A Brief Revision of Vector Calculus and Maxwell’s Equations

Debapratim Ghosh

Electronic Systems Group
Department of Electrical Engineering
Indian Institute of Technology Bombay

e-mail: dghosh@ee.iitb.ac.in
Outline

- Basics of vector calculus - scalar and vector point functions
- Gradients of scalars and vectors - divergence and curl
- Divergence and Stoke’s Theorems revisited
- Basic electric and magnetic quantities
- Gauss, Faraday and Ampere’s Laws
- Development of Maxwell’s Equations
- Boundary phenomena and boundary conditions
Scalar and Vector Point Functions

Consider, in the Cartesian coordinate system, any point $P(\mathbb{R})$ (simply means that the coordinates of $P$ can be expressed in terms of real numbers), which lies inside an arbitrary region $E$.

- If at that point $P$ there corresponds a definite scalar denoted by $f(\mathbb{R})$, then $f(\mathbb{R})$ is called a **scalar point function**.
  e.g. Temperature distribution, density and electric potential are scalar point functions.

- If at that point $P$ there corresponds a definite vector denoted by $F(\mathbb{R})$ then $F(\mathbb{R})$ is called a **vector point function**.
  e.g. Electric field, velocity etc. are vector point functions.
Derivative of a Vector Point Function

- The usual differentiation rules can be applied to vector functions as well. A vector can be a function of multiple variables. For a function say \( F(x, y, z, t) \), its rate of change is

\[
\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt}
\]  

\[\therefore dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz\]

\[= \left( \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right) F\]

- The RHS can be viewed as an operator on \( F \), and can be expressed as a dot product:

\[\left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right). (dx\hat{i} + dy\hat{j} + dz\hat{k})\]

- Let us name the operator \( \text{del} \) as

\[\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}\]

- The \( \text{del} \) operator \( \nabla \) is a vector and denotes the directional derivative of a function
Del Operation on Scalars

- If we have a scalar function $f$, the vector $\nabla f$, is defined as the gradient of $f$, and denotes directional changes i.e. derivatives along the Cartesian axes

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \quad (3)$$

- Recall the basic definition of electric potential, i.e.

$$V = - \int E \cdot dl$$

$$\therefore dV = -E \cdot dl$$

- Notice that both $E$ and $dl$ denote vectors (electric field and displacement), and can be resolved into three components: $E = E_x \hat{i} + E_y \hat{j} + E_z \hat{k}$, and $dl = dx \hat{i} + dy \hat{j} + dz \hat{k}$

- Considering movement along a 3 dimensional plane, the electric field can be expressed as a gradient of the scalar potential $V$, i.e.

$$E = - \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) V \quad (4)$$

$$= - \nabla V \quad (5)$$

- The displacement derivatives $dx$, $dy$, $dz$ are changed to partial derivatives as $V$ is a function of $x$, $y$, $z$, $t$
Del Operation on Vectors - Divergence

- There are two kinds of operations of $\nabla$ on vectors, since vectors can be multiplied in 2 different ways.
- The divergence of a vector $F = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$ is defined as a dot product, i.e.
  $$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

- Define any point $P$ on a closed surface in a vector field $F$. Then the divergence of $F$ at $P$ indicates the flow per unit volume through the surface at $P$.
  - $\nabla \cdot F > 0 \Rightarrow$ Outward flow
  - $\nabla \cdot F < 0 \Rightarrow$ Inward flow
  - $\nabla \cdot F = 0 \Rightarrow$ No flow

- This is quite simple to understand. Consider the surface $f_s$ shown below, through which a vector field $F$ is directed as shown.

- Clearly, the dot product of $\nabla$ and $F$ at point $P$ is positive, indicating outward flow. Had the normal to the surface $\nabla f_s$ been directed the other way, the divergence would have been negative.

