Microwave Transmission Lines

An Introduction to the Basics

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Abstract

This document presents an introduction to the basics of microwave transmission lines. It is important to understand the principles underlying the propagation and transmission of high-frequency signals, which are vital in areas such as communications circuit design as well as in high-frequency processor cores, and it is a well known fact that contemporary processors are clocked by frequencies as high as 3GHz! Designs at such high frequencies require careful consideration so as to minimize losses and to ensure maximum power transmission. This document starts by giving an insight into the basics of transmission lines and wave propagation theory. This is then followed by different transmission line technologies adopted in modern electronic systems for fabrication.

Chapter 1

Introduction

Microwaves are a part of the electromagnetic spectrum. Usually, waves with wavelengths ranging from as low as a few millimeters to almost a metre are classified as microwaves. Conventional definition for the microwave frequency range is from 300M$Hz - 300$GHz. A very important question is the reason behind studying microwaves. What do these have to offer, and how are they advantageous? The answer is that most of modern electronic communication engineering make use of microwaves. Then again, what do microwaves have that makes them suitable for use in comminication engineering? Let us consider for example, a mobile phone, an indispensable communication tool for all. Supposing that a mobile uses the GSM1800 band, i.e. it makes use of communication frequencies of about 1800MHz. For proper wireless transmission and reception, every device requires a transmission/reception antenna, tuned to the frequency of operation, and the antenna size is usually determined by the wavelength λ. A good antenna size approximation is $\lambda/4$, and for an 1800MHz wave, the antenna size required would be around 4.2cm. What would be the required antenna size if the transmission frequency would be low, say for example $100Hz$?

This leads to another interesting question. If antenna sizes can be reduced considerably by using high frequenices, why not use much higher frequencies? Above microwaves, there are the infra-red and visible light spectra, and even above, there are ultraviolet rays, X-rays and gamma rays. Many of the high-frequency waves such as the UV rays and above, are detrimental for health as they are ionizing radiations. This also means that for satellite communications, these waves would be afftected by the earth's atmosphere. Moreover, even non-ionizing waves such as infrared rays are easily affected by atmospheric constituents and do not have obstruction penetration strength. Would these make good candidates for wireless communication, especially mobile phones, where users are often inside their houses, surrounded by thick walls?

1.1 The High Frequency Circuit Analysis Problem

In this section, we will see the method of analysis for low-frequency circuits and from thereon, examine why this cannot be used at high frequencies.

Consider a circuit with a source $V_s(t)$ and load R_L, as in Figure 1.1. The connection between them is by means of two conductors (whose resistance is assumed to be low). An initial look at this circuit shows that at any instant of time t, the voltage across the load $V_o(t)$ will equal the source voltage i.e., $V_o(t) = V_s(t)$. In such cases, the wire length l has absolutely no role is determining $V_o(t)$ (assuming l is small enough).

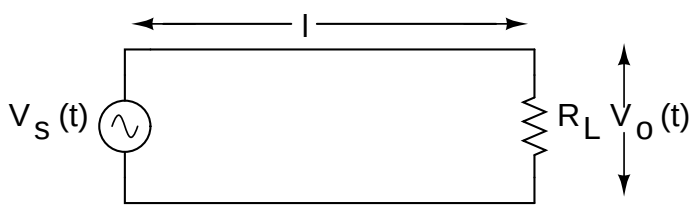

Figure 1.1: Simple source-load circuit

This approximation however, is true for low frequencies only. Consider the source sine wave traveling along the length l with a wave velocity v. Define the propagation time from source to load t_p as

$$t_p = \frac{v}{l}$$

Define a voltage along the line, which is a function of both time t and the position on the line x, where $0 \leq x \leq l$. Assume $x = 0$ denotes the source end and $x = l$ the load end. Consider the source voltage $V_s(t)$ waveform shown in Figure 1.2. Consider the source voltage at $t = 0$ i.e. $V_s(t = t_0, x = 0) = V_1$. Then, the load voltage will equal V_1 only when $t = t_0 + t_p$, i.e. after the line propagation delay time. But, then, $V_s(t = t_0 + t_p, x = 0) = V_2$. Voltages V_1 and V_2 are likely to e largely different, if the wave frequency is high, i.e. the time period T is small. Clearly, it is seen that the voltage varies along the length of the line when the frequency is high, or in general, when the wave dimensions become comparable to the dimensions of the circuit components. Hence, the lumped circuit theorems at low frequencies cannot directly be applied to analyzing circuits at high frequencies. The voltage at any given point on the line is now a function of time as well as space (or position).

The starting method of analyzing high frequency transmission lines is to consider a small section of the line, where the voltage is assumed not to change significantly over the length of the section, wherein the laws of lumped circuit theory can be applied.

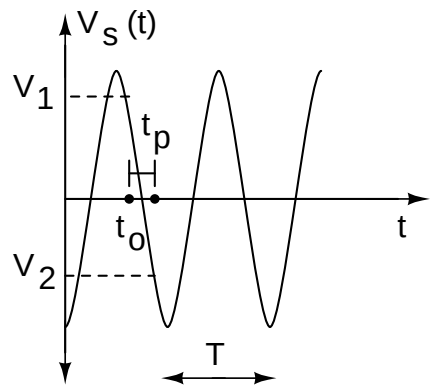

Figure 1.2: Sinusoidal source voltage

1.2 Voltage and Current along a Transmission Line

Let us now study the transmission line behaviour to an input, in greater detail. Consider a sinusoidal signal traveling along the line. Irrespective of the kind of signal traveling through the line, the conductors will have some resistance. Let R denote the resistance of the line per unit length. Also, this two-conductor connection is separated by a dielectric, which may have a parasitic conductance component, denoted per unit length as G. Due to the traveling sine wave, a time-varying magnetic field will be generated around each conductor, while a mutual electric field will interact between the two conductors. The magnetic field leads to a distributed inductance along the line, while the interacting electric field leads to a mutual capacitance. Denote the per unit length inductance and capacitance as L and C respectively. Consider an infinitesimally small section of the transmission line Δx, where the voltage variation over the length Δx is negligible. The section of such a size can then be expressed as an $R-L-G-C$ lumped circuit as shown in Figure 1.3.

