Lecture 6.

BSC: \( \Omega = \{ \text{set of decodable error patterns} \} \)

\( \mu_{\text{e}}(\Omega) = 1 - P_B(\varepsilon) \).

This \( \Omega \) is monotonically decreasing.

\[
\begin{align*}
\omega: & \quad 1111 \ldots 1 \ 000 \ 0 \\
\longrightarrow & \quad a \\
\omega': & \quad 000 \ 111110 \ldots 0 \\
\longrightarrow & \quad b \\
\end{align*}
\]

\( a + b - 2 \Delta > 0 \)

Boundary points: \( a + b - 2 \Delta = a+1 \) \( \Delta = \frac{b-1}{2} = d' - 1 \).
Reed-Muller codes.

\[ G_m = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \otimes m \]
\[ G_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ etc.} \]

\[ N = 2^m \]
\[ R = \frac{K}{N} \]

RM codes: Take the rows with the highest weight as generator
Polar codes: K rows chosen in a channel dependent way.

We know the number of intersections.

Do RM codes achieve capacity? [Look at slides for a better description]

Ingredients: RM codes are 2-transitive
Symmetric monotone sets have sharp thresholds.
EXIT functions satisfy the area theorem.

- 1-transitive: \( \forall i, j \in [N], \exists \pi: [N] \to [N] \text{ s.t. } \pi(i) = j \)
  and \( \pi(c) = c \)

- 2-transitive: \( \forall i, j, k, l, \exists \pi \text{ that takes } i \to j, k \to l \)
  and \( \pi(c) = c \).
Sharp Thresholds.

\( \Omega \): monotone, symmetric (1-transitive)

\( \mu_e \): Bernoulli product measure.

\( \mu_e(Q) \) goes from 0 to 1-8 within a window of size \( \log \frac{1/25}{\log N} \).

Recall area theorem.

\( \hat{x}^{\text{MAP}}(y_{\hat{x}}) = ? \)

Call these elements as erase patterns \( \Omega \). \( h_i(\Omega) = \mu_e(\Omega) = P \{ \hat{x}^{\text{MAP}}(y_{\hat{x}}) = ? \} \).

\( \Omega \) is monotone.

\( \Omega \) is symmetric. Needs to fix \( i \), and then 1-transitivity of \( \Omega \).

Hence \( C \) is 2-transitive implies \( \Omega \) is symmetric.
Independence.

Monotonicity + Symmetry + Friedgut-Kalai = $O\left(\frac{1}{\log n}\right)$

Add independence and area theorem to fix rate.

There is more symmetry, so we can do better than $O\left(\frac{1}{\log n}\right)$.

Block recurrent bit $\rightarrow$ Friedgut + Kalai $\rightarrow$ Bourgain + Kalai.