- Divergence also indicates expansion or compression (as in case of a fluid).
The second form of Del operation on a vector is called Curl, which is a vector product i.e.

\[ \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \]  

The curl gives an indication of a vector field’s rotational nature. For a rigid body rotating about an axis, the curl relates the angular velocity \( \omega \) of a point \( P \) to its linear velocity \( v \)

\[ v = \omega \times r. \]  

From rotational mechanics, \( v = \omega \times r \). Assuming \( \omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \), it can be shown that

\[ \omega = \frac{1}{2} (\nabla \times v) \]

\[ \therefore \text{If } \nabla \times v = 0 \Rightarrow v \text{ is irrotational.} \]
The Divergence and Stoke’s Theorems

- **Divergence Theorem** is a consequence of the Law of conservation of matter. **The total expansion/compression of a fluid inside a 3-D region equals the total flux of the fluid outward/inward.** If a vector \( \mathbf{v} \) represents fluid flow velocity,

\[
\iiint_V (\nabla \cdot \mathbf{v}) \, dV = \oiint_A \mathbf{v} \cdot d\mathbf{a}
\]  

(10)

- Where \( d\mathbf{a} \) is a small incremental area through which the fluid flows. **Discussion:** How can the above theorem be used to explain Gauss’ Law in electrostatics?

- **Stoke’s Theorem** can be better understood by revisiting Green’s Theorem, which states that **the amount of “circulation” of a vector in the boundary of a region equals the circulation in the enclosed area.** The curl indicates the circulation per unit area

- **Stoke’s Theorem** is a more generalized, 3-D form of Green’s Theorem which conveys a similar meaning i.e.

\[
\oint_C \mathbf{F} \cdot d\mathbf{l} = \oiint_A (\nabla \times \mathbf{F}) \cdot d\mathbf{a}
\]  

(11)

- **Discussion:** The curl of a vector indicates its rotational nature. Intuitively, one can think of magnetic field around a current-carrying conductor. How does Stoke’s Theorem fit in this scenario?
Electric Field and Displacement Density

- Let us define some electrostatic quantities we will be working with

\[
Q \quad \epsilon \quad q \\
\hline
r \quad \hat{r} \\
\hline
\]

- **Electric Field Intensity**: This indicates the electric “impact” in the vicinity of a charge \( Q \) placed in a medium

\[
\vec{E} = \frac{\vec{F}}{q} = \frac{Q}{4\pi \epsilon r^2} \hat{r}
\]  

(12)

- **Electric Displacement Density**: This indicates the electric field strength of a charge \( Q \) in its vicinity, irrespective of the medium characteristics

\[
\vec{D} = \epsilon \vec{E} = \frac{Q}{4\pi r^2} \hat{r}
\]  

(13)

- In both cases, \( \hat{r} \) denotes the unit vector on the axial line between both charges (or, can be along any line)

- \( \epsilon \) denotes the **permittivity** of the medium. Also known as **dielectric constant**, it is the measure of resistance offered by a medium to electric field formation inside it (unit- Farad/metre)
Electric Potential

- **Electric Potential**: If a charge is moved from one point to another, in presence of an electric field, some work is done. This work is characterized by the electric potential.

- If a unit charge is moved along a length $dl$ inside a field $E$, then the potential is

$$V = - \int_c E \cdot dl$$  \hspace{1cm} (14)

- $V$ is defined as moving a positive unit charge from infinity to a point inside the electric field. A negative sign exists because the work is done against the electric field.

- Infinity denotes a point where the effect of the electric field is negligible.

