1. (8 marks) Some short questions

(a) (1 marks) $X$ and $Y$ are mutually exclusive events. Find $\Pr(X|Y)$.
(b) (2 marks) $X$ is uniformly distributed in the interval $(0, 10)$. Find $\Pr(2 < X \leq 5 \mid X > 4)$.
(c) (2 marks) $X$ is uniformly distributed in $(0, 1)$ and $Y := -a \log_e X$. Obtain the probability density function of $Y$.
(d) (3 marks) $X$ is a uniformly distributed random variable in the interval $[0, y]$. However, $y$ itself is a uniformly distributed random variable in the interval $[a, b]$. Find the mean and variance of $X$.

Solution:

(a) Since $X$ and $Y$ are mutually exclusive $X \cap Y$ is empty, hence $\Pr(X \cap Y)$ is zero. Thus

\[ \Pr(X|Y) = \frac{\Pr(X \cap Y)}{\Pr(Y)} = 0. \]

(b) $\Pr(2 < X \leq 5 \mid X > 4) = \frac{\Pr\left(\{2 < X \leq 5\} \cap \{X > 4\}\right)}{\Pr(X > 4)} = \frac{\Pr(4 < X \leq 5)}{\Pr(X > 4)}$.

Since $X$ is uniformly distributed in the interval $(0, 1)$, we have the following: $\Pr(4 < X \leq 5) = 1/10$ and $\Pr(X > 4) = 6/10$. Therefore, $\Pr(2 < X \leq 5 \mid X > 4) = 1/6$.

(c) $Y := -a \log_e X$, therefore $Y$ takes values in $(0, \infty)$. Now the distribution function can be obtained as

\[ F_Y(y) = \Pr(Y \leq y) = \Pr(-a \log_e X \leq y) = \Pr\left(X \geq e^{-\frac{y}{a}}\right) = 1 - e^{-\frac{y}{a}}, \]
where the last equality follows from the fact that $X$ is uniformly distributed in $(0, 1)$. The density function then comes out to be

$$f_Y(y) = \frac{d}{dy}(F_Y(y)) = \frac{e^{-\frac{y}{a}}}{a} u(y)$$

For $a < 0$, the solution is identical, but the exponential is reversed and is distributed in the range $(-\infty, 0)$.

(d) We make use of the fact that

$$E(X) = \int_{y=a}^{b} E(X|Y = y) f_Y(y) dy.$$

The conditional expectation can be calculated as follows:

$$E(X|Y = y) = \int_{0}^{y} \frac{x}{y} dx = \frac{y^2}{2}.$$

Using this we get

$$E(X) = \int_{y=a}^{b} E(X|Y = y) f_Y(y) dy = \int_{y=a}^{b} \frac{y^2}{2} \left( \frac{1}{b-a} \right) dy = \frac{1}{2(b-a)} \left[ \frac{y^2}{2} \right]_{a}^{b} = \frac{b + a}{4}.$$

Similarly for the variance

$$E((X - E(X))^2) = E(X^2) - E(X)^2 = \int_{y=a}^{b} E(X^2|Y = y) f_Y(y) dy - E(X)^2.$$

The conditional second moment:

$$E(X^2|Y = y) = \int_{0}^{y} \frac{x^2}{y} dx = \frac{y^2}{3}.$$

Thus we have

$$E(X^2) = \int_{y=a}^{b} E(X^2|Y = y) f_Y(y) dy.$$
\[
= \int_{y=a}^{b} \frac{y^2}{3} \left( \frac{1}{b-a} \right) dy \\
= \frac{1}{3(b-a)} \left[ \frac{y^3}{3} \right]_a^b \\
= \frac{b^2 + a^2 + ab}{9}.
\]

Therefore

\[
\mathbf{E}((X - \mathbf{E}(X))^2) = \frac{b^2 + a^2 + ab}{9} - \frac{(b + a)^2}{16}.
\]

2. (6 marks) The newly deployed coast guard near Mumbai is using scanning equipment to detect for Kuber (or its clones) off Mumbai. The scanner receives a signal \( X \) whose distribution depends on the distance of Kuber. Let \( K \) denote the presence of Kuber at distance \( r \) and \( \bar{K} \) the event that Kuber is not present. An alert is sounded, denoted by event \( A \), if the signal level exceeds 0.5. The signal level has the following conditional probability density functions.

\[
f_{X|K}(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5(x-r)^2}
\]

\[
f_{X|\bar{K}}(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}
\]

(a) (1.5 marks) Obtain the false negative probability if Kuber is at a distance of 1 unit. Repeat for the case when Kuber is at a distance 2 units.

