Polyphase Conditions and Structures for 2-D Quincunx FIR Filter Banks Having Quadrantal or Diagonal Symmetries

Pushkar G. Patwardhan, Bhushan Patil, and Vikram M. Gadre

***Abstract*—In this brief, we derive conditions on the polyphase matrix of 2-D finite-impulse response (FIR) quincunx filter banks, for the filters in the filter bank to have quadrantal or diagonal symmetry. These conditions provide a framework for synthesizing polyphase structures which structurally enforce the symmetry. This is demonstrated by constructing examples of small parameterized matrix structures which satisfy the above conditions, thus giving perfect reconstruction FIR quincunx filter banks with quadrantal or diagonally symmetric short-kernel (i.e., short-support) filters. It is also shown that cascades of the above constructed small structures can be used to construct filters of higher order.**

***Index Terms*—2-D symmetry, quincunx filter banks.**

I. Introduction

LINEAR-PHASE filter banks are desirable for image-processing and coding applications. In the case of 2-D nonseparable filter banks, linear-phase corresponds to centro-symmetry of the 2-D filter impulse response. However, for the 2-D case, other filter symmetries are possible, like quadrantal symmetry and diagonal symmetry. Fig. 1 shows examples of signals with these symmetries. For the case of 2-D quincunx filter banks, filters having these symmetries are required to "preserve" signal symmetries in 2-D signal extension schemes [9], [10]. One of the requirements for preserving signal symmetries, as discussed in the above references, is to design quincunx filter banks with the filters having quadrantal or diagonal symmetries.

Various methods for designing linear-phase quincunx filter banks have been discussed in the literature [3]–[5], [8], [12]–[15]. Design methods using cascade structures are discussed in [4], [5]. [13]–[15] discuss design methods using the lifting factorization, and [7] uses generalized McClellan transformations to design quincunx filter banks from 1-D filter banks. All the above references consider the design of linear-phase (i.e., centro-symmetric) quincunx filter banks, but none of the references explicitly considers the case of quadrantal or diagonal symmetric filter banks. It is interesting to note that the design examples presented in [7] and [15] do have quadrantal and diagonal symmetries, though this was not explicitly intended in the development of the design methods. The presence of these symmetries has also not been explicitly mentioned in these references. In this brief, we derive conditions on the polyphase matrix of the quincunx filter bank, so that the filters have quadrantal or diagonal symmetries. These conditions provide a framework for synthesizing polyphase matrices which structurally enforce the filter symmetries.

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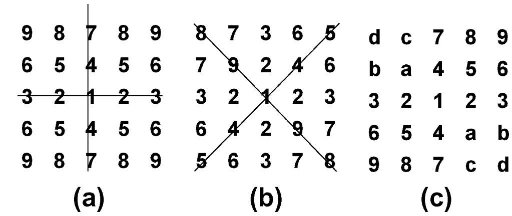

Fig. 1. Examples of sequences with (a) quadrantal symmetry, (b) diagonal symmetry, and (c) centro symmetry.

This brief is organized as follows. The rest of this section presents a brief review of the polyphase representation for the quincunx filter bank, and the characterization of quadrantal and diagonal symmetry. Polyphase conditions for the case of quadrantally symmetric filter bank are derived in Section II-A, and for diagonally symmetric filters in Section II-B. In Sections III-A and III-B, we construct small structures satisfying the above derived conditions, which gives perfect-reconstruction quincunx filter banks with short-kernel filters having symmetries. It is also shown that cascades of these structures can be used to design symmetric quincunx filter banks with filters having higher orders.

Notation: Boldfaced lower-case letters are used to represent vectors, and bold-faced upper case letters are used for matrices. $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A}$, $\mathbf{A}^{-1}$ denotes the inverse of $\mathbf{A}$, and $\det(\mathbf{A})$ denotes the determinant of $\mathbf{A}$. Following the notation of [2], a vector $\mathbf{z} = [z_0\ z_1]^{\mathrm{T}}$ raised to a matrix power $\mathbf{A} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}$ is defined as follows: $\mathbf{z}^{\mathbf{A}}$ is a vector whose ith entry is $z_0^{A_{0i}} z_1^{A_{1i}}$, where $i = 0, 1$.