Since we have assumed R, L, G, C to be the per unit length line parameters, the actual resistance, capacitance etc. for this section of the transmission line will be the product of these parameters and the length in consideration, i.e., Δx, which is assumed to be infinitesimally small. Note that we deal with the positional voltages and currents in this circuit as a consequence of the discussion in Section 1.1. The voltage and current at the left end of the line are denoted as $V(x)$ and $I(x)$, while at a distance of Δx towards the right are $V(x+\Delta x)$ and $I(x+\Delta x)$.

Assume that the frequency of operation is f, and the angular frequency $\Omega = 2\pi f$. Now, applying Kirchhoff's Voltage Law (KVL) in the loop, we have

$$V(x+\Delta x) = V(x) - I(x)(R + j\Omega L)\Delta x$$

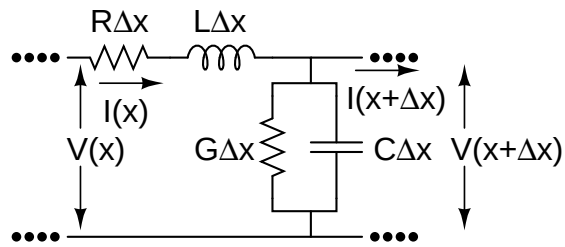

Figure 1.3: Small section of a transmission line- lumped circuit equivalent

$$\therefore \frac{V(x+\Delta x)-V(x)}{\Delta x} = -I(x)(R+j\Omega L)$$

taking the limiting case of $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{V(x+\Delta x)-V(x)}{\Delta x} = \frac{\mathrm{d}V}{\mathrm{d}x} = -(R+j\Omega L)I \tag{1.1}$$

Similarly, by applying Kirchhoff's Current Law (KCL) at the right node, we get

$$\frac{\mathrm{d}I}{\mathrm{d}x} = -(G+j\Omega C)V \tag{1.2}$$

We now have two differeintial equations in two variables, V and I. To make them easier to solve, differentiate both (1.1) and (1.2) by x again. We now have two equations

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = (R+j\Omega L)(G+j\Omega C)V \tag{1.3}$$

$$\frac{\mathrm{d}^2I}{\mathrm{d}x^2} = (R+j\Omega L)(G+j\Omega C)I \tag{1.4}$$

Let $\gamma^2 = (R+j\Omega L)(G+j\Omega C)$. The significance of the term γ will be dicussed later. The two differential equations now become

$$\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = \gamma^2 V \tag{1.5}$$

$$\frac{\mathrm{d}^2I}{\mathrm{d}x^2} = \gamma^2 I \tag{1.6}$$

Equations (1.5) and (1.6) are called the wave equations. Let us see why. These equations are now simplified, which are ordinary second order differential equations, which can be solved independently. Consider the voltage equation (1.5). Its solution is of the form

$$V(x) = V_1 e^{-\gamma x} + V_2 e^{\gamma x} \tag{1.7}$$

The solution for $V(x)$ is a function of the position on the transmission line alone. However, the complete solution for $V(x)$ is a function of both x and time t. Consider a sinusoidal input to the transmission line, which can be represented by a complex exponential $e^{j\Omega t}$. Thus, the complete voltage solution on the line is

$$V(x,t) = V_1 e^{-\gamma x} e^{j\Omega t} + V_2 e^{\gamma x} e^{j\Omega t} \tag{1.8}$$

Recall that $\gamma^2 = (R + j\Omega L)(G + j\Omega C)$. Thus γ in general, is a complex quantity. Therefore, let $\gamma = \alpha + j\beta$. $V(x,t)$ can now be represented as

$$V(x,t) = V_1 e^{-\alpha x} e^{j(\Omega t - \beta x)} + V_2 e^{\alpha x} e^{j(\Omega t + \beta x)} \tag{1.9}$$

This gives an interesting result. Consider, for simplicity, $\alpha = 0$. Let us focus on the first term of (1.9), i.e. $V_1 e^{j(\Omega t - \beta x)}$. This quantity is complex, comprising of a sine and a cosine component. Consider the real part of this term, i.e. $V_1 \cos(\Omega t - \beta x)$. If this term is now evaluated for various values of x in increasing order, the results are as shown in Figure 1.4.

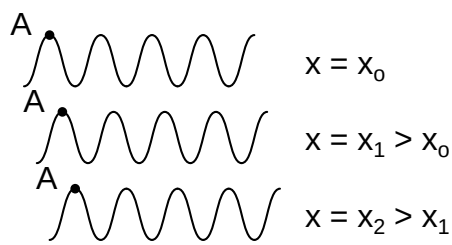

Figure 1.4: Wave propagation as a function of x

Observe the point A on the wave. As the value of x increases, the point A shifts to the right, indicating the wave "propagation" in the positive x direction. Similarly, the term $V_2 \cos(\Omega t + \beta x)$, which is the real part of the second term of the solution for $V(x,t)$, indicates a wave traveling in the negative x direction (work it out yourself, for different values of x). This indicates that when a transmission line is excited with an AC input, there are two waves traveling- from source to load and vice-versa. Likewise, the current will also propagate in two directions!

Chapter 2

Transmission Line Characteristics

In the previous chapter, we had an initial look at the lumped-component sectional representation of a transmission line, and that waves propagate in two directions, once the line is excited with a voltage input. In this chapter, we will study the transmission line characteristics in greater detail, such as the line behaviour with different kinds of load, issues of losses and different techniques for analysis, and applications of transmission lines.