(b) (1.5 marks) If Kuber is not present, determine the false positive probability.

(c) (1.5 marks) If the probability of Kuber being present is 0.01, find the conditional probability of Kuber being present at distance 1 unit if there is an alarm.

(d) (1.5 marks) If the probability of Kuber being present is 0.01, find the conditional probability of Kuber not being present at distance 1 unit if there is no alarm.

If necessary, make reasonable assumptions and state them clearly.

**Solution:** (a) For \( r = 1 \), probability of a false negative signal is

\[
\Pr (X \leq 0.5|K, r = 1) = \int_{-\infty}^{0.5} f_{X|K}(x, r = 1) dx = \int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-0.5(x-1)^2} dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.5} e^{-0.5z^2} dz
\]

\[
= 0.5 - \text{erf} (0.5) \approx 0.31.
\]

For \( r = 2 \)

\[
\Pr (X \leq 0.5|K, r = 2) = \int_{-\infty}^{0.5} f_{X|K}(x, r = 2) dx = \int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-0.5(x-2)^2} dx
\]
$$\Pr \left( X > 0.5 \mid \overline{K} \right) = \int_{0.5}^{\infty} f_{X \mid \overline{K}}(x) dx = \int_{0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} dx = 0.5 - \text{erf} \left( 0.5 \right) \simeq 0.07$$

(b) Given Kuber is not present, the probability of a false positive is

$$\Pr \left( X > 0.5 \mid K \right) = \int_{0.5}^{\infty} f_{X \mid K}(x) dx = \int_{0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} dx = 0.5 - \text{erf} \left( 1.5 \right) \simeq 0.07$$

We indeed need the density of $r, f_R(r)$ to evaluate the conditional probabilities, so assume $f_R(r) = \delta(1)$.

(c) The required conditional probability can be found out as

$$\Pr \left( K, r = 1 \mid X > 0.5 \right) = \frac{\Pr \left( K, r = 1, X > 0.5 \right)}{\Pr \left( X > 0.5 \right)}$$

where

$$\Pr \left( K, r = 1, X > 0.5 \right) = \left( \int_{0.5}^{\infty} f_{X \mid K}(x) dx \right) \Pr \left( K \right) = \left( \int_{0.5}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} dx \right) \times 0.01,$$

and

$$\Pr \left( X > 0.5 \right) = \left( \int_{0.5}^{\infty} f_{X \mid K}(x) dx \right) \Pr \left( K \right) + \left( \int_{0.5}^{\infty} f_{X \mid \overline{K}}(x) dx \right) \Pr \left( \overline{K} \right).$$

(d) Similar to the previous case, here the required conditional probability can be obtained as

$$\Pr \left( \overline{K} \mid X \leq 0.5 \right) = \frac{\Pr \left( \overline{K}, X \leq 0.5 \right)}{\Pr \left( X \leq 0.5 \right)},$$

where

$$\Pr \left( \overline{K}, X \leq 0.5 \right) = \left( \int_{-\infty}^{0.5} f_{X \mid \overline{K}}(x) dx \right) \Pr \left( \overline{K} \right) = \left( \int_{-\infty}^{0.5} \frac{1}{\sqrt{2\pi}} e^{-0.5x^2} dx \right) \times 0.99,$$

and

$$\Pr \left( X \leq 0.5 \right) = \left( \int_{-\infty}^{0.5} f_{X \mid K}(x) dx \right) \Pr \left( K \right) + \left( \int_{-\infty}^{0.5} f_{X \mid \overline{K}}(x) dx \right) \Pr \left( \overline{K} \right).$$

3. (7 marks) $X$ and $Y$ are random variables with the joint pdf

$$f_{XY}(x, y) = \begin{cases} 
xe^{-x(1+y)} & \text{for } x \geq 0 \text{ and } y \geq 0 \\
0 & \text{otherwise.}
\end{cases}$$
Obtain $E(X)$, $E(Y)$ and $E(Y|X = x)$.

**Solution:** We first calculate the marginal density functions as follows:

$$f_X(x) = \int f_{XY}(x, y)dy = \int_0^\infty xe^{-x}e^{-xy}dy = xe^{-x} \left[ -\frac{e^{-xy}}{x} \right]_0^\infty = e^{-x}.$$

$$f_Y(y) = \int f_{XY}(x, y)dx = \int_0^\infty xe^{-x(1+y)}dy = \left[ x \int e^{-x(1+y)}dx \right]_0^\infty - \int_0^\infty \left[ \frac{d}{dx} (x) \int e^{-x(1+y)}dx \right] dx = \int_0^\infty e^{-x(1+y)}dx = \frac{1}{1+y}.$$

