Lecture 3: Insensitive Bandwidth Sharing and Flow Based Models

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What is the best way to share resources that is robust to traffic statistics?

Main measures of the quality of service (QoS) are the allocated server rates, and duration of the task or sojourn in the system.

Being able to estimate congestion of a resource (when rates are below desired rates) is of great importance for user contracts.

Doing so that is both insensitive to traffic characteristics and tractable will lead to robust engineering and pricing rules in designing future networks.
The main focus of this talk will be on congestion and pricing in systems that operate under an **insensitive** fair allocation scheme for heterogeneous flows with differing maximum or peak bandwidth requirements.

Our aim is to understand relationship to pricing mechanisms for insensitive systems and calculation of revenue and user payments.

Design of pricing mechanisms-Pre or post payment- Charge upon arrival based on current state or charge at end of call.

Idea is to study second order characteristics or volatility.
Swiss Army Formula of Palm Calculus

**Theorem**

Let \( \{A(t)\} \) be a stationary point process whose points are denoted by \( \{T_n\} \) with \( \ldots < T_{-1} < T_0 \leq 0 < T_1 < \ldots \). Let \( \{W_n\} \) be a stationary sequence of non-negative random variables and define \( \tau_n = T_n + W_n \). Let \( D(t) \) be the point process associated with the sequence \( \{\tau_n\} \), i.e., \( D(t) = \sum_{n \in \mathbb{Z}} \mathbb{1}_{(0,t]}(\tau_n) \). Let \( X(t) = X(0) + A(t) - D(t) \), \( X(0) \) a positive integer, then \( \{X(t)\} \) is an integer-valued cadlag process. Let \( B(t) \) be a stationary non-decreasing cadlag process. Then for any stationary process \( Z(t) \) that is jointly stationary with \( A(.) \) and \( B(.) \):

\[
\lambda_A \mathbb{E}_A \left[ \int_{(0,W_0]} Z(s) dB(s) \right] = \frac{1}{t} \mathbb{E} \left[ \int_{(0,t]} X(s-)Z(s) dB(s) \right]
\]

\(^a\)Note the sequence \( \{\tau_n\} \) need not be ordered.
**Proof** By definition $X(t) - X(0) = A(t) - D(t)$. Note Lebesgue-Stieltjes integration by parts for càdlàg processes can be re-written as:

$$X_t Y_t = X_0 Y_0 + \int_{(0,t]} X_s dY_s + \int_{(0,t]} Y_s dX_s$$

Take as $Y_t = \int_{(0,t]} Z(s) dB(s)$, $X_t = X(t) - X(0)$ and noting $Y_0 = 0$ substituting we obtain the result.
Some applications:

1. **Palm inversion formula:** The above result can also be directly from the Swiss Army formula by taking $W_n = T_{n+1} - T_n$, $dB(s) = ds$, and noting by definition $X(t) = 1$ for this choice of $W_n$.

2. **Neveu's Cycle Formula**
   To obtain this from the Swiss Army formula, let $B(t)$ be a stationary point process $N_2$ with mean intensity $\lambda_2$. Take $W_n = T_{n+1}^1 - T_n^1$ where $T_n^1$ are the points of $N_1(t) = A(t)$. Then noting by definition of Palm probability that $\mathbb{E}\left[\int_{(0,1]} Z(s)dN_2^s\right] = \lambda_2 \mathbb{E}_{N_2}[Z(0)]$ we obtain Neveu's exchange formula.

3. **Little's Formula**
   
   \[
   \mathbb{E}[Q_0] = \lambda_A \mathbb{E}_{A}[W_0]
   \]

   where $W_0 = W_0^- + \sigma_0$ where $W_0^-$ denotes the workload seen by an arrival.

   From SAF by taking $Z(t) = 1$. 

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Insensitive bandwidth sharing
We can further extend Little’s formula to relate higher order moments (provided they exist) by choosing $Z(t) = X(t)^k$, $k \geq 1$ to obtain:

$$\lambda_A \mathbb{E}_A \left[ \int_{(0, W_0]} X^k(s) ds \right] = \mathbb{E}[X(0)^{k+1}]$$

In a similar way if we take $B(t) = A(t)$ then we obtain:

$$\mathbb{E}_A[A(0, W_0)] = \mathbb{E}_A[X(0-)]$$
We will consider a flow level model of tasks arriving of random sizes or volumes.