2.1 Wave Propagation in Transmission Lines

We have seen that in general, the complete solution for the voltage and current along a transmission line is given as

$$\begin{aligned} V(x,t) &= V_1 e^{-\alpha x} e^{j(\Omega t - \beta x)} + V_2 e^{\alpha x} e^{j(\Omega t + \beta x)} \\ I(x,t) &= I_1 e^{-\alpha x} e^{j(\Omega t - \beta x)} + I_2 e^{\alpha x} e^{j(\Omega t + \beta x)} \end{aligned} \tag{2.1}$$

Note that $\gamma = \alpha + j\beta$. Let us investigate the quantity γ in more detail. Suppose $\gamma = 0$. This is then, a trivial case of a non-propagating sinusoidal wave, as it exists due to the mere presence of the input $e^{j\Omega t}$ itself. Thus clearly, γ is a quantity that determines the wave propagation. Hence, it is called the *propagation constant*. Let us now look into the quantities α and β more closely. To do this, we again select the voltage wave component that is traveling toward the positive x direction as we had in Section 1.2, i.e. $V_1 e^{-\alpha x} \cos(\Omega t - \beta x)$. Now what does α indicate? Had α been zero, the sine wave peak would be V_1. Due to a nonzero α, the sine wave amplitude (envelope) changes exponentially over the distance x, depending on the value of α. As α causes change in wave amplitude over the length of the transmission line, it is called the *attenuation constant*. The term βx in (2.1) denotes a phase component of the wave, hence β is called the *phase constant*. Note that a distance of λ, i.e. the wavelength of the propagating wave on the

transmission line would mean a phase change of 2π. Thus, for $x = \lambda$, $\beta\lambda = 2\pi$. Hence

$$\beta = \frac{2\pi}{\lambda} \tag{2.2}$$

What do you think will be the units for α and β? A first look tells us that since the term αx is dimensionless (why?), α would have the units m^{-1}. This indicates the absolute attenuation affecting the wave per unit length of the transmission line. However, since α in some sense, relates to a change in voltage or current, it is treated in the same way as voltage or current gain and is sometimes expressed in decibels (dB/m). Often, microwave engineers deal with a term named "effective travel distance", which is the distance along the transmission line, at which the wave amplitude becomes $1/e$ times the starting amplitude (at $x = 0$). Clearly, at this point, $\alpha x = 1$. The effective dB gain is $20 \log_{10} e^{-1} = 8.68$dB. We define here a new unit, named Neper (Np), where $1Np = 8.68$dB. Thus, α is often expressed in the unit Np/m^{-1}. Likewise, the units for β would be rad/m.

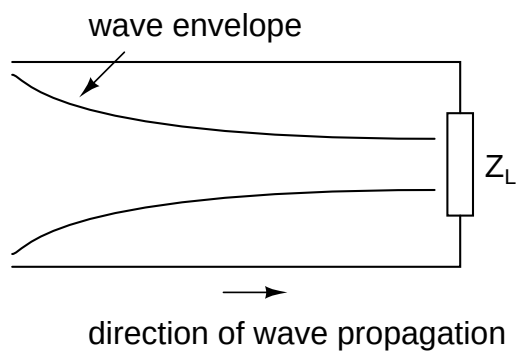

Figure 2.1: Propagating wave envelope for $\alpha > 0$

As seen in Figure 2.1, for $\alpha > 0$, the wave amplitude decreases exponentially as it propagates towards the load. This clarifies the reaon why α is called the attenuation constant.

2.2 Line and Load Impedance

Let us consider the positional solutions for the voltage and current on the transmission line, i.e. $V(x)$ and $I(x)$. Now we know that the voltage and current have two components propagating in opposite directions. Considering moving towards the load a positive x direction, equation (1.7) can be re-written for both voltage and current as

$$\begin{aligned} V(x) &= V_+ e^{-\gamma x} + V_- e^{\gamma x} \\ I(x) &= I_+ e^{-\gamma x} + I_- e^{\gamma x} \end{aligned} \tag{2.3}$$

As seen in (2.3), there are four unknowns, i.e. V_+, V_-, I_+ and I_-. However, the current equation may also be expressed in terms of V_+ and V_-. Let us see how. We will make use of the knowledge of γ as a function of line parameters, i.e. $\gamma^2 = (R + j\Omega L)(G + j\Omega C)$. Consider equation (1.1), which is

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}x} &= -(R + j\Omega L)I \\ &= -\sqrt{(R + j\Omega L)(G + j\Omega C)}\sqrt{\frac{R + j\Omega L}{G + j\Omega C}}I\end{aligned}$$

Observe the above expression carefully. The term $\sqrt{\frac{R+j\Omega L}{G+j\Omega C}}$ has the units of impedance, so let us call it Z_0. We now have

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -\gamma Z_0 I \tag{2.4}$$

Now let us differentiate the voltage expression in (2.3) as per the results obtained in (2.4).

$$-\gamma Z_0 I(x) = -\gamma V_+ e^{-\gamma x} + \gamma V_- e^{\gamma x}$$

$$\therefore I(x) = \frac{1}{Z_0}(V_+ e^{-\gamma x} - V_- e^{\gamma x}) \tag{2.5}$$

In some sense, the impedance Z_0 seems to govern the line voltage and current much like Ohm's Law. Z_0 is seen to depend on the per unit length line parameters R, L, G and C, and also on the frequency of operation Ω. Thus, Z_0 is an impedance not physically present anywhere on the line, but rather, a distributed impedance that is characteritic to the line parameters themselves. Z_0 is thus called the *characteristic impedance* of the transmission line and is a very important quantity governing microwave-based designs.

Let us now change our analysis slightly. So far, we have now focused more on the transmission line parameters than anything else. It will be soon clear that the load connected at the end of the line plays a very important factor in determining the voltage and current variation along the length of the line. Let us therefore, consider a length axis l, where $l = 0$ is the position of the load and movement twoards the voltage source indicates movement along the positive l direction. Thus, substituting $x = -l$ in (2.3) gives us

$$\begin{aligned}V(l) &= V_+ e^{\gamma l} + V_- e^{-\gamma l} \\ I(l) &= I_+ e^{\gamma l} + I_- e^{-\gamma l}\end{aligned} \tag{2.6}$$

Using the results obtained in (2.5), we have

$$\begin{aligned} V(l) &= V_+ e^{\gamma l} + V_- e^{-\gamma l} \\ I(l) &= \frac{1}{Z_0}(V_+ e^{\gamma l} - V_- e^{-\gamma l}) \end{aligned} \tag{2.7}$$

The impedance at any point on the line is thus given by

$$Z(l) = \frac{V(l)}{I(l)} = Z_0 \frac{V_+ e^{\gamma l} + V_- e^{-\gamma l}}{V_+ e^{\gamma l} - V_- e^{-\gamma l}} \tag{2.8}$$

