

Chapter 3: The Uncertainty Principle & Time-Bandwidth Product

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Introduction

For a long time Fourier transforms have been an invaluable tool for signal processing and its applications. The Fourier transform is essentially required to get the information of the signal in the frequency domain.

To get a high resolution of the frequency data of a signal, one needs to observe the signal in time domain for a long interval. If one observes the signal in the time domain for only a short interval, then the (frequency data) Fourier transform of the signal is more spreader out. E.g.: In the DFT computation if one wishes to have closely spaced frequency components i.e. high resolution of samples (n) in frequency domain, then one would require an equal number of samples (n) in the time domain for analysis. This is equivalent to saying that one should observe the signal for a longer interval.

However this was not a problem as long as one was restricted to analysis in a single domain (i.e. time domain or frequency domain). But a problem arises when one starts to analyze signals in Composite domains. Composite domain analysis is studying the given data in 2 or more domains simultaneously. The need for such analysis arises when one wants to extract the frequency components at different instants of time. For e.g. if one wants to analyze the composition of the notes played by a solo instrumental performance, then composite domain analysis is required.

Composite domains in general do not pose a difficulty in analysis. This happens only when the 2 domains are inversely related to each other. E.g.: Time and frequency domains have a relation ($t = 1/f$). In this case it is seen that if a signal has compact support in time then its frequency equivalent does not have it, and vice versa.

What is Uncertain?

If we analyze a signal in time and frequency together, then more we zoom in into time, the equivalent amount we zoom out in frequency, and vice versa.

- 1) A sine wave from $-\infty$ to ∞ has FT as impulse at frequency ω . That is we are certain in frequency domain but interval in time is large.
- 2) An impulse signal has FT as constant 1 over frequency $-\infty$ to ∞ . That is we are certain in time but in frequency the spread is large.

Hence if one wishes to zoom into time and frequency simultaneously, one comes across a fundamental limit. This limit to the highest resolution that one can achieve in the time-frequency composite domain is the uncertainty principle. This is a limit imposed by nature.

Since compact support in any one-domain dictates that representation in the other domain is of non-compact support, one cannot use support as measure in composite domain analysis. It would anyway be of little use. For analysis in composite domain it is therefore the spread or variance (i.e. where approx 98 % of the energy concentration lies) that is measured.

Time Bandwidth Product

Time-bandwidth product is the product of the duration of a signal and its spectral width. Generally, signals of short duration have wide spectral width and vice versa. The time-bandwidth product for actual signals will vary but there will always be a minimum time-bandwidth product for a certain desired effect. In communications over a channel, transmitting a certain amount of data over a given bandwidth requires a certain time. The time-bandwidth product measures how well we use the available bandwidth for a given channel.

Considering the predefined formulae for variances in time and frequency of the signal $x(t)$ the uncertainty principle derivation leads to a lower limit of 0.25. In other words the smallest area covered in the composite domain by the product of variance in frequency and variance in time is 0.25.

If one performs the calculation for a triangular wave then the time-bandwidth product is observed to be 0.3. However to reach the fundamental limit, one can work out the expression for the optimal waveform from the derivation itself. It turns out that the optimal waveform is the **Gaussian Waveform**. The time and frequency variance is 0.5 each, and hence the time-bandwidth product is = 0.25.

Uncertainty Principle:

For a finite energy function $x(t)$, the following relation holds for its time-bandwidth product,

$$\boxed{\therefore \sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}}$$

I.e. 0.25 is the lower bound for the time-bandwidth product of a finite energy signal.

Making the use of famous Schwarz inequality derives this inequality.

The equality holds for a Gaussian distributed signal, whose time and frequency variances Are 0.5 each. So a Gaussian waveform is an optimal waveform.

Proof:

Let us consider a finite energy signal $x(t)$ with centre in time and frequency,

$$t_0 = 0$$

$$\omega_0 = 0$$

$$\sigma_t^2 = \frac{\int t^2 |x(t)|^2 dt}{\|x\|_2^2} \quad (\text{Time Variance})$$

$$\sigma_\omega^2 = \frac{\int \omega^2 |x(\omega)|^2 d\omega}{\|x(\omega)\|_2^2} \quad (\text{Frequency Variance})$$

The frequency variance can also be written as (using duality property of Fourier transform),

$$\sigma_{\omega}^2 = \frac{\int \left| \frac{d}{dt} x(t) \right|^2 dt}{\|x\|_2^2}$$

$$\sigma_t^2 \cdot \sigma_{\omega}^2 = \frac{\int t^2 |x(t)|^2 dt \int \left| \frac{d}{dt} x(t) \right|^2 dt}{\|x\|_2^4}$$