- The potential $V$ is a scalar quantity. We have already shown that

$$\bar{E} = -\nabla V$$
Conduction Current Density

- When the current variation in a medium is spatial, the term **conduction current density** is useful as it denotes current flow per unit area

\[
I = \iint_A \vec{J} \cdot d\vec{a} \tag{15}
\]

- \( \vec{J} \) is the conduction current density and \( d\vec{a} \) is a small cross-section area

- If we assume a conductor with a uniform geometry, a relation between \( \vec{J} \) and electric field \( \vec{E} \), is a direct consequence of Ohm’s Law

\[
V = IR \Rightarrow E I = JA \frac{1}{\sigma} \frac{I}{A} \tag{16}
\]

\[
\Rightarrow \vec{J} = \sigma \vec{E} \tag{17}
\]

in vector form, \( \vec{J} = \sigma \vec{E} \tag{18} \)

where \( \sigma \) is the electrical conductivity, considering a conductor with length \( I \) and cross-section area \( A \)

- It is interesting to note that in a good conductor, where \( \sigma \to \infty \), for finite current to exist (i.e. finite \( \vec{J} \)), then the electric field \( E \) inside the conductor must be zero

- This will be proved in a more detailed manner later
Magnetic Field and Flux Density

- Static charges produce electric field, while magnetic field is a result of movement of charges. This is evident, as a current carrying conductor generates a magnetic field around it (Biot-Savart Law)

- **Magnetic Field Intensity**: This indicates the impact of a small conductor of length $dl$ carrying a current $I$, at a distance $r$ from it. As per Biot-Savart Law,

$$dH = \frac{I}{4\pi r^2} \times \hat{r}$$  \hspace{1cm} (19)

- If your right thumb points towards the direction of current, then the direction of curled fingers show the field direction, i.e. into the plane of the paper

- Else, if your curled right hand fingers point from the $I$ to the $r$ vector (if fingers are aligned through the common plane), then the thumb shows the field direction at that point, again, going into the paper

- **Magnetic Flux Density**: This is a measure of the number of “lines of force” passing through a unit area in the medium

$$\overline{B} = \mu \overline{H}$$  \hspace{1cm} (20)

- $\mu$ denotes the **permeability** of the medium, which is defined as the ability of the medium to support a magnetic field formation within it (unit-Henry/metre)
Types of Media Based on Direction Dependency

- **Isotropic Media:** Properties of the media are not direction dependent, i.e. magnetic field and flux directions are parallel (likewise with electric field and flux). Here, the quantities $\epsilon$, $\mu$ and $\sigma$ are scalars

- **Anisotropic Media:** Properties of the media are direction dependent. Here, the quantities $\epsilon$ and $\mu$ are **tensors**. Assume a Cartesian coordinate system, and the vectors $\vec{H}$ and $\vec{B}$ have components on all 3 axes

> Since $\vec{B} = \mu \vec{H}$, $\mu$ is now a $3 \times 3$ matrix, i.e.

$$
\begin{bmatrix}
B_x \\
B_y \\
B_z
\end{bmatrix} =
\begin{bmatrix}
\mu_{xx} & \mu_{xy} & \mu_{xz} \\
\mu_{yx} & \mu_{yy} & \mu_{yz} \\
\mu_{zx} & \mu_{zy} & \mu_{zz}
\end{bmatrix}
\begin{bmatrix}
H_x \\
H_y \\
H_z
\end{bmatrix}
$$

(21)

- In such cases, permeability and permittivity are denoted like tensors, as $\overline{\mu}$ and $\overline{\epsilon}$
Gauss's Law: The total outward electric displacement through a closed surface equals the total charge enclosed. i.e.

\[ \iiint_A \bar{D} \cdot d\bar{a} = Q \] (22)

The spatial charge distribution inside the surface can be expressed in terms of \( \rho \), the charge density (per unit volume)

\[ \therefore \iiint_A \bar{D} \cdot d\bar{a} = \iiint_V \rho \, dV \] (23)

Using the Divergence Theorem, this gives

\[ \nabla \cdot \bar{D} = \rho \] (24)

Eq. (23) and (24) are the integral and differential forms of Gauss’s Law, respectively.

The first of Maxwell’s equations is therefore, the Gauss’ Law in the two different forms.
Basic Laws II- Gauss’s Law for Magnetic Flux

- The total magnetic flux coming out of a closed surface equals the total magnetic charge (poles) enclosed

- This is a direct conclusion drawn from Gauss’s Law in the electrostatic scenario. However, unlike electric charges, magnetic charges always exist in dipoles.