Recall that the quantities $V_+ e^{\gamma l}$ and $V_- e^{-\gamma l}$ denote wave propagation in opposite directions- towards the load and source respectively. Intuitively, we can say that the voltage component $V_+ e^{\gamma l}$ travels towards the load due to excitation from the source at the other end of the transmission line. However, the voltage component $V_- e^{-\gamma l}$, which travels back to the source, is not what one would have expected to exist. In some sense, this quantity is a voltage "reflected" back from the load. Let us define a quantity Γ, called *voltage reflection coefficient*, as follows

$$\Gamma(l) = \frac{V_- e^{-\gamma l}}{V_+ e^{\gamma l}} = \frac{V_-}{V_+} e^{-2\gamma l} \tag{2.9}$$

Thus, the reflection coefficient at any point along the line is the ratio of the backward moving wave to the forward moving wave in terms of voltage magnitude, at that point. The original idea of designing a transmission line is to transmit the entire power from the source to the load, without any power reflected back from the load. Moreover, reflection towards the source can degrade the performance of the source itself over time. Hence, we aim to have the reflection coefficient Γ as small as possible. Consider equation (2.8), where

$$Z(l) = Z_0 \frac{V_+ e^{\gamma l} + V_- e^{-\gamma l}}{V_+ e^{\gamma l} - V_- e^{-\gamma l}}$$

At $l = 0$, i.e. at the load, we have the load impedance $Z(0) = Z_L = V(0)/I(0)$. Thus, at $l = 0$, taking V_+ common from both numerator and denominator,

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{2.10}$$

where Γ_L is the reflection coefficient at the load end. Now, Γ_L can be expressed from (2.10) by

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.11}$$

Equation (2.11) gives some interesting information. It is clear that Γ_L (subsequently, the Γ at any point l) takes values such that $0 \leq |\Gamma(l)| \leq 1$, which is true since the maximum power that can be reflected back to the source can equal the input power itself (principle of conservation of energy). To ensure zero reflected power, clearly the load impedance has to be equal to the line characteristic impedance, i.e. $Z_L = Z_0$. Such a condition is known as a *matched condition* and is critical in ensuring that the load absorbs all the indcident power and reflects back nothing. What happens when the load is (a) an open circuit and (b) a short circuit?

2.3 Design Issues in Transmission Lines

So far, we have studies some important characteristics in a transmission line- the wave propagation, line and load impedances and how they affect the wave characteristics in the line. Here, we look briefly into certain related issues that are important for engineers who wish to design transmission lines at very high frequencies.

2.3.1 Transmission Line Losses

We have seen that the quantity γ governs the wave propagation characteristics in terms of both amplitude and phase. As $\gamma = \alpha + j\beta$, we have seen in section 2.1 that α affects the wave amplitude variation in the line and β is related to the phase of the sinusoid. We would like the transmission line to be a lossless line, i.e. the power that reaches the load should be the same as the power generated by the source. It is evident that the quantity α governs the power delivered to the load. If $\alpha \neq 0$, we have seen that the wave amplitude changes (more specifically, decreases) exponentially as we move towards the load. This directly indicates a loss; the voltage amplitude that reaches the load is not the same as that generated by the source. Thus, for the line to be lossless, it must have $\alpha = 0$.

What is the physical implication of all this? How does the fact that a lossless line having $\alpha = 0$ translate physically? Let us revert to the expression for the propagation constant γ. We know that,

$$\gamma = \sqrt{(R + j\Omega L)(G + j\Omega C)}$$

To have $\alpha = 0$, as a result, we need $\gamma = j\beta$. This can only happen when $R = G = 0$. With this, we now have,

$$\begin{aligned} \gamma &= j\Omega\sqrt{LC} \\ \implies \beta &= \Omega\sqrt{LC} \end{aligned} \tag{2.12}$$

We have seen that, $\beta = 2\pi/\lambda$. Substituting in (2.12), we get

$$\frac{2\pi}{\lambda} = 2\pi f\sqrt{LC}$$

The wave velocity v is the product of frequency and wavelength, which can be expressed as

$$v = \frac{1}{\sqrt{LC}} \tag{2.13}$$

For a lossless line the characteristic impedance Z_0 is purely real, since both $R = G = 0$. Thus,

$$Z_0 = \sqrt{\frac{L}{C}} \tag{2.14}$$

This indicates the physical properties of a lossless line. Resistance and conductance are lossy elements and must be zero to minimize the losses. Since the AC impedance of R and G are frequency-independent, these are lossy components, unlike the line inductance and capacitance.

2.3.2 Matching in Transmission Lines

We have discussed that when a line is excited by an AC source, there are voltage and current waves traveling on the line, in both directions- to and from the load. The waves traveling back to the source are termed as reflected waves. The waves reflected back to the source are highly undersirable as they tend to load the source and reduces its life. Thus, the second important design consideration is matching, in order to minimize reflections from the load.

Recall our discussion in section 2.2, where the concept of reflection coefficient Γ was introduced. It is the ratio of the amplitudes of the reverse-going wave to the forward-going gave, at any point l along the line. The matched condition requires that $\Gamma = 0$. From (2.9), we have,

$$\Gamma(l) = \Gamma_L e^{-2\gamma l}$$

Since the term $e^{-2\gamma l}$ is difficult to be made zero (as neither l nor γ is zero), we need to make $\Gamma_L = 0$. By definition, Γ_L is the reflection coefficient at the load, which is given by (2.11), as

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$\Gamma_L = 0$ would imply that

$$Z_L = Z_0 \tag{2.15}$$

This is a very important result and is known as the *matched condition*. Under this condition, there will be no reflected wave from the load to the source. The matched condition is somewhat analogous to the maximum power transfer theorem as in circuit theory and is a highly desirable condition in the design of transmission lines. We would like to design lossless (zero attenuation) and matched transmission lines.