- Introduced by Roberts and Massoulié (Infocom 2000)
- Ignores the packet level dynamics and models the files as fluid flows
- The bandwidth allocated to tasks of the same class are shared equally
- This can be modeled as letting each class of flow go to separate processor sharing queues but with variable capacity depending on number of flows in system
The system is a single link with $M$ classes of traffic

- **Link capacity** $C$
- **Traffic intensity** $\alpha_i = \lambda_i / \mu_i$, $i = 1 \ldots M$
- **Average load** $\rho = \sum_j \alpha_j / C$
- **Allocated bandwidth** $\phi_i(x)$, $i = 1 \ldots M$
Markov Process

- Let $X$ be the state process, where the state is the numbers of flows of each class.
- $X$ is modeled as a continuous time jump Markov process.
- Let $\phi_i(x)$ depends on the system state.
- State transition rates: $q(x, y) = \begin{cases} 
\lambda_i & y = x + \vec{e}_i \\
\mu_i \phi_i(x) & y = x - \vec{e}_i \\
0 & Otherwise
\end{cases}$
Bandwidth Allocation

- Bandwidth allocation is a fundamental, well studied problem
- Most popular and studied class of allocations are the *Utility* based allocations
- Let \( x \) be the state vector whose components \( x_i \) are the number flows of class \( i \)

\[
\max_{\phi} \sum_j x_j U(\phi_j(x)/x_j)
\]

s.t. \( \sum_j \phi_j(x) \leq C \)

when \( U_i(x) = \log x \) it is termed *proportional fairness*.

If we solve this problem for a single link of capacity \( C \) then flows of class \( i \) are allocated:

\[
\phi_i(X) = \frac{x_i}{\sum_j x_j} C
\]

that corresponds to Processor Sharing.
Insensitive Allocations

Insensitive allocations are those that result in *insensitivity* of the stationary distribution to task characteristics.

- Characterized by Balance Function $\Phi(.)$
- Balance Property

$$\phi_i(x)\phi_j(x - e_i) = \phi_j(x)\phi_i(x - e_j), \ \forall i, j \in N, \ \forall x : X_i > 0, x_j > 0$$

- Allocation is defined as $\phi_i(x) = \frac{\Phi(x - e_i)}{\Phi(x)}$
- Insensitive allocations have the advantage that the stationary distribution $\pi(x)$ depends on the flow size distribution only through its mean

$$\pi(x) = \pi(\bar{0})\Phi(x) \prod_{i=1}^{M} \alpha_i^{x_i}$$

- For Processor Sharing

$$\Phi(x) = \frac{|x|!}{x_1!x_2! \cdots x_N!}, \ x_1, x_2, \ldots, x_N > 0, \ |x| = \sum_{i=1}^{N} x_i$$

This result is due to Whittle (JAP 1974). (See Kelly’s book 1976).
Balanced Fairness

- Introduced by Bonald and Proutière (QUESTA 2003)
- Most efficient insensitive allocation is Balanced Fairness

**Lemma**

Consider another positive function $\tilde{\Phi}$ such that $\tilde{\Phi}(0) = 1$ and the rate and capacity constraints are satisfied. Then

$$\tilde{\Phi}(x) \geq \Phi(x) \quad \forall x \in \mathbb{Z}_+^M. \quad (1)$$

- The Balance Function for a single link is:

$$\Phi(x) = \left( \frac{1}{C} \sum_{i=1}^{M} \Phi(x - \vec{e}_i) \right)$$

- **Theorem** Stable iff $\rho < 1$

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Balanced Fairness and Proportional Fairness