- Hence inside a closed surface, the total magnetic charge is always zero. Thus,

\[
\oiint_{A} \mathbf{B} \cdot d\mathbf{a} = 0
\]  

(25)

- The differential form is again obtained by the Divergence theorem as,

\[
\nabla \cdot \mathbf{B} = 0
\]  

(26)
Faraday’s Law: *The net electromotive force in a closed loop equals the rate of change of magnetic flux enclosed by the loop* i.e.

\[ V = \oint_c \overline{E} \cdot d\overline{l} = -\frac{\partial \phi}{\partial t} \]  

(27)

The negative sign is a manifestation of Lenz’ law, which states that *the nature of the induced EMF is such that it opposes the source producing it*

In terms of the flux density, the total magnetic flux can be expressed as

\[ \phi = \oiint_A \overline{B} \cdot d\overline{a} \]  

(28)

\[ \therefore \oint_c \overline{E} \cdot d\overline{l} = -\frac{\partial}{\partial t} \oiint_A \overline{B} \cdot d\overline{a} \]  

(29)

Using Stoke’s theorem, the differential form is obtained as

\[ \nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \]  

(30)
Basic Laws IV- Ampere’s Circuital Law

**Ampere’s Law**: The total magnetomotive force through a closed loop equals the current along the loop

Similar to electric potential (≡ EMF), the magnetomotive force (MMF) can be expressed using the magnetic field intensity as,

\[ \oint_c \mathbf{H} \cdot d\mathbf{l} = I \]  

Expressing current in terms of current density \( \mathbf{J} \), we obtain

\[ \oint_c \mathbf{H} \cdot d\mathbf{l} = \iint_A \mathbf{J} \cdot d\mathbf{a} \]  

This can be converted to differential form by using Stoke’s theorem, i.e.

\[ \nabla \times \mathbf{H} = \mathbf{J} \]  

Notice that Ampere’s law is an electromagnetic dual of the Faraday’s Law. Just like the equivalence of Gauss’ Law for electric and magnetic fields, these two laws are also inter-related

Ampere’s Law sounds simple enough. However, Maxwell faced a problem while working with it. Let us see what it was
The Inconsistency in Ampere’s Law

- Assume a closed surface that encloses a charge density $\rho$. If there is a flow of charge leaving the surface, the rate of change of charge equals the conduction current, i.e.

$$\iiint_A \vec{J}.d\vec{a} = -\frac{\partial}{\partial t} \iiint_V \rho dV \quad (34)$$

- Using the Divergence theorem, this simplifies to

$$\nabla.\vec{J} = -\frac{\partial \rho}{\partial t} \quad (35)$$

- An extension of the law of conservation of charges, Maxwell termed Eq. (35) as the equation of continuity

- Maxwell found that Ampere’s Law in its present mathematical form was not consistent with the continuity equation. How? Recall Ampere’s Law $\nabla \times \vec{H} = \vec{J}$

- Taking divergence on both sides, this changes to

$$\nabla.(\nabla \times \vec{H}) = \nabla.\vec{J} = 0 \quad (36)$$

- Clearly, this result is inconsistent with Eq. (35)
Maxwell’s Correction to Ampere’s Law

Thus, Maxwell incorporated the result from Eq. (35) into Ampere’s Law, i.e.

\[ \nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t} (\nabla \cdot \bar{D}) \]

\[ \therefore \nabla \cdot \left( \bar{J} + \frac{\partial \bar{D}}{\partial t} \right) = 0 \] (37)

The term \( \frac{\partial \bar{D}}{\partial t} \) is called displacement current density.

The displacement current arises from the time-varying electric field/flux density due to movement of charges.

