On solutions of bounded-real LMI for singularly bounded-real systems

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European Control Conference, Limassol
June 15, 2018
- System: controllable and observable

\[
\frac{d}{dt} x = Ax + Bu, \ y = Cx + Du,
\]

where \( A \in \mathbb{R}^{n \times n}, \ B, C^T \in \mathbb{R}^{n \times p}, \ D \in \mathbb{R}^{p \times p}. \)
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• Bounded-real system: \( \|G(s)\|_{\mathcal{H}_\infty} \leq 1. \)
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• Bounded-real system: \(\|G(s)\|_{\mathcal{H}_\infty} \leq 1\).

• Bounded-real system \(\Leftrightarrow \exists \ K = K^T\) such that

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\begin{bmatrix}
A^T K + KA + C^T C & KB + C^T D \\
B^T K + D^T C & -(I - D^T D)
\end{bmatrix} \leq 0.
\]
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where \( A \in \mathbb{R}^{n \times n}, \ B, C^T \in \mathbb{R}^{n \times p}, \ D \in \mathbb{R}^{p \times p} \).

Bounded-real system: \( \|G(s)\|_{\mathcal{H}_\infty} \leq 1 \).

Bounded-real system \( \iff \exists K = K^T \) such that

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\begin{bmatrix}
    A^TK + KA + C^TC & KB + C^TD \\
    B^TK + D^TC & -(I - D^TD)
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\( \mathcal{H}_\infty \) synthesis problem, \( \mathcal{H}_2 \) synthesis problem, design of filters, etc.
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• \( \mathcal{H}_\infty \) synthesis problem, \( \mathcal{H}_2 \) synthesis problem, design of filters, etc.

• LMI solved using: LMI solvers (iterative), ARE solvers.
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- Bounded-real system: \( \| G(s) \|_{\mathcal{H}_\infty} \leq 1 \).

- Solved using ARE:

\[
A^T K + KA + C^T C + (KB + C^T D)(I - D^T D)^{-1}(B^T K + D^T C) = 0.
\]
- System: controllable and observable

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\frac{d}{dt} x = Ax + Bu, \ y = Cx + Du,
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- ARE does not exist if \( I - D^T D \) is singular.
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- ARE does not exist if \( I - D^T D \) is singular.

\[ G(s) = \frac{s-2}{s+2} \]

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Motivation

- $(I - D^T D)$ is singular: How do we solve bounded-real LMI?
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- For this talk: Bounded-real SISO systems with $I - D^T D = 0$. 

Reformulated problem: find $K$ such that 

$$A^T K + K A + C^T C \leq 0$$

and $KB + C^T = 0$. 

Will the algorithm used to find ARE solution work? Let's review the algorithm [P. van Dooren, SSC 1981].
Motivation

- $(I - D^TD)$ is singular: How do we solve bounded-real LMI?
- For this talk: Bounded-real SISO systems with $I - D^TD = 0$.
- The bounded-real LMI becomes

$$\begin{bmatrix}
A^T K + KA + C^T C & KB + C^T \\
B^T K + C & 0
\end{bmatrix} \leq 0.$$
• $(I - D^T D)$ is singular: How do we solve bounded-real LMI?

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• The bounded-real LMI becomes

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• Will the algorithm used to find ARE solution work?

• Let’s review the algorithm [P. van Dooren, SSC 1981].
**Hamiltonian matrix pair**

\[ E := \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H := \begin{bmatrix} A & 0 & BD^T \\ -C^T C & -A^T & -C^T \\ -C & -DB^T & I_p - DD^T \end{bmatrix} \in \mathbb{R}^{(2n+p) \times (2n+p)}. \]
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- \( \sigma(E, H) \): Set of roots of \( \det(sE - H) \) (with multiplicity). Also called eigenvalues of \( (E, H) \).
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Also called eigenvalues of \((E, H)\).

Assumption: \(\sigma(E, H) \cap j\mathbb{R} = \emptyset\) (for simplicity).
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• ARE existence \( \Leftrightarrow |\sigma(E, H)| = 2n \).
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ARE existence \(\iff |\sigma(E, H)| = 2n\).

Partition \(\sigma(E, H)\) in two disjoint sets based on certain rules. Each of these sets are called Lambda-set of \((E, H)\).