- Balanced Fairness coincides with proportional fairness on many topologies and has been empirically shown to approximate Proportional Fairness well in many cases. For single links it corresponds to processor sharing.
- Massoulié (Ann. App. Prob 2007) proved some very useful theoretical connections between Balanced Fairness and Proportional Fairness
  - **Theorem** If there exists $\tilde{\phi}$ s.t. $\phi_{BF}^i (nx) \to \tilde{\phi}_i (x)$ as $n \to \infty$, then $\tilde{\phi}(x) = \phi_{PF}^i (x)$
  - **Theorem** $\lim_{n \to \infty} \frac{1}{n} \log \pi_{BF} (nx) \Rightarrow - \max \sum_{j} x_j \log (\phi_j / \alpha_j)$ s.t.
    $$\phi \in C$$
    Where $C$ is the set of feasible allocations.
  - **Conjecture** $\phi_{BF}^i (nx) \to \phi_{PF}^i (x)$ as $n \to \infty$
- Walton (Ann. App Prob 2010) has generalized the results of Massoulié to any max stable (ie. stability condition $\rho < 1$) insensitive allocation
Consider a Processor sharing queue with $M$ classes defined by $(\nu_i, \sigma_i)$. Let $x = \text{col}(x_1, x_2, \ldots, x_M)$ denote the state of the system, where $x_j$ is the number of flows of type $j$ in equilibrium. In this case the rate received by a given source of type or class $m$ is:

$$
\phi_m(x) = \begin{cases} 
  x_m r_m & \text{if} \sum_{j=1}^{M} x_j r_j \leq C \\
  \frac{x_m}{\sum_{j=1}^{M} x_j} C & \text{otherwise}
\end{cases}
$$

Defining the balance function $\Phi(x)$ as

$$
\Phi(x) = \prod_{m=1}^{M} \frac{1}{(x_m!) r_m^{x_m}} \sum_{j=1}^{M} r_j x_j \leq C
$$

$$
= \frac{|x|!}{C^{|x|} \prod_{m: x_m > 0} x_m!} = \frac{1}{C} \sum_{m} \Phi(x - e_m) \text{ if } \sum_{j=1}^{M} x_j r_j > C
$$
Let us first show some properties of the model. Let us define congestion as the regime in which a flow receives less than the desired rate $r_i$.

**Lemma**

*For the flow model described above, if $\sum_{j=1}^{M} x_j r_j > C$ then $\phi_m(x) < x_m r_m$ for all classes that are present in the system, i.e. $x_m > 0, m = 1, 2, \ldots, M$*

Let $C_m = \{ x : \phi_m(x) < r_m x_m \}$ be the set of states for which congestion occurs for flows of type $m \in \mathcal{M}$. The first congestion metric, the probability of congestion $P_m$, is defined in straightforward manner,

$$P_m \triangleq \pi (X \in C_m).$$

There are two equivalent interpretations for $P_m$. It can be seen as the long term average the flows of class-$m$ are congested. Alternatively, from the PASTA property, it is the steady-state probability that a flow of class-$m$ enters a congested network.
The other congestion metric of interest, the time-average congestion rate, is a measure of the average fraction of time that an arrival does not receive its desired rate during its time in the system. Let $W_m$ be the sojourn time of class-$m$ arrivals in the system. Define

$$F_m \triangleq \frac{\mathbb{E}_m \left[ \int_0^{W_m} \mathbb{1}_{\{X(t) \in C_m\}} \, dt \right]}{\mathbb{E}_m[W_m]},$$

(2)

where the expectation is taken with respect to the Palm measure for the point process of arrivals of class-$m$ and $x$ is the stationary state process. Then $F_m$ denotes the ratio of the average time that a class-$m$ flow spends in a congested state during its sojourn to the average sojourn time.
It follows from Little’s formula and the Swiss Army formula of Palm calculus,

\[ \mathbb{E}_\pi[X_m(0)] = \nu_m \mathbb{E}_m[W_m] \]

and

\[ \mathbb{E}_\pi[\mathbb{I}\{X(0) \in C_m\} X_m(0)] = \nu_m \mathbb{E}_m \left[ \int_0^{W_m} \mathbb{I}\{X(t) \in C_m\} \, dt \right]. \]

Therefore

\[ F_m = \frac{\sum_{x \in C_m} x_m \pi(x)}{\sum_{x} x_m \pi(x)}. \]
It has been previously established that congestion will not occur for any class if the system state $x$ satisfies the condition $x \cdot r \leq C$. Since the states for which congestion occurs is the same for all flow classes, the probability of congestion will just be written as $P$. One can now write the probability of congestion as

$$P = \sum_{x : x \cdot r > C} \pi(x).$$

(3)

The following result expressions can actually be written as a function of far fewer states in terms of the blocking probability.