Thus, Ampere’s Law was restated by Maxwell as the net magnetomotive force around a closed loop equals the net current, which is a sum of conduction and displacement current.

\[ \nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \] (38)

Thus, Ampere’s Law was modified to be consistent with the continuity equation.
## Summary of Maxwell’s Equations

<table>
<thead>
<tr>
<th>Law</th>
<th>Integral form</th>
<th>Differential form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gauss’ Law (Electric)</td>
<td>( \iiint_A \vec{D} \cdot d\vec{a} = \iiint_V \rho , dV )</td>
<td>( \nabla \cdot \vec{D} = \rho )</td>
</tr>
<tr>
<td>Gauss’ Law (Magnetic)</td>
<td>( \iiint_A \vec{B} \cdot d\vec{a} = 0 )</td>
<td>( \nabla \cdot \vec{B} = 0 )</td>
</tr>
<tr>
<td>Faraday’s Law</td>
<td>( \oint_c \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \iint_A \vec{B} \cdot d\vec{a} )</td>
<td>( \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} )</td>
</tr>
<tr>
<td>Ampere’s Law</td>
<td>( \oint_c \vec{H} \cdot d\vec{l} = \iint_A \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{a} )</td>
<td>( \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} )</td>
</tr>
</tbody>
</table>

The integral form of the Maxwell’s equations may be used in any scenario. If the medium and fields are continuous and spatial derivatives of the fields exist, then the differential form may be more convenient. However, at media boundaries and discontinuities, the integral form must be used.
Boundary Phenomena- Surface Charges and Surface Current

- Suppose a medium exists with uniform charge and current distribution within. The quantities charge density $\rho$ (Coulomb/metre$^3$) and current density $J$ (Ampere/metre$^2$) are used to denote these.

- At the boundary of the medium, the surface quantities are noteworthy i.e. the charge and current density and the surface. Suppose, from the boundary, we define a depth $d$ into the interior of the medium.

- The charge under a surface of unit area is given as $\rho d$. Thus, the surface charge density is given as

$$\rho_s = \lim_{d \to 0} \rho d \quad (39)$$

- Similarly, the current flowing under a strip of unit width is given as $Jd$. The surface current is then

$$J_s = \lim_{d \to 0} Jd \quad (40)$$

- The units of $\rho_s$ and $J_s$ are Coulomb/metre$^2$ and Ampere/metre, respectively.

- The concepts of surface charge and surface currents are useful when dealing with the propagation of fields through medium boundaries.
Relation Between Fields at Media Interfaces

- Consider a continuous boundary between two media with different permittivity, permeability, and conductivity.
- Consider a uniform rectangular loop of size $L \times W$ around the boundary. Later, the boundary condition $W \to 0$ shall be put.
- Suppose the electric field is directed from medium 1 to medium 2. The electric field in each medium can be resolved into the normal and tangential components (w.r.t. the boundary), denoted as $E_n$ and $E_t$, respectively.

Now, use Faraday’s Law around the closed rectangular loop, which encloses a magnetic flux density $B$. Thus,

$$E_{t1}L + (E_{n1} + E_{n2}) \frac{W}{2} - E_{t2}L - (E_{n2} + E_{n1}) \frac{W}{2} = - \frac{\partial B}{\partial t} (L \times W) \tag{41}$$

where the direction of $B$ is perpendicular to the plane of the paper.
Electric Field and Displacement Density at Media Boundary

- Putting the limit $W \to 0$ in Eq. (41), the RHS reduces to zero. Therefore,
  \[ E_{t1} = E_{t2} \] (42)

- Thus, the **tangential component of the electric field is always continuous across boundaries**

- Consider the displacement density $D$ through a cross-section area $A$ between the two media, where charge density $\rho$ (plus surface charge density $\rho_s$) is enclosed

\[ D_{n1} - D_{n2} = \rho_s \] (44)

- Applying Gauss’ Law through this closed surface,
  \[ D_{n1}A - D_{n2}A = \rho Ad + \rho_s A \] (43)