2.3.3 Line Impedance Revisited

We have already derived the expressions for voltage and current as a function of position l on the transmission line. Thus, the impedance at any point l on the line is given by (2.8) as

$$Z(l) = Z_0 \frac{V_+ e^{\gamma l} + V_- e^{-\gamma l}}{V_+ e^{\gamma l} - V_- e^{-\gamma l}}$$

We know that $\Gamma_L = V_+/V_-$. Substituting, we get

$$Z(l) = Z_0 \frac{e^{\gamma l} + \Gamma_L e^{-\gamma l}}{e^{\gamma l} - \Gamma_L e^{-\gamma l}}$$

Also, $\Gamma_L = (Z_L - Z_0)/(Z_L + Z_0)$. Putting this we get

$$\begin{aligned} Z(l) &= Z_0 \frac{e^{\gamma l} + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma l}}{e^{\gamma l} - \frac{Z_L - Z_0}{Z_L + Z_0} e^{-\gamma l}} \\ &= Z_0 \frac{(Z_L + Z_0)e^{\gamma l} + (Z_L - Z_0)e^{-\gamma l}}{(Z_L + Z_0)e^{\gamma l} - (Z_L - Z_0)e^{-\gamma l}} \\ &= Z_0 \frac{Z_L(e^{\gamma l} + e^{-\gamma l}) + Z_0(e^{\gamma l} - e^{-\gamma l})}{Z_L(e^{\gamma l} - e^{-\gamma l}) + Z_0(e^{\gamma l} + e^{-\gamma l})} \end{aligned}$$

Expressing the sum of exponentials as hyperbolic cosine and sine functions, this equation becomes

$$Z(l) = Z_0 \frac{Z_L \cosh \gamma l + Z_0 \sinh \gamma l}{Z_L \sinh \gamma l + Z_0 \cosh \gamma l} \tag{2.16}$$

Equation (2.16) is a very important result. It indicates that if we know the characteristic impedance Z_0 and the load impedance Z_L, then we can calculate the impedance Z at any other point on the line l. In fact, this can be generalized even further- if we know the characteristic impedance Z_0 and the impedance at *any* point on the line l_1, then we can find the impedance at any other point on the line $l_2 \neq l_1$! Hence, this relation is known as the *impedance transformation relation* and is a very handy tool.

Let us now bring in a new term called *normalized impedance*, wherein the impedance at any point on the line is taken with reference to Z_0. Consider the impedance $Z(l)$ whose normalized impedance is expressed as

$$\bar{z}(l) = \frac{Z(l)}{Z_0} \tag{2.17}$$

As is clear, the $\bar{z}_L$ term is unitless. The normalized impedance is a useful term, whose significance shall be discussed later. In terms of this, the impedance transformation relation can be written as

$$\bar{z}(l) = \frac{\bar{z}_L \cosh \gamma l + \sinh \gamma l}{\bar{z}_L \sinh \gamma l + \cosh \gamma l} \tag{2.18}$$

Let us now focus our discussion on lossless transmission lines. In (2.18), we can then replace $\gamma = \alpha + j\beta$. Doing so gives us the impedance transformation relation for lossless lines, as

$$\bar{z}(l) = \frac{\bar{z}_L \cos \beta l + j \sin \beta l}{j\bar{z}_L \sin \beta l + \cos \beta l} \tag{2.19}$$

Based on this, let us now examine some interesting properties of a lossless transmission line. Suppose we know the normalized impedance on a line at point l, which is, say, $\bar{z}(l)$. Let us now move along the length of the line by a distance $\lambda/4$, where λ is the operating wavelength. The normalized impedance at this point is then

$$\bar{z}(l + \lambda/4) = \frac{\bar{z}_L \cos \beta(l + \lambda/4) + j \sin \beta(l + \lambda/4)}{j\bar{z}_L \sin \beta(l + \lambda/4) + \cos \beta(l + \lambda/4)}$$

We know that $\beta = 2\pi/\lambda$. Putting this in the above expression, we have

$$\begin{aligned} \bar{z}(l + \lambda/4) &= \frac{\bar{z}_L \cos(\beta l + \pi/2) + j \sin(\beta l + \pi/2)}{j\bar{z}_L \sin(\beta l + \pi/2) + \cos(\beta l + \pi/2)} \\ &= \frac{-\bar{z}_L \sin \beta l + j \cos \beta l}{j\bar{z}_L \cos \beta l - \sin \beta l} \end{aligned}$$

Multiplying both numerator and denominator by $-j$, we have

$$\begin{aligned} \bar{z}(l + \lambda/4) &= \frac{j\bar{z}_L \sin \beta l + \cos \beta l}{\bar{z}_L \cos \beta l + j \sin \beta l} \\ \implies \bar{z}(1 + \lambda/4) &= \frac{1}{\bar{z}(l)} \end{aligned} \tag{2.20}$$

We can thus state the first result as

Result 1 *The normalized line impedance inverts itself every $\lambda/4$ distance along the line.*

Now, let us move a distance of $\lambda/2$ along the transmission line. The normalized impedance is then

$$\begin{aligned}\bar{z}(l+\lambda/2) &= \frac{\bar{z}_L\cos(\beta l+\pi)+j\sin(\beta l+\pi)}{j\bar{z}_L\sin(\beta l+\pi)+\cos(\beta l+\pi)}\\ &= \frac{\bar{z}_L\cos\beta l+j\sin\beta l}{j\bar{z}_L\sin\beta l+\cos\beta l}\\ &= \bar{z}(l)\end{aligned}$$

This brings us to the second result.

Result 2 *The line impedance repeats itself every $\lambda/2$ distance along the line.*

As far as matching is concerned, the absolute load impedance does not have much meaning by itself. All line impedances are with respect to the characteristic impedance Z_0. Hence, in many discussions in future, we will find ourselves discussing more with the normalized impedance rather than the absolute impedance. This brings us to another obvious, yet indispensable result.