**Lemma**

*The probability of congestion can be written as*

$$P = \sum_{m=1}^{M} \frac{\rho_m B_m}{1 - \rho},$$

*with*

$$B_m = \sum_{x : C - r_m < x \cdot r \leq C} \pi(x).$$
Noting that the stationary distributions $\pi$ and $\pi^B$ where $\pi^B$ is the stationary distribution of an Erlang loss system given by

$$\pi^B(x) = G^B \prod_{k=1}^{M} \frac{\rho^{x_k}}{x_k!}$$

are proportional on those states $x$ such that $x \cdot r \leq C$, it follows that

$$B_m = \frac{G^B}{G} P^B_m.$$  

Thus the probability of congestion is:

$$P = \frac{G^B}{G} \sum_{m=1}^{M} \frac{\rho_m P^B_m}{1 - \rho}.$$
In the following we will discuss pricing of a single shared resource which performs insensitive bandwidth allocation, i.e., Processor sharing for example. What we will see is that most commonly used pricing mechanisms have very simple functional forms in terms of quantities that can be calculated.
There are three commonly used pricing mechanisms.

- Fixed cost per unit bandwidth per unit time (old telephone type pricing).

- The second is the notion of *efficiency optimal* pricing that corresponds to a social optimal solution. The way that is achieved is through a Vickrey-Clarke-Groves (VCG) auction with the notion of the revelation principle - get users or players to reveal "true" value or utility. Here we need an extension to divisible resources.

  - second price auction
  - not necessarily revenue maximizing but *practical* in markets with multiple sellers
  - used in online keyword auctioning for advertising and VCG type auctions have been used in spectrum auctions.
Revenue maximization pricing or congestion pricing. These correspond to Lagrangian methods called Lagrangian shadow prices or congestion prices.

- useful from distributed implementation perspective
- price associated with maximizing social welfare.

There is an important result due to Bulow and Kemperer (The American Economic Review 1996), which states that revenue from efficiency optimization with $k + 1$ players is at least as high as the revenue from revenue maximization with $k$ players.
Interest in analyzing revenue

Projecting earning allows dimensioning resources
  - eg., cloud-computing, network bandwidth

To analyze a dynamic system, we focus on *insensitive allocations*:
  - fairness (processor-sharing discipline),
  - efficient allocations (maximizes social welfare), and
  - product-form stationary distribution!
- Stationary distribution independent of higher-order statistics of users’ jobs

- Insensitive to
  - job size distribution keeping mean $\nu$ the same
    - exponential distribution $\rightarrow$ phase-type distribution
  - arrival process keeping rate $\lambda$ the same
    - Poisson $\rightarrow$ Poisson-session
  - time-scale
    - $\lambda \rightarrow N\lambda$ and $\nu \rightarrow \nu/N$ has no effect—scale independent.

- Price of insensitivity: restriction to processor-sharing on a single link.
If a network operator charges more by congestion in the network, it gives them greater incentive to keep the network congested!

This also implies that users are satisfied with arbitrarily small resource allocations

QoS: Users pay only when their minimum rate requirement $r_{min}$ is met

We can perform certain forms of admission control while preserving insensitivity. However any discriminatory type processing leads to the system not being insensitive.
Consider the following problem of resource allocation

- Infinitely divisible resource: $C$ capacity rate
  eg., cloud-computing, modeling large file transfers over TCP

- $K$ classes of users

- Let $\mathbf{x}$ represent the system state where
  $x_k = $ number of class $k$ users present

- Each user with an identical log utility function
Problem formulation (contd)