Symbol for a Lambda-set: \(\Lambda\).
Hamiltonian matrix pair

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\( \Lambda: \) Subset of \( \sigma(E, H) \) with \( |\Lambda| = n \).
Preliminaries

Algorithm: strictly bounded-real systems

- Hamiltonian matrix pair
  \[ E = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} A & 0 & BD^T \\ -C^T C & -A^T & -C^T \\ -C & -DB^T & I_p - DD^T \end{bmatrix} \in \mathbb{R}^{(2n+p)\times(2n+p)}. \]

- Assumption: \( \sigma(E, H) \cap j\mathbb{R} = \emptyset \) (for simplicity).

- \( \Lambda \): Subset of \( \sigma(E, H) \) with \( |\Lambda| = n \).

- \( n \) eigenvectors corresponding to the elements of \( \Lambda \):
  \[ V_1, V_2 \in \mathbb{R}^{n\times n} \text{ and } V_3 \in \mathbb{R}^{p\times n} \]
  \[ \begin{bmatrix} A & 0 & BD^T \\ -C^T C & -A^T & -C^T \\ -C & -DB^T & I_p - DD^T \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \Gamma, \]

  where \( \sigma(\Gamma) = \Lambda \).
- Hamiltonian matrix pair

\[ E = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} A & 0 & BD^T \\ -C^TC & -A^T & -CT \\ -C & -DB^T & I_p - DD^T \end{bmatrix} \in \mathbb{R}^{(2n+p) \times (2n+p)}. \]
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- \(\Lambda\): Lambda-set of \(\text{det}(sE - H)\).
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- \(\Lambda\): Lambda-set of \(\det(sE - H)\).

- ***n***-dimensional eigenspaces corresponding to \(\Lambda\)

\[
\mathcal{V} := \text{img} \begin{bmatrix}
V_1 \\
V_2 \\
V_3
\end{bmatrix}, \quad V_1, V_2 \in \mathbb{R}^{n \times n} \text{ and } V_3 \in \mathbb{R}^{p \times n}.
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Then, the following statements hold

1. \( V_1 \) is invertible.
2. \( K := V_2 V_1^{-1} \) is symmetric.
3. \( K \) is a solution to the ARE:

\[ A^T K + K A + C^T C + (K B + C^T D) (I - D^T D)^{-1} (B^T K + D^T C) = 0. \]
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- \( \Lambda: \) Lambda-set of \( \det(sE - H). \) \( (\deg \det(sE - H) = 2n) \)

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Classification of bounded-real systems based on

- \( \Delta := \deg \det(sE - H) \).

Bounded-real systems
Classification of bounded-real systems based on:

- \( \Delta := \deg \det(sE - H) \).

- **Strictly bounded-real** \( \Delta = 2n \)
  - \( 0 \leq \Delta < 2n \)
  - Allpass : \( \det(sE - H) = 0 \)

Algorithm exists: For \( \Delta = 2n \) with \( \sigma(E, H) \cap j\mathbb{R} = \emptyset \).
Classification of bounded-real systems based on:
- \( \Delta := \deg \det(sE - H) \).

- **Strictly bounded-real**: \( \Delta = 2n \)
- **0 \leq \Delta < 2n**
- **Allpass**: \( \Delta := -\infty \)

Algorithm exists: \( \Delta = 2n \) with \( \sigma(E, H) \cap j\mathbb{R} = \emptyset \).
\( \Delta = -\infty \) [Bhawal et.al. TCAS 2018].
Classification of bounded-real systems based on:

- $\Delta := \deg \det(sE - H)$.

- **Strictly bounded-real**
  - $\Delta = 2n$

- **Singularly bounded-real**
  - $\Delta = 0$

- **Allpass**
  - $\Delta = -\infty$

- **$\Delta \geq 0$**
  - $0 \leq \Delta < 2n$

- **$\Delta < 0$**
  - $0 < \Delta < 2n$
Classification of bounded-real systems based on:

- $\Delta := \text{deg} \det(sE - H)$.