The marginal expectations can now be calculated as

$$E(X) = \int_0^\infty xe^{-x}dx = 1.$$

$$E(Y) = \int_0^\infty \frac{y}{(1+y)^2}dy = \infty.$$

In order to calculate $E(Y|X = x)$ we first note that the conditional pdf

$$f_{Y|X=x}(x, y) = \frac{f_{XY}(x, y)}{f_X(x)} = xe^{-xy}$$

for $x \geq 0$, $y \geq 0$ and zero otherwise. Hence we obtain

$$E(Y|X = x) = \int_0^\infty yxe^{-xy}dy = \frac{1}{x}.$$

4. (9 marks) $X_n$ are iid random variables with density

$$f_X(x) = \begin{cases} \frac{1}{a} & \text{for } 0 \leq x \leq a \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_n := \max_{1 \leq i \leq n}\{X_i\}$.

(a) (2 marks) Find the probability distribution function of $Y_n$. Denote this by $F_n(y)$.

(b) (3 marks) Find the limiting distribution for the sequence of distributions $F_n(y)$.
(c) (1 mark) Identify the limiting random variable for the sequence of random variables $Y_n$. Denote this by $Y$.

(d) (3 marks) $Y_n$ converges to $Y$ in probability. Prove or disprove.

**Solution:**

(a) 

$$F_n(y) := \Pr (Y_n \leq y) = \Pr \left( \max_{1 \leq i \leq n} \{X_i\} \leq y \right)$$

$$= \prod_{1 \leq i \leq n} \Pr (X_i \leq y)$$

$$= \begin{cases} 
  \left( \frac{y}{a} \right)^n & \text{for } 0 \leq y \leq a \\
  1 & \text{for } y > a \\
  0 & \text{otherwise.}
\end{cases}$$

(b) Since the sequence of real numbers $\{(y/a)^n\}$ with $0 \leq y < a$ converges to zero, the sequence of functions $\{F_n(y)\}$ converges pointwise to the following function:

$$F(y) = \begin{cases} 
  1 & \text{for } y \geq a \\
  0 & \text{otherwise.}
\end{cases}$$

This function is the limiting distribution function (see Figure 1).

(c) The candidate limiting random variable, say $Y$, has the unit step function at the point $a$ as its distribution function. In other words, $Y$ attains the value $a$ with probability 1.
(d) As elaborated in part (c) \( Y \) can be replaced by the constant \( a \). Thus

\[
\lim_{n \to \infty} \Pr (|Y_n - Y| > \epsilon) = \lim_{n \to \infty} \Pr (|Y_n - a| > \epsilon) = \lim_{n \to \infty} \Pr (a - Y_n > \epsilon) \text{ (since } Y_n < a) = \lim_{n \to \infty} \Pr (Y_n < a - \epsilon) = \lim_{n \to \infty} \left( a - \epsilon \right)^n = 0 \text{ for all } 0 < \epsilon < a.
\]

5. (8 marks) \( W_1(t) \) and \( W_2(t) \) are two Wiener processes with parameters \( \alpha_1 \) and \( \alpha_2 \) respectively and defined over the interval \([0, \infty)\). Define a new random process \( X(t) = W_1(t) - W_2(t) \).

(a) (2.5 marks) For \( W_1(t) \), obtain the joint density function \( f_{W_1}(a_1, a_2; t_1, t_2) \).

(b) (3 marks) Obtain the density function for the random process \( X(t) \), \( f_X(s; t) \).

(c) (2.5 marks) For \( t_1, t_2 \geq 0 \), obtain \( R_{XX}(t_1, t_2) \).

Solution:

(a) We may assume without loss of generality that \( t_2 > t_1 \). The joint density then turns out to be

\[
f_{W_1}(a_1, a_2; t_1, t_2) = f_{W_1}(a_2; t_2|W_1(t_1) = a_1)f_{W_1}(a_1; t_1).
\]

Clearly,

\[
f_{W_1}(a_1; t_1) = \frac{1}{\sqrt{2\pi \alpha_1 t_1}} e^{-\frac{a_1^2}{2\alpha_1 t_1}}
\]

because \( W_1 \) is a Wiener process. In order to find out the conditional pdf, we observe that this probability is equal to the probability that the increment is \( (a_2 - a_1) \) as \( t \) goes from \( t_1 \) to \( t_2 \). As the increments are independent, we get (see Stark and Woods eqn (7.2-15))

\[
f_{W_1}(a_2; t_2|W_1(t_1) = a_1) = \frac{1}{\sqrt{2\pi \alpha_1 (t_2 - t_1)}} e^{-\frac{(a_2 - a_1)^2}{2\alpha_1 (t_2 - t_1)}}.
\]

