there exists a stationary regime for $Q$, i.e. there is a unique $\{\theta_t\}$ consistent solution defined on the same probability space $(\Omega, \mathcal{F}, P)$. 
Assume that $A$ is a continuous, stationary, increasing random measure with $E[A(0, 1)] = \lambda_A$ and $\rho_A = \lambda_A c^{-1} < 1$. Then,

1) For all continuous functions $\varphi$

$$cE\varphi(Q(0))1_{\{Q(0) > 0\}} = \lambda_A E_A\varphi(Q(0))1_{\{Q(0) > 0\}}.$$

2) For all Borel sets $B$ of $\mathbb{R}$ which do not contain the origin 0,

$$P_A[Q(0) \in B] = \rho_A^{-1} P[Q(0) \in B].$$

3) At the origin i.e. when $B = \{0\}$,

$$P_A[Q(0) = 0] = \rho_A^{-1} (P[Q(0) = 0] - 1) + 1.$$
Under the hypotheses above:

\[ E[Q(0)] = \rho_A E_A[Q(0)] \]

Note unlike the classical Little's law that relates the average number to the average waiting time here we just have a kind of Mecke formula.
Now consider an input of the type ON-OFF given by the following description:

Let $N$ be a stationary marked point process with points $\{T_n; n \in \mathbb{Z}\}$ and marks $\{(L_n, S_n, F_n), n \in \mathbb{Z}\}$ such that

- $T_0 \leq 0 < T_1$,
- The random marks $(L_n)$ are positive,
- Each triplet $(T_n, L_n, S_n)$ satisfies $T_{n+1} - T_n = L_n + S_n$,
- The marks $F_n$ are continuous increasing processes null at 0, constant on $]L_n, +\infty[$ and such that $F_n(t) \geq ct$ on $\{0 \leq t \leq L_n\}$ (burstiness assumption).
Define the following random measure:

\[ A(B) \equiv \sum_{n \in \mathbb{Z}} \int_{B-T_n} F_n(dt) \]

where \( B \) is a Borel set in \( \mathbb{R} \).
Then \( A(t) \) is a continuous stationary increasing process that specifies the cumulative input up to time \( t \).
We assume that, under $P_N$, the Palm measure associated with $N$, the sequences $(F_n)$, $(L_n)$ and $(S_n)$ are i.i.d. and mutually independent and in addition the r.v’s $S_n$ are exponentially distributed. Defining $m \equiv E_N[T_1]$, $n \equiv E_N[F_0(L_0)] = E_N[A(0, T_1)] = \lambda_A m$ and $q = P[T_0 + L_0 < 0] = \frac{E_N[S_0]}{m}$.
With respect to the filtration $\mathcal{F}_t$ generated by the process $A([u, t])_{t \geq 0}$, $u < t$, the stochastic intensity of the point process $(N_t)$ is given by

$$\lambda_t \equiv (qm)^{-1}1\{\xi_t=0\}$$

where $\xi_t \in \{0, 1\}$ and takes the value 1 if the source is ON at time $t$ and 0 otherwise.
Pollaczek-Khinchine Formula for Fluid Inputs

\[
E[Q(0)] = \frac{1}{c - \lambda_A} \frac{1}{m} E_N^0 [F_0(L_0) - \lambda_A L_0]^2 - \frac{1}{m} E_N^0 \left[ \int_0^{L_0} t(F_0(dt) - \lambda_A dt) \right]
\]

where \(m = E_N^0[T_1]\), \(F_0(t)\) is denotes the cumulative input on \([0, t]\) for the source when ON under \(P_{N_0}\), \(L_0\) is the length of an ON period of the source and \(\lambda_A = E[A(0, 1)]\).

Note the difference with the Pollaczek-Khinchine formula in the point process case.
RCL for processes of unbounded variation

We can actually extend the RCL to processes of unbounded variation (Levy processes for example).
In order to do so we first formulate the general class of processes.
We consider the class of semi-martingales given by:

\[ X_t = X_0 + \int_0^t \Delta X_s dN_s + \int_0^t a(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \]

where \( \Delta X_t = X_t - X_{t-} \). and

\[ N_t = \sum_{s \leq t} \mathbb{1}(X_s \neq X_{s-}) \]

In addition we assume the following regularity condition on the diffusion

\[ \limsup_t \frac{1}{t} \int_0^t \sigma^2(s, X_s) ds < \infty \]

Define the process \( \{ Y_t \} \) by :

\[ Y_t = \sum_{n=0}^{\infty} (X_{t_n} - X_{t_n-}) \mathbb{1}_{[t_n, t_{n+1})}(t) \]
Then \( \{Y_t\} \) is a right continuous process which measures the jump size. We assume that \( \sup_{t_n} E|X_{t_n} - X_{t_n^-}|^2 < \infty \) implying that the process \( \{Y_t\} \) satisfies \( \sup_t E|Y_t|^2 < \infty \). It is also assumed that \\
\( \lim \sup_t \frac{1}{t} \int_0^t Y_s^2 d\Lambda_s < \infty. \)
Pathwise rate conservation