A. Quincunx Filter Bank

A 2-D quincunx filter bank is shown in Fig. 2. Throughout this brief, we will use $\mathbf{Q}$ to refer to the particular quincunx matrix $\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Using the 2-D polyphase decomposition [1], [2], we can represent each analysis and synthesis filter of the quincunx filter bank in the form

$$H_k(\mathbf{z}) = E_{k,0}(\mathbf{z}^{\mathbf{Q}}) + \mathbf{z}^{-\mathbf{k}_1} E_{k,1}(\mathbf{z}^{\mathbf{Q}}) \tag{1a}$$

$$G_k(\mathbf{z}) = \mathbf{z}^{-\mathbf{k}_1} R_{0,k}(\mathbf{z}^{\mathbf{Q}}) + R_{1,k}(\mathbf{z}^{\mathbf{Q}}) \tag{1b}$$

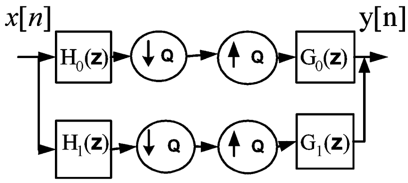

Fig. 2. Quincunx filter bank.

where $\mathbf{k_1} = [1\ 0]^{\mathrm{T}}$.

We will use the above polyphase representation in this brief.

B. Characterization of Quadrantal and Diagonal Symmetry

A quadrantally symmetric 2-D sequence, with center of symmetry $\mathbf{c} = [c_1\ c_2]^T$, can be characterized as [10], [11]

$$\mathrm{X}(\mathbf{z}) = \gamma_1 \mathbf{z}^{-2\mathbf{A_1 c}} \mathrm{X}(\mathbf{z}^{\mathbf{T_1}}) = \gamma_2 \mathrm{z}^{-2\mathbf{A_2 c}} \mathrm{X}(\mathbf{z}^{\mathbf{T_2}}) = \gamma_1 \gamma_2 \mathbf{z}^{-2\mathbf{c}} \mathrm{X}(\mathbf{z}^{-\mathbf{I}}) \tag{2a}$$

where $\mathbf{T_1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathbf{T_2} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \gamma_1 = \pm 1, \gamma_2 = \pm 1$

$$\mathbf{A_i} = \frac{1}{2}(\mathbf{I} - \mathbf{T_i}), \qquad \text{for i} = 1, 2 \tag{2b}$$

A diagonally symmetric 2-D signal, with center of symmetry $\mathbf{c} = [\mathrm{c}_1\ \mathrm{c}_2]^{\mathrm{T}}$, can be characterized as follows [10], [11]:

$$\mathrm{X}(\mathbf{z}) = \gamma_3 \mathbf{z}^{-\mathbf{A_3 c}} \mathrm{X}(\mathbf{z}^{\mathbf{T_3}}) = \gamma_4 \mathrm{z}^{-\mathbf{A_3 c}} \mathrm{X}(\mathbf{z}^{\mathbf{T_4}}) = \gamma_3 \gamma_4 \mathbf{z}^{-2\mathbf{c}} \mathrm{X}(\mathbf{z}^{-\mathbf{I}}) \tag{3a}$$

where $\mathbf{T_3} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \mathbf{T_4} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \gamma_3 = \pm 1, \gamma_4 = \pm 1$
and

$$\mathbf{A_i} = (\mathbf{I} - \mathbf{T_i}), \qquad \text{for i} = 3, 4. \tag{3b}$$

II. Polyphase Conditions for Filter Symmetry

Considering the case of each symmetry separately, we first derive a condition on the polyphase matrix for the analysis filters to have the symmetry. Further constraints are then imposed so that the determinant of the analysis polyphase matrix is a 2-D monomial, which is necessary for the filters to be finite-impulse response (FIR).