- $\Lambda_k \geq 0$ total allocation to class $k$, such that
  - $\sum_k \Lambda_k \leq C$
  - $\Lambda_k$ is shared equally between all $x_k$ users

- Model system as a server with $M/G$ inputs, i.e.,
  - Poisson arrival with rate $\lambda_k$
  - general independent service requirement with mean $\nu_k$

- Processor sharing discipline
Problem formulation 3

- QoS. User of class $k$ pays only if $\Lambda_k / x_k \geq r_{min}$

- Equivalently, user pays only when number of users in the system, $|\vec{x}| \leq n^*$ where

$$n^* = \left\lfloor \frac{C}{r_{min}} \right\rfloor$$

- Define $\rho_k = \frac{\lambda_k \nu_k}{C}$ and $\rho = \sum_k \rho_k$
Resource allocation 1

- Maximizing social welfare

\[
\max \sum_{k: x_k > 0} x_k \log(\Lambda_k / x_k)
\]

subject to \( \Lambda_k \geq 0 \)

\[
\sum_k \Lambda_k \leq C
\]

- Optimal allocation is

\[
\Lambda_k = C \frac{x_k}{|\vec{x}|}
\]
Writing the Lagrangian

\[ L(\tilde{\Lambda}; z) = \sum_{k: x_k > 0} x_k \log(\Lambda_k / x_k) + z(C - \sum_k \Lambda_k) \]

and solving the dual problem gives

\[ z = \frac{|\bar{x}|}{C}. \]
Fixed rate pricing: mean revenue per unit-time

- $\beta$, per unit-resource per unit-time price
  - $\Lambda$ allocation for time $T \implies \beta \Lambda T$ charge
- Like a *pay-what-you-use* plan
- In state $\vec{x}$ for $1 \leq |\vec{x}| \leq n^*$,
  - user price $c_k^F(\vec{x}) = \beta C / |\vec{x}|$ per unit-time
  - operator’s revenue $R_F(\vec{x}) = \beta C$ per unit-time
VCG pricing

- Second price auction
- Not revenue maximizing; Revenue inefficiency means:

\[
\max_{\phi} \sum_{i=1}^{K} p_i^V \phi_i \geq \sum_{i=1}^{K} p_i^V \phi_i^V
\]

- used in online keyword auctioning for advertising
- User pays the decrease in social welfare by its presence\(^1\)

\[
c_k^V(\vec{x}) = \max_{\vec{\Lambda}} \sum_{s \neq r | \Lambda_s = 0} \log(\Lambda_s) - \sum_{s \neq r} \log(\Lambda_s^{PS})
\]

\[
\approx 1 - \frac{1}{2|\vec{x}|} \quad \text{ (for user } r, \text{ a member of class } k)\]

- Operator’s revenue \(R_V(\vec{x}) \approx |\vec{x}| - \frac{1}{2}\) per unit-time
  - payments only if \(2 \leq |\vec{x}| \leq n^* \) (QoS)

VCG: revenue approximation

Figure: The approximation used.

- Exact expression for revenue in state $\vec{x}$ is

$$|\vec{x}|(|\vec{x}| - 1) \log \frac{|\vec{x}|}{|\vec{x}| - 1}, \quad \text{for } 2 \leq |\vec{x}| \leq n^*$$

- Approximation used: $|\vec{x}| - \frac{1}{2}$
Consider
\[
\max \sum_{k=1}^{K} x_k \log \left( \Lambda_k / x_k \right)
\]
subject to
\[
\sum_{k=1}^{K} \Lambda_k \leq C
\]

- Shadow price interpretation of dual variable of the constraint
- User pays \( c_k^L(\vec{x}) = |\vec{x}| / C \) per unit-time per user
- Operator's revenue \( R_L(\vec{x}) = |\vec{x}|^2 / C \) per unit-time
  - payments only if \( 1 \leq |\vec{x}| \leq n^* \) (QoS)
Key observation

- Link between mean revenue and moments of the $\ell_1$ norm of the number of users in the system