Bounded-real systems

- Strictly bounded-real $\Delta = 2n$
- $0 \leq \Delta < 2n$
- Allpass $\Delta = -\infty$
- Singly bounded-real $\Delta = 0$
- $0 < \Delta < 2n$

Singularly bounded-real systems: Bounded-real systems with $\Delta = 0$. 
Strictly bounded-real system versus Singularly bounded-real system:

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Problem statement

Find an algorithm to compute the **unique** solution of bounded-real LMI for **singularly bounded-real** systems.
Known result (Doo’81)

- *Hamiltonian matrix pair (Assumption: $\sigma(E, H) \cap j\mathbb{R} = \emptyset$)*

\[
E = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} A & 0 & BD^T \\ -C^T C & -A^T & -C^T \\ -C & -DB^T & I_p - DD^T \end{bmatrix} \in \mathbb{R}^{(2n+p) \times (2n+p)}.
\]

- $\Lambda$: Lambda-set of $\det(sE - H)$.
- *$n$-dimensional eigenspaces corresponding to $\Lambda$*

\[
\mathcal{V} := \text{img} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \quad V_1, V_2 \in \mathbb{R}^{n \times n} \text{ and } V_3 \in \mathbb{R}^{p \times n}.
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- Then, the following statements hold.

1. $V_1$ is invertible.
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Known result (Doo’81)

- **Hamiltonian matrix pair** \( (D = I) \)

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E = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} A & 0 & B \\ -C^T C & -A^T & -C^T \\ -C & -B^T & 0 \end{bmatrix} \in \mathbb{R}^{(2n+p) \times (2n+p)}.
\]

- **\( \Lambda \): Lambda-set of** \( \det(sE - H) \). **No Lambda-set here.**
Theorem

- Hamiltonian matrix pair

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E = \begin{bmatrix}
I_n & 0 & 0 \\
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0 & 0 & 0 
\end{bmatrix},
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Theorem

- **Hamiltonian matrix pair**

\[ E = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} A & 0 & B \\ -C^T C & -A^T & -C^T \\ -C & -B^T & 0 \end{bmatrix} \in \mathbb{R}^{(2n+p) \times (2n+p)}. \]

- Define \( \hat{A} := \begin{bmatrix} A \\ -C^T C \\ -A^T \end{bmatrix} \) and \( \hat{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix} \).
Theorem

- **Hamiltonian matrix pair**

\[ E = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} A & 0 & B \\ -C^T C & -A^T & -C^T \\ -C & -B^T & 0 \end{bmatrix} \in \mathbb{R}^{(2n+p) \times (2n+p)}. \]

- Define \( \hat{A} := \begin{bmatrix} A \\ -C^T C & -A^T \end{bmatrix} \) and \( \hat{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix}. \)

- \( W := [\hat{B} \quad \hat{A}\hat{B} \quad \ldots \quad \hat{A}^{n-1}\hat{B}] \in \mathbb{R}^{2n \times n}. \)
Theorem

- **Hamiltonian matrix pair**

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- Define \( W := \begin{bmatrix} \hat{B} & \hat{A} \hat{B} & \ldots & \hat{A}^{n-1} \hat{B} \end{bmatrix} \in \mathbb{R}^{2n \times n} \).

- Define \( W =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \), where \( X_1, X_2 \in \mathbb{R}^{n \times n} \).
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- Define \( \hat{A} := \begin{bmatrix} A \\ -C^T C & -A^T \end{bmatrix} \) and \( \hat{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix} \).

- \( W := [\hat{B} \quad \hat{A}\hat{B} \quad \ldots \quad \hat{A}^{n-1}\hat{B}] \in \mathbb{R}^{2n \times n} \).

- Define \( W := [X_1 \quad X_2] \), where \( X_1, X_2 \in \mathbb{R}^{n \times n} \).

Then, the following statements hold.

1. \( X_1 \) is invertible.
2. \( K := X_2X_1^{-1} \) is symmetric.
3. \( KB + C^T = 0 \) and \( A^TK + KA + C^TC \leq 0 \).
- Singularly bounded real system: \((\det(sE - H) = 30)\).

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.5 & -6 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = -\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} x + u.
\]
Singularly bounded real system: \((\det(sE - H) = 30)\).

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.5 & -6 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \ y = -\begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} x + u.
\]

\[
W = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 3 \\ 5 & 1 & -1 \\ 4 & -1 & -5 \\ 2 & -2 & -1 \end{bmatrix} =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.
\]
- **Singularly bounded real system:** \((\text{det}(sE - H) = 30)\).