For the process $\{X_t\}$ defined by if $\frac{X_t}{t} \to 0$ a.s. as $t \to \infty$ then:

$$\lim_{t \to \infty} \left[ \frac{1}{t} \int_0^t Y_s \, dA_s + \frac{1}{t} \int_0^t a(s, X_s) \, ds \right] = 0 \text{ a.s.}$$
Let $x_0 \in \mathbb{R}$ and suppose the process $\{X_t\}$ satisfies $\frac{X_t}{t} \to 0$ a.s. as $t \to \infty$ then:

$$
\lim_{t \to \infty} \left[ \frac{1}{t} \int_0^t I_{[X_s > x_0]} a(s, X_s) ds + \frac{1}{t} \int_0^t Z_{s-} \, dA_s + \frac{L(t, x_0)}{2t} \right] = 0 \text{ a.s.} \quad (1.8)
$$

where the process $\{Z_t\}$ is defined by

$$
Z_t = \sum_{n=0}^{\infty} [(X_{t_n} - x_0)^+] - (X_{t_n-} - x_0)^+ I_{[t_n, t_{n+1}]}(t) \quad \forall \ t \geq 0
$$

and $\{L(t, x_0)\}$ denotes the local time process of the semi-martingale $X_t$ at $x_0$ defined by

$$
L(t, y) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \lambda_{[y, y+\epsilon]}(X_s) d < X^c, X^c >_s
$$

and $< X^c, X^c >_t = \sigma_t^2$
Suppose the process \( \{X_t\} \) is stationary and ergodic and the underlying point process has an average intensity \( \lambda_N = E[\lambda_0] \). Furthermore let \( F^+_X(x) \) denote the right derivative of the distribution function associated with the invariant distribution of \( X \). Then for any \( a \in \mathbb{R} \):

\[
2E[\lambda[X_0 > a]a(X_0)] + 2\lambda_N E^0_N[\triangle(X_0 - a)^+] + \sigma^2(a)F^+_X(a) = 0
\]

where \( E^0_N \) denotes expectation w.r.t. the Palm measure associated with the stationary point process and \( E \) denotes expectation w.r.t. invariant distribution.
Consider the process \( \{X_t\} \) defined as follows:

\[
dX_t = -\rho X_t dt + \sigma dW_t; \quad t \in [\tau_n, \tau_{n+1})
\]

where \( \{\tau_n\} \) is the sequence of hitting times to 0 of \( \{X_t\} \) defined by:

\[
\tau_0 = \inf\{t \geq 0 : X_t = 0\}
\]

\[
\tau_{n+1} = \inf\{t \geq \tau_n : X_t = 0\}
\]

At hitting times the process \( \{X_t\} \) is reflected above the origin by \( \eta_n \) where \( \{\eta_n\} \) are i.i.d. non-negative random variables with common distribution function denoted by \( F(x) \).
Following the technique of removing the drift in we first establish that indeed \( \{X_t\} \) is ergodic and converges to a stationary process. 
Define the nonlinear transformation:

\[
Y_t = f(X_t)
\]

where

\[
f(x) = \int_0^x e^{\frac{\rho z^2}{\sigma^2}} dz
\]

Then since \( \lim_{x \to \infty} f(x) = \infty \) and \( \lim_{x \to -\infty} f(x) = -\infty \), with \( f'(x) = e^{\frac{\rho x^2}{\sigma^2}} > 0 \) and \( f(0) = 0 \). This implies that \( f(.) \) defines an invertible 1 : 1 transformation from \( \mathbb{R} \to \mathbb{R} \).
Then clearly:

\[
\int_0^\infty \frac{1}{\tilde{\sigma}^2(z)} dz = \frac{1}{\sigma^2} \int_0^\infty e^{-\frac{2\rho y^2}{\sigma^2}} df(y) = \frac{1}{\sigma^2} \int_0^\infty e^{-\frac{\rho y^2}{\sigma^2}} dy < \infty
\]

Hence, it follows that the process \( \{Y_t\} \) is ergodic and so is \( \{X_t\} \).
Let \( G(x) \) denote the stationary distribution of \( \{X_t\} \). Using the level crossing formula we obtain
\[
\frac{\sigma^2}{2} G^+(x) - \rho \int_x^\infty yG^+(y)dy + 1_N \int_x^\infty (1 - F(y))dy = 0
\]
which shows that \( G^+(x) \) is absolutely continuous in \( x \) and hence differentiating w.r.t \( x \),
\[
\frac{\sigma^2}{2} \frac{dG^+(x)}{dx} + \rho xG^+(x) - 1_N(1 - F(x)) = 0
\]
Using the fact that \( G^+(0) = 0 \) we obtain:
\[
G^+(x) = \frac{21_N}{\sigma^2} \int_0^x e^{-\frac{\rho}{\sigma^2}(x^2-y^2)}(1 - F(y))dy
\]
Using the fact that \( \int_0^\infty G^+(x)dx = 1 \) we can calculate \( 1_N \) which gives:
\[
G^+(x) = \frac{\int_0^x e^{-\frac{(x^2-y^2)^2}{\sigma^2}}(1 - F(y))dy}{\int_0^\infty \int_0^z e^{-\frac{(z^2-y^2)^2}{\sigma^2}}(1 - F(y))dydz}
\]
This requires 10 pages of computation via usual techniques. See for example the book of Gihman and Skorohod.
RCL is a very useful technique to compute stationary quantities associated with càdlàg processes.

Palm approach helps us precisely identify the measures we need for a computation.

Martingale properties can be used under Palm since stochastic intensity remains same under Palm.
Insensitive queueing networks.
Bandwidth allocation and utility optimization
Flow based architectures.
Computation of performance quantities.
Pricing for shared resource systems
Large systems and randomized load balancing. Chaoticity of system.
References


- M. Miyazawa; The derivation of invariance relations in complex queueing systems with stationary inputs, Advances in Applied Probability 15 (1983), pp. 875-885