Zeroth moment

$$\bar{R}_F = \beta C \mathbb{E}[|\bar{x}|^0 1(1 \leq |\bar{x}| \leq n^*)]$$

First moment

$$\bar{R}_V \approx \mathbb{E} \left[ \left( |\bar{x}| - \frac{1}{2} \right) 1(1 < |\bar{x}| \leq n^*) \right]$$

Second moment

$$\bar{R}_L = \frac{1}{C} \mathbb{E}[|\bar{x}|^2 1(1 \leq |\bar{x}| \leq n^*)]$$
Operator’s revenue

- \( \pi \), the stationary distribution
- Operator’s mean revenue with QoS: \( \bar{R}_F \), \( \bar{R}_V \), \( \bar{R}_L \) (per unit-time)

\[
\bar{R}_{(\cdot)} = \mathbb{E} \left[ R_{(\cdot)}(\vec{x}) \right] \\
= \sum_{\vec{x}} R_{(\cdot)}(\vec{x}) \mathbf{1}_{(1 \leq |\vec{x}| \leq n^*)} \pi(\vec{x})
\]

- Fixed \( C \) and \( r_{min} \), varying \( \rho \) (i.e., \( \lambda \), \( \nu \))
Operator’s revenue: fixed rate pricing

Figure: Mean revenue under fixed-rate pricing. ($\beta = C = 1$)

$$\hat{R}_F = \beta C \rho$$
(without QoS)

$$\bar{R}_F = \beta C \rho (1 - \rho^{n^*})$$
(with QoS)
Operator’s revenue: VCG auctions

(a) $\hat{R}_V$, without QoS.

(b) $\bar{R}_V$, with QoS.

\[\hat{R}_V = \rho^2 (3 - \rho_1 - \rho)\]

\[\bar{R}_V = \rho^2 (3 - \rho_1 - \rho) - \rho n^* + 1 (n^* + 1 - \rho - 1)^2\]
Operator’s revenue: congestion based pricing

\[ \hat{R}_L = \frac{\rho(1+\rho)}{C(1-\rho)^2} \]  
(without QoS)

\[ \bar{R}_L = \frac{1-\rho}{C} \sum_{n=1}^{n^*} n^2 \rho^n \]  
(with QoS)

Figure: Mean revenue under congestion based pricing with QoS. \((C = 1)\)
Mean job level payments

- $W_k$, sojourn time of class $k$ job
- $A_k$, arrival process for class $k$
- Mean payment by a class $k$ user $\bar{c}_k^F, \bar{c}_k^V, \bar{c}_k^L$

$$
\bar{c}_k^{(\cdot)} = \mathbb{E}_{A_k} \left[ \int_0^{W_k} c_k^{(\cdot)}(\vec{x}(t)) \mathbf{1}_{(1 \leq |\vec{x}(t)| \leq n^*)} \, dt \right]
$$

- Using Swiss Army formula\(^2\)

$$
\bar{c}_k^{(\cdot)} = \frac{1}{\lambda_k} \mathbb{E} \left[ x_k c_k^{(\cdot)}(\vec{x}) \mathbf{1}_{(1 \leq |\vec{x}| \leq n^*)} \right]
$$

---

Mean job payments: fixed rate pricing

Figure: Mean user payment under fixed rate pricing. \((\nu_k = 1, \beta = 1)\)

\[
\hat{c}_k^F = \nu_k \beta \\
(\text{without QoS})
\]

\[
\bar{c}_k^F = \nu_k \beta (1 - \rho^{n^*}) \\
(\text{with QoS})
\]
Figure: Mean user payment under VCG auctions. \( (\nu_k = 1, \ C = 1) \)

\[
\hat{c}_k^V = \frac{\nu_k \rho (3-\rho)}{2C (1-\rho)} \\
(\text{without QoS})
\]

\[
\bar{c}_k^V = \frac{\nu_k}{C} \left( \rho \frac{1-\rho^{n^*}}{1-\rho} + \frac{\rho}{2} - (n^* - \frac{1}{2}) \rho^{n^*} \right) \\
(\text{with QoS})
\]
Mean job payments: congestion based pricing