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.5 & -6 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = -\begin{bmatrix} 5 & 4 & 2 \end{bmatrix} x + u.
\]

- **\(W\)**:
\[
W = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 3 \\ 5 & 1 & -1 \\ 4 & -1 & -5 \\ 2 & -2 & -1 \end{bmatrix} =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.
\]

- **\(K\)**:
\[
K = X_2X_1^{-1} = \begin{bmatrix} 32 & 16 & 5 \\ 16 & 11 & 4 \\ 5 & 4 & 2 \end{bmatrix}.
\]
- Singularly bounded real system: \( \det(sE - H) = 30 \).

\[
\frac{d}{dt} x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5.5 & -6 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \ y = - \begin{bmatrix} 5 & 4 & 2 \end{bmatrix} x + u.
\]

- \( W = [\hat{B} \ \hat{A}\hat{B} \ \hat{A}^2\hat{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 3 \\ 5 & 1 & -1 \\ 4 & -1 & -5 \\ 2 & -2 & -1 \end{bmatrix} =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \)

- \( K = X_2X_1^{-1} = \begin{bmatrix} 32 & 16 & 5 \\ 16 & 11 & 4 \\ 5 & 4 & 2 \end{bmatrix}. \)

- \( A^TK + KA + C^TC = \text{diag}(-30, 0, 0) \leq 0 \) and \( KB + C^T = 0. \)
• Singularly bounded real system: \( \det(sE - H) = 30 \).

\[
\frac{dx}{dt} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-5.5 & -6 & -3
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} u, \quad y = -\begin{bmatrix} 5 & 4 & 2 \end{bmatrix} x + u.
\]

\[
W = \begin{bmatrix}
\hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & -3 \\
1 & -3 & 3 \\
5 & 1 & -1 \\
4 & -1 & -5 \\
2 & -2 & -1
\end{bmatrix} =: \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.
\]

\[
K = X_2X_1^{-1} = \begin{bmatrix}
32 & 16 & 5 \\
16 & 11 & 4 \\
5 & 4 & 2
\end{bmatrix}.
\]

\[
A^T K + KA + C^T C = \text{diag}(-30, 0, 0) \leq 0 \text{ and } KB + C^T = 0.
\]

\[
K \text{ satisfies } \begin{bmatrix}
A^T K + KA + C^T C & KB + C^T \\
B^T K + C & 0
\end{bmatrix} \leq 0.
\]
Hamiltonian system

\[
\begin{bmatrix}
I_n & 0 & 0 \\
0 & I_n & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{z} \\
\dot{u}
\end{bmatrix}
= \begin{bmatrix}
A & 0 & B \\
-C^T C & -A^T & -C^T \\
-C & -B^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z \\
u
\end{bmatrix}.
\]
- Hamiltonian system

\[
\begin{bmatrix}
I_n & 0 & 0 \\
0 & I_n & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{z} \\
\dot{u}
\end{bmatrix} =
\begin{bmatrix}
A & 0 & B \\
-C^TC & -A^T & -C^T \\
-C & -B^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z \\
u
\end{bmatrix}.
\]

- Output nulling representation:

\[
\frac{d}{dt} \begin{bmatrix}
x \\
z
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
-C^TC & -A^T
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} +
\begin{bmatrix}
B \\
-C^T
\end{bmatrix} u,
\quad
0 =
\begin{bmatrix}
-C & -B^T
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix}.
\]
Hamiltonian system

\[
\begin{bmatrix}
I_n & 0 & 0 \\
0 & I_n & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{z} \\
\dot{u}
\end{bmatrix}
= 
\begin{bmatrix}
A & 0 & B \\
-C^T C & -A^T & -C^T \\
-C & -B^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z \\
u
\end{bmatrix}.
\]

Output nulling representation:

\[
\frac{d}{dt}
\begin{bmatrix}
x \\
z
\end{bmatrix}
= 
\begin{bmatrix}
A & 0 \\
-C^T C & -A^T
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix}
+ 
\begin{bmatrix}
B \\
-C^T
\end{bmatrix}
\hat{u},
\quad
0 = 
\begin{bmatrix}
-C & -B^T
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix}.
\]

\[\text{deg det}(sE - H) = 0 \Rightarrow \text{num}(\hat{C}(sI - \hat{A})^{-1}\hat{B}) \in \mathbb{R} \setminus 0.\]
- Hamiltonian system

\[
\begin{bmatrix}
I_n & 0 & 0 \\
0 & I_n & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{z} \\
\dot{u}
\end{bmatrix}
= 
\begin{bmatrix}
A & 0 & B \\
-C^T C & -A^T & -C^T \\
-C & -B^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z \\
u
\end{bmatrix}.
\]