Result 3 *The normalized load impedance for a matched load is unity.*

2.4 Wave Patterns in Transmission Lines

Let us now revisit the solutions for voltage and current on transmission lines as a function of the position l. These are given by (2.7).

$$\begin{aligned}V(l) &= V_+e^{\gamma l}+V_-e^{-\gamma l}\\ I(l) &= \frac{1}{Z_0}(V_+e^{\gamma l}-V_-e^{-\gamma l})\end{aligned}$$

For lossless lines, these become

$$\begin{aligned}V(l) &= V_+e^{j\beta l}+V_-e^{-j\beta l}\\ I(l) &= \frac{1}{Z_0}(V_+e^{j\beta l}-V_-e^{-j\beta l})\end{aligned} \tag{2.21}$$

This set of lines may be rewritten by taking the $V_+e^{j\beta l}$ term common. This is done in order to find the maximan and minima of the voltage and current along the line. Since we already know the definition of reflection coefficient Γ,

$$\begin{aligned}V(l) &= V_+e^{j\beta l}(1+\Gamma_Le^{-j2\beta l})\\ I(l) &= \frac{V_+}{Z_0}e^{j\beta l}(1-\Gamma_Le^{-j2\beta l})\end{aligned} \tag{2.22}$$

The load reflection coefficient in general, may be a complex quantity depending on the load impedance Z_L. Hence Γ_L can be expressed in polar form with a magnitude and a phase component as $\Gamma_L = |\Gamma_L|e^{j\phi_L}$. Equation (2.22) now becomes

$$\begin{aligned} V(l) &= V_+ e^{j\beta l}(1 + |\Gamma_L|e^{j(\phi_L - 2\beta l)}) \\ I(l) &= \frac{V_+}{Z_0} e^{j\beta l}(1 - |\Gamma_L|e^{j(\phi_L - 2\beta l)}) \end{aligned} \tag{2.23}$$

Let us focus on the voltage equation term $1 + |\Gamma_L|e^{j(\phi_L - 2\beta l)}$. This can be visualized as a vector sum of a constant unit vector and a vector whose phase is decided by l. As l increases, the phase angle $(\phi_L - 2\beta l)$ decreases, implying a clockwise rotation. Thus, as we move from the load towards the source, the vector locus rotates clockwise. Figure 2.2 illustrates this.

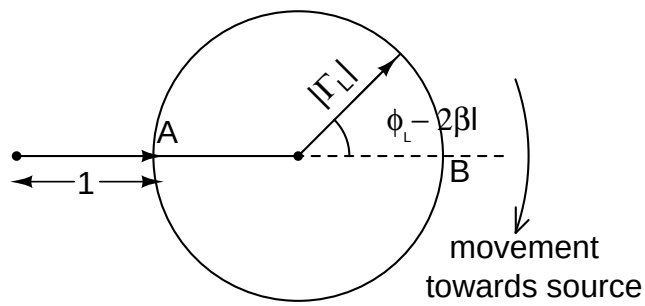

Figure 2.2: Locus of the vector $1 + |\Gamma_L|e^{j(\phi_L - 2\beta l)}$

Clearly, the voltage maxima is achieved at point B, when $(\phi_L - 2\beta l) = 0$ and the voltage minima is at A, where $(\phi_L - 2\beta l) = \pi$. Consequently, the current maxima and minima occur at A and B respectively (convince yourself). Therefore, we can generalize the following in case of voltage along the line:

Voltage magnitude	Phase condition
Maxima	$(\phi_L - 2\beta l) = 2n\pi$
Minima	$(\phi_L - 2\beta l) = (2n+1)\pi$

Here, n is any integer (we assume non-negative, for the sake of simplicity) i.e. $n\epsilon\{0, 1, ...\}$. We can thus, very easily compute the magnitudes of the maximum and minimum voltages using Figure 2.2.

$$\begin{aligned} |V_{max}| &= V_+ e^{j\beta l}(1 + |\Gamma_L|) \\ |V_{min}| &= V_+ e^{j\beta l}(1 - |\Gamma_L|) \end{aligned} \tag{2.24}$$

Define a *Voltage Standing Wave Ratio* (VSWR) ρ, as

$$\rho = \frac{|V_{max}|}{|V_{min}|} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \tag{2.25}$$

We know by now, that $0 \leq |\Gamma_L| \leq 1$, which means that $1 \leq \rho \leq \infty$. For a perfectly matched line, the load VSWR should be unity. This quantity VSWR is merely an extension to our knowledge of the reflection coefficient, but is a highly useful tool for measurements at microwave frequencies. For instance, we can very easily compute the maximum and minimum impedance along the transmission line. At the point of maximum impedance, the voltage magnitude peaks and the current magnitude reaches a minimum. From (2.23) and (2.24), you can show that

$$\begin{aligned} Z_{max} &= \rho Z_0 \\ Z_{min} &= \frac{Z_0}{\rho} \end{aligned} \tag{2.26}$$

And in terms of normalized impedance, the Z_{max} is simply equal to the VSWR! Thus, this analysis shows us the pattern of the standing waves along the line. Voltage and current waves are out of phase by 180° (from the fact that voltage maxima/minima coincide with current minima/maxima respectively). Also, consecutive maxima or minima are apart from each other by a distance of $\lambda/2$. Can you prove this?

2.5 The Smith Chart

We have seen that the impedance at any point on the line can be calculated provided we know the load impedance and the characteristic impedance. We can also do these operations the other way round, i.e. the load impedance can be caluclated provided we know the impedance at some other point on the line. Such comptutations can be done by means of the impedance transformation relation. We shall, for now, limit our discussion to lossless transmission lines i.e. $\alpha = 0$. However, the computation of impedances using the transformation relation, though straightforward, is rather tedious and non-intuitive. In this section, we shall study a simple graphical or figurative tool, the Smith Chart, which will aid us in solving problems on transmission lines. This tool was developed by Philip H. Smith in 1939 and is still widely in use.

Consider the normalized impedance at any point on the line, denoted as $\bar{z}$. The reflection coefficient at this point is then given by

$$\Gamma = \frac{\bar{z} - 1}{\bar{z} + 1} \tag{2.27}$$

Recall that Γ is a complex quantity, and the expression in (2.27) indicates that every value of $\bar{z}$ can be

mapped into a unique value of Γ (how?). Let us then, try to map every value of normalized impedance $\bar{z}$ onto a point in the complex-Γ plane. Let $\bar{z} = r + jx$ and $\Gamma = u + jv$. From (2.27), we have

$$\begin{aligned}
\bar{z} &= \frac{1+\Gamma}{1-\Gamma} \\
\therefore r + jx &= \frac{1+u+jv}{1-(u+jv)} \\
&= \frac{(1+u)+jv}{(1-u)-jv}
\end{aligned}$$

Let us rationalize the R.H.S., i.e. multiply both numerator and denominator by $(1-u)+jv$.