Figure: Mean user payment under congestion based pricing. 
\( \nu_k = 1, C = 1 \)

\[
\hat{c}_k^L = \frac{\nu_k(1-\rho)}{C^2} \sum_{n=1}^{\infty} n^2 \rho^{n-1} \]  
(without QoS)

\[
\bar{c}_k^L = \frac{\nu_k(1-\rho)}{C^2} \sum_{n=1}^{n^*} n^2 \rho^{n-1} \]  
(with QoS)
Prices are proportional to $1/|\vec{x}|$, to $\approx 1$, and to $|\vec{x}|$

Revenue is proportional to 1, to $|\vec{x}|$, and to $|\vec{x}|^2$

Mean revenue related to different moments of $|\vec{x}|$

- Zeroth moment
  \[ \bar{R}_F = \beta C \mathbb{E}[|\vec{x}|^0 1(1 \leq |\vec{x}| \leq n^*)] \]
- First moment
  \[ \bar{R}_V \approx \mathbb{E} \left[ (|\vec{x}| - \frac{1}{2}) 1(1 < |\vec{x}| \leq n^*) \right] \]
- Second moment
  \[ \bar{R}_L = \frac{1}{C} \mathbb{E}[|\vec{x}|^2 1(1 \leq |\vec{x}| \leq n^*)] \]

Decomposition of mean user payment

\[ \bar{c}_k(\cdot) = \nu_k F(\cdot), n^*(\rho) \]
A pre-payment mechanism
Freezing prices on arrival

So far,

- users pay the exact charge accrued
- pay on service completion
- this mechanism is equivalent to a *post-payment mechanism*

Consider a mechanism where the fee is charged on arrival

- price per unit-time fixed on arrival
- should reflect the underlying pricing model
- should depend on perceived state of congestion

*A pre-payment mechanism!*
Payment charged \( (p_k^x(\vec{x})) \)

\[
p_k^x(\vec{x}) = \gamma_k^x(\vec{x} + \vec{e}_k) \mathbb{E}_{\vec{x}}[W_k]
\]

\( \vec{x} \) state observed on arrival
\( \gamma_k^x(\cdot) \) price per unit-time (reflects the pricing model)
Proposition

Conditional sojourn time $\mathbb{E}_x[W_k]$ is linear in number of users in the system on arrival, i.e.,

$$\mathbb{E}_x[W_k] = A_{k,0} + \sum_{m=1}^{K} A_{k,m} x_m.$$
A pre-payment scheme
Defining the price per unit time $\gamma^X_k(\cdot)$

- Fixed rate pricing
  $$\gamma^F_k(\vec{x}) = \frac{\sigma^F_k}{|\vec{x}|}$$

- VCG auctions
  $$\gamma^V_k(\vec{x}) = \sigma^V_k$$

- Congestion-based pricing
  $$\gamma^L_k(\vec{x}) = \sigma^L_k|\vec{x}|$$

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Insensitive bandwidth sharing
Comparing second moments

- Server ($r_{\text{min}} = 0.1$ bits/second)
- Single class of traffic ($\lambda = 0.3$ packets/second, $\nu = 1$ bits)
- Varying capacity
Second moment: fixed-rate pricing

Figure: Comparison of second moment under fixed rate pricing.
Figure: Comparison of second moments under VCG auctions.
Second moment: congestion-based pricing

Figure: Comparison of second moments under congestion-based pricing.
Consider a large network consisting of $N$ servers each of capacity $C$. We assume $N$ is large. Consider the homogeneous case.

- Jobs arrive at rate $N\lambda$ on arrival they are routed to an appropriate server.

- Let us analyze 2 different strategies:
  - Uniform random routing to a server w.p. $\frac{1}{N}$.
  - Routing to a server with least cost.

How do the two schemes compare?
Note that under both VCG and Congestion Based pricing the price is monotonically increasing in the server congestion. Hence routing to least price server amounts to JSQ routing.
Hence we compare:

- **Pricing under uniform random routing**: Because of Poisson thinning this amounts to analyzing the single PS system.

