Thus: As $\#\text{iterations} \to \infty$, $\varepsilon_{\text{MAP}} \leq 0.4885 = \text{MAP threshold for (3,6)-code}$

To get the best bound, look for a fixed point:

$$x = f(\varepsilon, x), \text{ where } x_{t+1} = \varepsilon \left(1 - (1 - x_t)^{r-1}\right)^{\ell-1} = f(\varepsilon, x_t)$$

$$\varepsilon(x) = \frac{x}{(1 - (1 - x)^{r-1})^{\ell-1}}$$

So, we could study the limiting curve $(\varepsilon(x), x)$.

More appropriately, study $O(\varepsilon(x), (1 - (1 - x)^{r-1})^\ell)$

Another fact: the area also has area = rate of the code.
1 + l(r-1) variable nodes.
l constraints.
So 1 + l(r-2) free variables = #degrees of freedom.

Natural channel at root: \( \varepsilon(x) \)
Natural channel at previous level: \( x \)

Area A:

\[
A + Bl(r-1) = 1 + l(r-2).
\]
\[ B = \frac{r-1}{r} \]

Solving, we get \[ A = 1 - \frac{d}{r} \] - rate of the code.

These ideas generalize to irregular LDPC codes to spatially coupled codes.

\[ y = \left( 1 - (1-x)^{r-1} \right)^{\frac{1}{r}} \]

Want to argue that \( z_1 = z_2 \).

Spatially coupled codes analysis.

Take each edge and retain or switch with equal probability.

Then connect uniformly to equivalent node up to depth \( w \) to the right.
Protograph:

Before:
\[ X_0 = \varepsilon \]
\[ X_L = f(\varepsilon, X_{L-1}) = \varepsilon (1 - (1 - X_{L-1})^{r-1})^{l-1} \]

Now:
\[ X_0^{(i)} = \varepsilon ; \quad i = -L \text{ to } L \]
\[ X_t^{(i)} = f(\varepsilon, X_{t-1}^{(i)}, \ldots, X_{t-1}^{(i+w-1)}, X_{t-1}^{(i-1)}, X_{t-1}^{(i-w+1)}) \]