$$\begin{aligned}
\therefore r + jx &= \frac{(1+u)+jv}{(1-u)-jv}\frac{(1-u)+jv}{(1-u)+jv} \\
&= \frac{1-u^2-v^2+j2v}{(1-u)^2+v^2}
\end{aligned}$$

Equating the real part of the R.H.S. to r and the imaginary part to x, we have the following equations.

$$\begin{aligned}
r &= \frac{1-u^2-v^2}{(1-u)^2+v^2} \\
x &= \frac{2v}{(1-u)^2+v^2}
\end{aligned} \tag{2.28}$$

2.5.1 Constant Resistance Solution

Consider the resistance expression in (2.28). Since we are expressing r as a function of the complex Γ plane axes u and v, we need to arrive at a particular expression in terms of u and v to determine the nature of the mapping.

$$\begin{aligned}
r &= \frac{1-u^2-v^2}{(1-u)^2+v^2} \\
\therefore\ & r(1-u)^2 + rv^2 = 1-u^2-v^2 \\
\therefore\ & r - 2ru + ru^2 + rv^2 + u^2 + v^2 - 1 = 0 \\
\therefore\ & (r+1)u^2 - 2ru + r - 1 + (r+1)v^2 = 0
\end{aligned}$$

Dividing throughout by $(r+1)$ we get

$$u^2 - \frac{2ru}{r+1} + v^2 + \frac{r-1}{r+1} = 0$$

Completing the square, we get

$$u^2 - \frac{2ru}{r+1} + \left(\frac{r}{r+1}\right)^2 + v^2 + \frac{r-1}{r+1} = \left(\frac{r}{r+1}\right)^2$$

$$\therefore \left(u - \frac{r}{r+1}\right)^2 + v^2 = \left(\frac{r}{r+1}\right)^2 - \frac{r-1}{r+1}$$

$$\therefore \left(u - \frac{r}{r+1}\right)^2 + v^2 = \left(\frac{r}{r+1}\right)^2 - \frac{r^2-1}{(r+1)^2}$$

$$\therefore \left(u - \frac{r}{r+1}\right)^2 + v^2 = \left(\frac{1}{r+1}\right)^2 \tag{2.29}$$

Clearly, (2.29) shows the equation of a circle, with centre $(r/(r+1), 0)$ and radius $1/(r+1)$. Based on various values of the normalized resistance component r, a set of cricles can be formed, as shown in Figure 2.3. Each circle corresponds to a unique values of r and hence are called *constant resistance circles*. The coordinates indicate the corresponding u- and v-values.

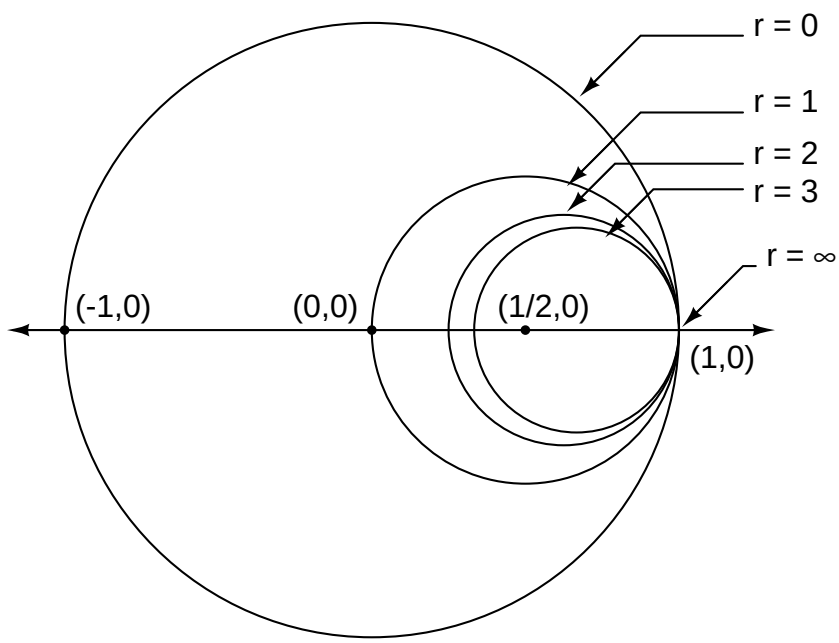

Figure 2.3: Constant Resistance Circles on the Complex Γ-plane

2.5.2 Constant Reactance Solution

Consider the reactance expression in (2.28). Solving in a similar procedure to the constant resistance solution, we have

$$x = \frac{2v}{(1-u)^2 + v^2}$$
$$\therefore\ x(1-u)^2 + xv^2 = 2v$$
$$\therefore\ x(1-u)^2 + xv^2 - 2v = 0$$

Dividing by x and completing the square, we get

$$(u-1)^2 + v^2 - \frac{2v}{x} + \frac{1}{x^2} = \frac{1}{x^2}$$

$$\therefore\ \ (u-1)^2 + \left(v - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \tag{2.30}$$

Clearly, (2.30) is the equation of a circle centered at $(1, 1/x)$ and radius $1/x$. The set of constant reactance circles which can thus be obtained are shown in Figure 2.4.

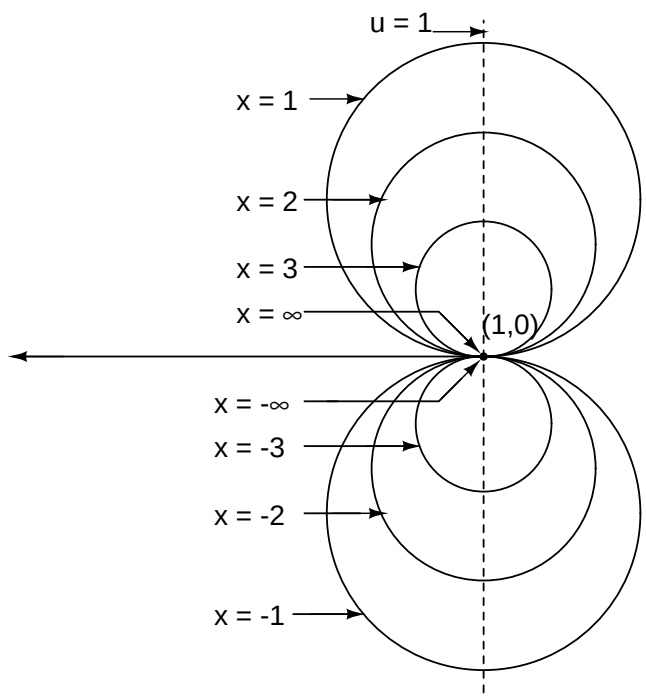

Figure 2.4: Constant Reactance Circles on the Complex Γ-plane

The circles on the positive half of the Γ-plane correspond to the inductive reactance component while