(b) In a Wiener process \( W(t) \), for every \( t \), \( W(t) \) has a Gaussian distribution with zero mean and \( at \) variance. It follows that \( X(t) = W_1(t) - W_2(t) \), for every time \( t \) is Gaussian with zero mean and variance equal to \( (\alpha_1 t + \alpha_2 t) \), because the sum of two Gaussian random variables is again Gaussian. Thus \( X(t) \) is again a Wiener process with parameter \( (\alpha_1 + \alpha_2) \). Hence the following density function:

\[
f_X(s; t) = \frac{1}{\sqrt{2\pi (\alpha_1 + \alpha_2) t}} e^{-\frac{s^2}{2(\alpha_1 + \alpha_2)t}}.
\]
(c) 
\[
R_{XX}(t_1, t_2) = E(X(t_1)X(t_2)) \\
= E((W_1(t_1) - W_2(t_1))(W_1(t_2) - W_2(t_2))) \\
= R_{W_1W_1}(t_1t_2) + R_{W_2W_2}(t_1t_2) \text{ (assuming } W_1(t) \text{ and } w_2(t) \text{ are independent}) \\
= (\alpha_1 + \alpha_2) \min\{t_1, t_2\}.
\]

6. (12 marks) Some more short questions

(a) (3 marks) \(X\) and \(Y\) are two events, not necessarily independent of each other. \(\Pr(X) = 0.8\) and \(\Pr(Y|X) = 0.5\). Which of the following is true: (1) \(\Pr(X) \leq \Pr(Y)\) or (2) \(\Pr(X) \geq \Pr(Y)\). Drawing the Venn diagrams indicating \(\Omega, X\) and \(Y\) might help.

(b) (3 marks) Numbers 1, 2, 3, \ldots, \(n\) are permuted and arranged in a random order. Find the probability that numbers 1, 2, \ldots, \(k\) are neighbours, i.e., the subsequence 1, 2, \ldots, \(k\) appears in the random order. Note that the probability is a function of \(k\).

(c) (3 marks) \(X\) and \(Y\) are uniformly distributed in \((0, 1)\). For \(0 \leq \alpha \leq 1\), find the probability of the event \(\{X \leq \alpha \& Y \leq \alpha \& X + Y \leq \alpha\}\).

(d) (3 marks) \(X\) is a zero mean random variable and \(Y\) is a random variable that is not independent of \(X\). Claim: The variance of \(X\) is obtained as \(E(E(X|Y)X)\). Prove or disprove.

Solution:

(a) From given \(\Pr(X) = 0.8, \Pr(XY) = 0.5\). Which implies \(\Pr(XY) = 0.8 \times 0.5 = 0.4\).

![Figure 2: Events X, Y and their probabilities.](image)

\[
\Pr(X \cup Y) = \Pr(X) + \Pr(Y) - \Pr(X \cap Y)
\]
\[
\Pr(X \cup Y) \leq 1
\]
\[
\Pr(X) + \Pr(Y) - \Pr(XY) = 0.8 + \Pr(Y) - 0.4 = 0.4 + \Pr(Y)
\]
\[
\Pr(Y) \leq 0.6
\]
\[
\Pr(Y) < \Pr(X)
\]

(b) The total number of outcomes is \(n!\), whereas the total number of permutations in which the subsequence 1, 2, 3, ..., \(k\) appears is \((n - k + 1)!\) (think of the subsequence as a single entity). Thus the probability is given by \((n - k + 1)!/n!\).

(c) \(X\) and \(Y\) are uniformly distributed in \((0, 1)\). It follows that \(\{X \leq \alpha\} \cap \{Y \leq \alpha\} \supseteq \{X + Y \leq \alpha\}\) (see Figure 3). Therefore \(\Pr(X \leq \alpha, Y \leq \alpha, X + Y \leq \alpha) = \Pr(X + Y \leq \alpha)\). Since \(X\) and \(Y\) are uniform and \(0 \leq \alpha \leq 1\), we have \(\Pr(X + Y \leq \alpha) = \alpha^2/2\).

(d) The claim is not true. This can be seen from the following.

\[
\mathbf{E}(\mathbf{E}(X|Y)X) = \int_X \int_Y x \mathbf{E}(X|Y = y) f_{XY}(x, y) dx dy
\]
\[
= \int_X x \left(\int_Y \mathbf{E}(X|Y = y) f_{XY}(x, y) dy\right) dx,
\]
which is clearly not equal to the variance, which is equal to \(\mathbf{E}(X^2) = \int_X x^2 \left(\int_Y f_{XY} dy\right) dx\) because \(X\) has zero mean.