Schwarz Inequality says: $|\langle f(\cdot), g(\cdot) \rangle|^2 \leq \|f\|^2 \|g\|^2$

Thus, 1st integral in the numerator,

$$\int t^2 |x(t)|^2 dt = \int |tx(t)|^2 dt = \|tx(t)\|_2^2$$

$$\int \left| \frac{dx(t)}{dt} \right|^2 dt = \left\| \frac{dx(t)}{dt} \right\|_2^2$$

From Schwarz Inequality,

$$\begin{aligned} \text{Numerator of } \sigma_t^2 \sigma_{\omega}^2 &\geq \left| \int tx(t) \overline{\frac{dx(t)}{dt}} dt \right|^2 \\ &\geq \left| \operatorname{Re} \int tx(t) \overline{\frac{dx(t)}{dt}} dt \right|^2 \quad (\because |\operatorname{Re}(x)| \leq |x|) \end{aligned} \quad \text{(a)}$$

$$\text{Now, } \operatorname{Re}(z) = \frac{1}{2}(z + z^*)$$

$$\text{Thus, } \operatorname{Re} \int x(t) \overline{\frac{dx(t)}{dt}} dt = \frac{1}{2} \left\{ x(t) \overline{\frac{dx(t)}{dt}} + \overline{x(t)} \frac{dx(t)}{dt} \right\} = \frac{1}{2} \frac{d}{dt} |x(t)|^2$$

Using this in (a),

$$\begin{aligned} \text{Numerator of } \sigma_t^2 \sigma_{\omega}^2 &\geq \left| \int \frac{1}{2} t \frac{d}{dt} |x(t)|^2 dt \right|^2 \\ &= \frac{1}{4} \left| \int t \frac{d}{dt} |x(t)|^2 dt \right|^2 \end{aligned}$$

Integrating by parts we get,

$$t|x(t)|^2 \Big|_{-\infty}^{+\infty} - \int |x(t)|^2 dt \quad \text{(b)}$$

Consider the integral,

$$\int_{-\infty}^{+\infty} t^2 |x(t)|^2 dt, \text{ is finite, } \sigma_t^2 \text{ is finite}$$

$$\therefore t^2 |x(t)|^2 \rightarrow 0 \text{ as } t \rightarrow \infty$$

And hence,

$$t|x(t)|^2 \rightarrow 0$$

Therefore, 1st part of the integral (d) goes to zero.

Numerator of $\sigma_t^2 \sigma_\omega^2$

$$\geq \frac{1}{4} \left| \int |x(t)|^2 dt \right|^2$$

$$= \frac{1}{4} \left(\|x\|_2^2 \right)^2$$

$$= \frac{1}{4} \|x\|_2^4$$

$$\therefore \sigma_t^2 \sigma_\omega^2 \geq \frac{\frac{1}{4} \|x\|_2^4}{\|x\|_2^4}$$

$$\boxed{\therefore \sigma_t^2 \sigma_\omega^2 \geq \frac{1}{4}}$$

Optimal function (Gaussian)

When would this equality be reached, i.e. $\sigma_t^2 \cdot \sigma_\omega^2 = 1/4$, and if so then what is the corresponding "optimal" function?

From Schwartz Inequality we can see that equality is achieved when $x(t) = \gamma_0 \frac{dx(t)}{dt}$

where γ_0 is a constant i.e. the two functions are linearly dependant on each other

$$\frac{dx(t)}{x(t)} = \frac{1}{\gamma_0} dt$$

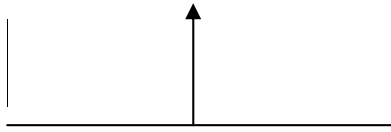
Integrating on both sides, we get

$$\ln(x) = \frac{1}{\gamma_0} \frac{t^2}{2} + C_0$$

$$x(t) = \tilde{C}_0 \cdot e^{\frac{t^2}{2\gamma_0}} \quad \text{Where } \tilde{C}_0 = e^{C_0}$$

Since we want finite energy, γ_0 must be negative real

\tilde{C}_0 Is a constant, leaving it does not matter since it just scales the function. Hence we get the Optimal function as the Gaussian function, given by



Where $\sigma_0 \in \mathfrak{R}$, is its standard deviation.

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