those on the negative half of the Γ-plane correspond to the capacitive reactance component. Combining the results of the constant resistance and reactance circles, we get what is called the Smith Chart, as shown in Figure 2.5. Simply by superimposing the resistance and reactance circles, the Smith Chart is obtained as a result. The outermost circle points to $r = 0$. As an exrecise, point out the resistance and reactance circles for discrete values.

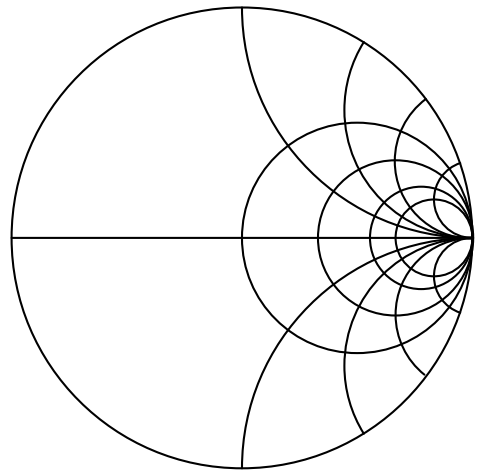

Figure 2.5: Smith Chart as a superposition of resistance and reactance circles

Let us now see some interesting results from the Smith Chart in Figure 2.5. Consider the point $\Gamma = 0$. This would mean that this point corresponds to $\bar{z} = 1$, i.e. the corresponding impedance is equal to the characteristic impedance. A short-circuit impedance, i.e. $\bar{z} = 0$, would be the point where the $r = 0, x = 0$ circles intersect, and an open-circuit impedance, i.e. $\bar{z} = \infty$ would be the point where the $r = \infty, x = \infty$ circles intersect. Thus, we see that every impedance is uniqeuly identitfied by intersection of a constant resistance and a constant reactance circle. Recall from our discussion in section 2.4 that a clockwise movement on the Γ-plane corresponds to movement towards the source and vice-versa (as per our conventions on length l). Figure 2.6 shows the results.

Recall from our discussion in section 2.3.3 that the normalized line impedance inverts itself after every $\lambda/4$ length along the line and repeats every $\lambda/2$ length along the line. This means that half the Smith Chart circumference corresponds to a movement of $\lambda/4$ on the line, and one full circumference implies a $\lambda/2$ movement. You can verify this easily by considering the open- and short-circuit points on the line. Commonly used Smith Charts don't just have a few circles- they are calibrated much like a stationery graph sheet- with accurately labeled resistance and reactance circles. Figure 2.7 shows a commercially available Smith Chart that is commonly used to solve transmission line problems.

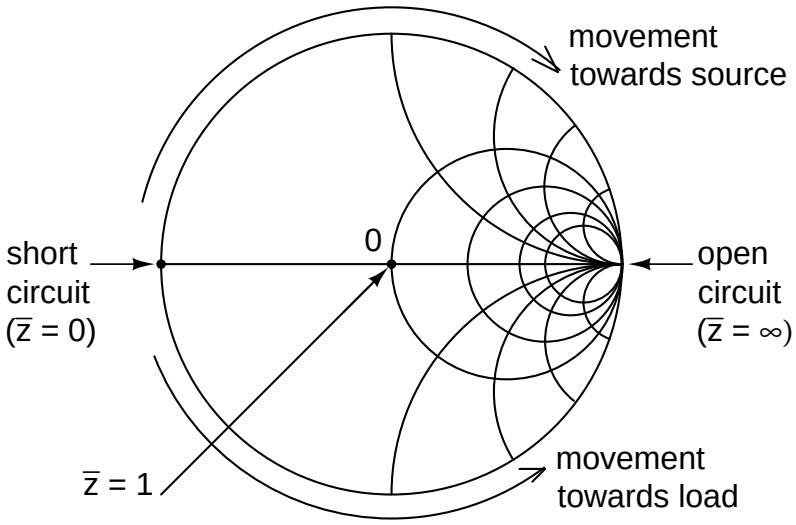

Figure 2.6: Various results using Smith Chart

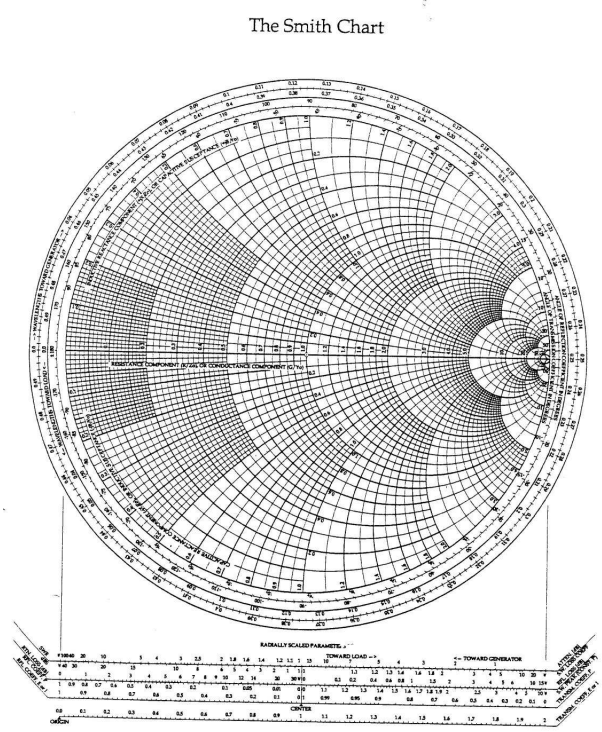

Figure 2.7: A commonly used Smith Chart