A. Polyphase Conditions for Quadrantally Symmetric Filters

We require the analysis filters to be quadrantally symmetric as in (2a), i.e.,

$$\begin{aligned} \mathrm{H_k}(\mathbf{z}) &= \gamma_{\mathrm{k}1} \mathbf{z}^{-2\mathbf{A_1 c_k}} \mathrm{H_k}(\mathbf{z}^{\mathbf{T_1}}) \\ &= \gamma_{\mathrm{k}2} z^{-2\mathbf{A_2 c_k}} H_{\mathrm{k}}(\mathbf{z}^{\mathbf{T_2}}) = \gamma_{\mathrm{k}1}\gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{c_k}} H_{\mathrm{k}}(\mathbf{z}^{\mathbf{T_1 T_2}}). \end{aligned} \tag{4}$$

Writing (1a) in matrix form we have

$$H_k(\mathbf{z}) = \left[E_{k,0}(\mathbf{z^Q}) \quad E_{k,1}(\mathbf{z^Q})\right] \cdot \begin{bmatrix} 1 \\ \mathbf{z}^{-\mathbf{k_1}} \end{bmatrix}. \tag{5}$$

Using (5) in (4), we get

$$\begin{aligned} &\left[E_{k,0}(\mathbf{z^Q}) \quad E_{k,1}(\mathbf{z^Q})\right] \cdot \begin{bmatrix} 1 \\ \mathbf{z}^{-\mathbf{k_1}} \end{bmatrix} \\ &= \gamma_{\mathrm{k}1} \mathbf{z}^{-2\mathbf{A_1 c_k}} \left[\mathrm{E_{k,0}}(\mathbf{z^{T_1 Q}}) \quad \mathrm{E_{k,1}}(\mathbf{z^{T_1 Q}})\right] \cdot \begin{bmatrix} 1 \\ \mathbf{z}^{-\mathbf{T_1 k_1}} \end{bmatrix} \\ &= \gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{A_2 c_k}} \left[E_{k,0}(\mathbf{z^{T_2 Q}}) \quad E_{k,1}(\mathbf{z^{T_2 Q}})\right] \cdot \begin{bmatrix} 1 \\ \mathbf{z}^{-\mathbf{T_2 k_1}} \end{bmatrix} \\ &= \gamma_{\mathrm{k}1}\gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{c_k}} \left[E_{k,0}(\mathbf{z^{-Q}}) \quad E_{k,1}(\mathbf{z^{-Q}})\right] \cdot \begin{bmatrix} 1 \\ \mathbf{z}^{\mathbf{k_1}} \end{bmatrix}. \end{aligned}$$

Noting that $\mathbf{T_1 k_1} = \mathbf{k_1}$ and $\mathbf{T_2 k_1} = -\mathbf{k_1}$, and comparing terms

$$\begin{aligned} E_{\mathrm{k},0}(\mathbf{z^Q}) &= \gamma_{\mathrm{k}1} \mathbf{z}^{-2\mathbf{A_1 c_k}} E_{\mathrm{k},0}(\mathbf{z^{T_1 Q}}) \\ &= \gamma_{\mathrm{k}2} z^{-2\mathbf{A_2 c_k}} E_{\mathrm{k},0}(\mathbf{z^{T_2 Q}}) \\ &= \gamma_{\mathrm{k}1}\gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{c_k}} E_{\mathrm{k},0}(\mathbf{z^{-Q}}) \end{aligned} \tag{6a}$$

$$\begin{aligned} E_{\mathrm{k},1}(\mathbf{z^Q}) &= \gamma_{\mathrm{k}1} \mathbf{z}^{-2\mathbf{A_1 c_k}} E_{\mathrm{k},1}(\mathbf{z^{T_1 Q}}) \\ &= \gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{A_2 c_k}} \mathbf{z}^{2\mathbf{k_1}} E_{\mathrm{k},1}(\mathbf{z^{T_2 Q}}) \\ &= \gamma_{\mathrm{k}1}\gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{c_k}} \mathbf{z}^{2\mathbf{k_1}} E_{\mathrm{k},1}(\mathbf{z^{-Q}}). \end{aligned} \tag{6b}$$