- Output nulling representation:

\[
\frac{d}{dt} \begin{bmatrix} x \\ z \end{bmatrix} = 
\begin{bmatrix}
A & 0 \\
-C^T C & -A^T
\end{bmatrix}
\begin{bmatrix} x \\ z \end{bmatrix} + 
\begin{bmatrix}
B \\
-C^T
\end{bmatrix}
u,
0 = 
\begin{bmatrix}
-C & -B^T
\end{bmatrix}
\begin{bmatrix} x \\ z \end{bmatrix}.
\]

- \( \text{deg det}(sE - H) = 0 \Rightarrow \text{num}(\hat{C}(sI - \hat{A})^{-1}\hat{B}) \in \mathbb{R} \setminus 0. \)

- Relative degree = 2n. The initial few Markov parameters are zero.
- Hamiltonian system

\[
\begin{bmatrix}
I_n & 0 & 0 \\
0 & I_n & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{z} \\
\dot{u}
\end{bmatrix}
= \begin{bmatrix}
A & 0 & B \\
-C^T C & -A^T & -C^T \\
-C & -B^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z \\
u
\end{bmatrix}.
\]

**Lemma**

- \( \frac{d}{dt} x = Ax + Bu, \ y = Cx + Du \) (singly bounded-real).

- Define \( \hat{A} := \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} \) and \( \hat{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix} \).

Then,

\[
\hat{C} \hat{A}^k \hat{B} = 0 \quad \text{for} \quad k \in \{0, 1, 2, \ldots, 2n - 2\}.
\]
For allpass systems: bounded-real LMI becomes
\[
\begin{bmatrix}
A^T K + KA + C^T C & KB + C^T \\
B^T K + C & 0
\end{bmatrix} = 0 \Rightarrow \begin{cases}
A^T K + KA + C^T C = 0 \\
KB + C^T = 0
\end{cases}
\]
For allpass systems: bounded-real LMI becomes
\[
\begin{bmatrix}
A^T K + KA + C^T C & KB + C^T \\
B^T K + C & 0
\end{bmatrix} = 0 \Rightarrow \begin{cases}
A^T K + KA + C^T C = 0 \\
KB + C^T = 0
\end{cases}
\]

**Corollary (Allpass systems)**

- $\Sigma_{\text{all}}$: $\frac{d}{dt}x = Ax + Bu$ and $y = Cx + u$.
- Define $\hat{A} = \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix}$ and $\hat{B} = \begin{bmatrix} B \\ -C^T \end{bmatrix}$.
- $W := \begin{bmatrix} \hat{B} & \hat{A}\hat{B} & \ldots & \hat{A}^{n-1}\hat{B} \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}^{2n \times n}$.

Then, the following statements hold.

1. $X_1$ is invertible.
2. $K := X_2X_1^{-1}$.
3. $KB + C^T = 0$ and $A^T K + KA + C^T C = 0$.

**Reason:** For allpass systems $\hat{C}\hat{A}^k\hat{B} = 0$ for all $k \in \mathbb{N}$.
Conclusion

Bounded-real systems

Strictly bounded-real
$\Delta = 2n$

Singularly bounded-real
$\Delta = 0$

$0 \leq \Delta < 2n$

Allpass
$\Delta = -\infty$

$0 < \Delta < 2n$

Algorithms already present for $\Delta = 2n$. 
Bounded-real systems

- Strictly bounded-real
  \[ \Delta = 2^n \]
- \[ 0 \leq \Delta < 2^n \]
- Allpass
  \[ \Delta = -\infty \]
- Singly bounded-real
  \[ \Delta = 0 \]
  \[ 0 < \Delta < 2^n \]

- Markov parameters of Hamiltonian system crucial.
Markov parameters of Hamiltonian system crucial.

Flop count $O(n^3)$: better than LMI solvers $O(n^{4.5})$. 
Markov parameters of Hamiltonian system crucial.
Flop count $\mathcal{O}(n^3)$: better than LMI solvers $\mathcal{O}(n^{4.5})$.
Algorithm works for LQR, passivity, as well.
Future work

Bounded-real systems

- Strictly bounded-real \( \Delta = 2n \)
- \( 0 \leq \Delta < 2n \)
- Allpass \( \Delta = -\infty \)
- Singularity bounded-real \( \Delta = 0 \)
- \( 0 < \Delta < 2n \)

- **Algorithms required for** \( 0 < \Delta < 2n \). (Paper under review)
Thank you

Questions?