- In the limiting boundary condition, $d \to 0$. Therefore,
  \[ D_{n1} - D_{n2} = \rho_s \] (44)

- For good dielectrics, $\rho_s \to 0$. In this case, $D_{n1} - D_{n2} = 0$. Here, the **normal component of the displacement density is continuous across boundaries**
Magnetic Flux and Field Intensity at Media Boundary

- Extending Gauss’ Law for the normal component of the magnetic field, we obtain
  \[ B_{n1} - B_{n2} = 0 \] (45)
- Thus, **the normal component of the flux density is continuous across boundaries**
- Consider the normal and tangential components of the magnetic field along a loop at the boundary, enclosing current density \( J \) (plus surface current \( J_s \))

\[
\begin{array}{c}
\text{Medium 1} \\
\text{Normal} \\
\text{Medium 2}
\end{array}
\]

\[
H_{n1} \quad H_{n2} \quad H_{t1} \quad H_{t2}
\]

- Applying Ampere’s Circuital Law around the loop, we obtain
  \[ H_{t1}L - H_{t2}L = J_sL + J(L \times W) + \frac{\partial D}{\partial t} (L \times W) \] (46)
- Using the boundary condition \( W \rightarrow 0 \), this simplifies to \( H_{t1} - H_{t2} = J_s \). But since both \( H \) and \( J_s \) are vectors, the directions must be included
Magnetic Field and Surface Current

- According to Ampere’s Law, $H$ and $J$ can be related as $\nabla \times H = J$. Thus, $H$ and $J$ are perpendicular

- The boundary condition can then be rewritten as

$$\hat{n} \times (H_{t1} - H_{t2}) = J_s$$

(47)

- Thus, the surface current inside the loop will be directed into the plane of the paper (use Right Hand Thumb Rule)

- Since the normal components $H_{n1}$ and $H_{n2}$ lie along the normal, the boundary condition can be simplified to include the complete field, as

$$\hat{n} \times (H_1 - H_2) = J_s$$

(48)

- For a perfect dielectric, surface current $J_s = 0$. Therefore,

$$H_{t1} = H_{t2}$$

(49)

- In this case, the tangential component of the magnetic field is continuous across boundaries
Boundary Conditions at Dielectric-Conductor Interface

- The boundary conditions discussed so far are for two perfectly dielectric media, or those which are largely dielectric with weak conductivity.

- Suppose medium 2 is a perfect conductor with $\sigma \to \infty$. How will the boundary conditions change?

- It has already been established that for finite current to exist in a conductor, the electric field has to be zero inside the conductor. As per Maxwell's equations, a zero electric field also means a zero magnetic field.

- Hence, the boundary conditions for dielectric-dielectric, and dielectric-conductor boundaries may be summarized as:

<table>
<thead>
<tr>
<th>General Boundary Condition</th>
<th>Dielectric-dielectric</th>
<th>Dielectric-conductor</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{t1} = E_{t2}$</td>
<td>$E_{t1} = E_{t2}$</td>
<td>$E_{t1} = 0$</td>
</tr>
<tr>
<td>$D_{n1} - D_{n2} = \rho$</td>
<td>$D_{n1} = D_{n2}$</td>
<td>$D_{n1} = \rho$</td>
</tr>
<tr>
<td>$B_{n1} = B_{n2}$</td>
<td>$B_{n1} = B_{n2}$</td>
<td>$B_{n1} = 0$</td>
</tr>
<tr>
<td>$\hat{n} \times (H_{t1} - H_{t2}) = J_s$</td>
<td>$H_{t1} = H_{t2}$</td>
<td>$\hat{n} \times H_{t1} = J_s$</td>
</tr>
</tbody>
</table>
References

- Electromagnetic Waves by R. K. Shevgaonkar
- Microwave Engineering by D. M. Pozar
- Electromagnetic Waves and Radiating Systems by Jordan and Balmain