- **Pricing under JSQ**
Pricing under JSQ

- JSQ is very difficult to analyze except when $N$ is large because of asymptotic independence (*propagation of chaos*).
- Since knowing state of $N$ servers would require too much overhead, idea pick a small number of $L$ at random and perform JSQ.
- Basic idea is under JSQ on comparing $L$ picked at random, when $N$ is large, the individual servers behave in a statistically independent manner.
Let $\pi_N(.)$ denote the stationary distribution under JSQ (necessary and sufficient condition is $\lambda E[\sigma] < C$),
Then, for any finite set of queues the following result due to Vvedenskaya, Dobrushin, Karpelevich 1996 (PIT), Graham 2000 (JAP).

**Proposition**

*Consider a system of $N$ identical servers, Poisson arrivals, and JSQ routing on arrival.*
*Then for any finite number of $L$ queues picked at random and performing JSQ we have:*

$$\lim_{N \to \infty} E_{\Pi^N}(r_0(k)) = P_K = \rho \frac{L^{k-1}}{L-1}, \quad k \in \mathbb{Z}$$
or equivalently

Let $\Pi^{(L)}$ denote the restriction of $\Pi^N$ to $L$ coordinates, with $\pi = \Pi^{(1)}$ being the one dimensional marginal of $\Pi$. Then:

$$\Pi^{(L)} = \bigotimes_{i=1}^{L} \pi$$

In other words, there is asymptotic independence and moreover the tail distribution is super-exponential as opposed to simply being exponential in the case of pure random routing.
Sketch of Proof

First let us compute the average potential arrival rate to a given server, say 1. By the independent sampling assumption, the probability that the server is not chosen in two trials is \( \left( \frac{N-2}{N} \right) \) and hence the probability of a potential arrival to the queue is

\[
N \nu \left( 1 - \frac{N - 2}{N} \right) = 2 \nu
\]

. In other words the potential arrival rate to the server is \( 2 \nu \). However an arrival only enters the server if it is the smaller of the two chosen. By independence (chaos) let \( \pi_k \) be the stationary probability that a server has \( k \) jobs and \( P_k = \sum_{j=k}^{\infty} \pi_j \) denote the tail probability of having at least \( k \) jobs. Then the probability that the arrival joins queue 1 is

\[
0.5 \times P(second \ server = k) + 1 \times P(second \ server > k) = 0.5\pi_k + P_{k+1}
\]

where \( \pi_k \) denotes the probability that the second server has \( k \) jobs too and \( P_{k+1} \) is the probability that the second server has more than \( k \) jobs by independence.
Noting that 
\[ \pi_k = P_k - P_{k+1} \]
we obtain the probability that an arrival joins server 1 is 
\[ 0.5(P_k - P_{k+1}) + P_{K+1} = 0.5(P_k + P_{k+1}) \]
. But the potential arrival rate is \(2\nu\) and hence the arrival rate to server 1 is \(\nu_k = \nu(P_k + P_{k+1})\). From balance equations:
\[ \nu_k \pi_k = \mu C \pi_{k+1} \]
and hence substituting for \(\nu_k\) we obtain:
\[ P_k = \rho^{2^k-1} \]
\[ \pi_k = \rho^{2^k-1}(1 - \rho^2) \]
In a similar way if we take \(L > 2\) then we obtain using similar arguments (except that there are more events that result in ties):
\[ \lambda_k = \lambda \frac{(P_k)^L - (P_{k+1})^L}{P_k - P_{k+1}}, \]
\[ P_k = \rho^{L^{k-1}} \]
Figure: Mean user payments for VCG auction pricing mechanism

Figure: Mean user payments for Lagrange shadow pricing mechanism
Figure: Mean user payments for VCG auction pricing mechanism
Conclusion

- Assumed usage-based billing and elastic users
  - e.g., Amazon Elastic Cloud Compute (EC2)
- Structure of the pricing models
- Obtained analytical expressions for revenue, payments
- Results allow operators to dimension capacity
- Analyzed a pre-payment scheme
  characterized volatility of payments explicitly
Thank you very much for your attention and thanks to the organizers.
P. Brémaud.
A Swiss Army Formula of Palm Calculus

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Thank you