Using the relations $\mathbf{2A_1 c} = \mathbf{QA_3 Q^{-1} c}$, $\mathbf{2A_2 c} = \mathbf{QA_4 Q^{-1} c}$, $\mathbf{T_1 Q} = \mathbf{QT_3}$, and $\mathbf{T_2 Q} = \mathbf{QT_4}$, (6a) can be simplified as

$$\begin{aligned} E_{\mathrm{k},0}(\mathbf{z^Q}) &= \gamma_{\mathrm{k}1} \mathbf{z}^{-\mathbf{QA_3 Q^{-1} c_k}} E_{\mathrm{k},0}(\mathbf{z^{QT_3}}) \\ &= \gamma_{\mathrm{k}2} z^{-\mathbf{QA_4 Q^{-1} c_k}} E_{\mathrm{k},0}(\mathbf{z^{QT_4}}) \\ &= \gamma_{\mathrm{k}1}\gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{QQ^{-1} c_k}} E_{\mathrm{k},0}(\mathbf{z^{-Q}}). \end{aligned}$$

With $\mathbf{d_k} = \mathbf{Q^{-1} c_k}$, and $\mathbf{z^Q} \to \mathbf{z}$, we can write the above as

$$\begin{aligned} E_{\mathrm{k},0}(\mathbf{z}) &= \gamma_{\mathrm{k}1} \mathbf{z}^{-\mathbf{A_3 d_k}} E_{\mathrm{k},0}(\mathbf{z^{T_3}}) \\ &= \gamma_{\mathrm{k}2} z^{-\mathbf{A_4 d_k}} E_{\mathrm{k},0}(\mathbf{z^{T_4}}) \\ &= \gamma_{\mathrm{k}1}\gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{d_k}} E_{\mathrm{k},0}(\mathbf{z^{-I}}). \end{aligned} \tag{7a}$$

Using similar simplifications as above, and also noting that $\mathbf{2k_1} = \mathbf{Q}[1\ 1]^{\mathrm{T}}$, (6b) can be written as

$$\begin{aligned} E_{\mathrm{k},1}(\mathbf{z}) &= \gamma_{\mathrm{k}1} \mathbf{z}^{-\mathbf{QA_3 d_k}} E_{\mathrm{k},1}(\mathbf{z^{QT_3}}) \\ &= \gamma_{\mathrm{k}2} z^{-\mathbf{QA_4 d_k}} \mathbf{z}^{\mathbf{Q}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} E_{\mathrm{k},1}(\mathbf{z^{QT_4}}) \\ &= \gamma_{\mathrm{k}1}\gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{Qd_k}} \mathbf{z}^{\mathbf{Q}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} E_{\mathrm{k},1}(\mathbf{z^{-Q}}). \end{aligned}$$

With $\mathbf{d'_k} = \mathbf{d_k} - [1/2\ \ 1/2]^{\mathrm{T}}$ and $\mathbf{z^Q} \to \mathbf{z}$, and noting that $\mathbf{A_4}[1/2\ \ 1/2]^{\mathrm{T}} = [1\ 1]^{\mathrm{T}}$ and $\mathbf{A_3}[1/2\ \ 1/2]^{\mathrm{T}} = [0\ 0]^{\mathrm{T}}$, we can write the above as

$$\begin{aligned} E_{\mathrm{k},1}(\mathbf{z}) &= \gamma_{\mathrm{k}1} \mathbf{z}^{-\mathbf{A_3 d'_k}} E_{\mathrm{k},1}(\mathbf{z^{T_3}}) \\ &= \gamma_{\mathrm{k}2} z^{-\mathbf{A_4 d'_k}} E_{\mathrm{k},1}(\mathbf{z^{T_4}}) \\ &= \gamma_{\mathrm{k}1}\gamma_{\mathrm{k}2} \mathbf{z}^{-2\mathbf{d'_k}} E_{\mathrm{k},1}(\mathbf{z^{-I}}). \end{aligned} \tag{7b}$$

Thus, (7a) and (7b) give conditions on the polyphase components of the analysis filters, for the analysis filters to have quadrantal symmetry. And, in fact, (7) says that $E_{\mathrm{k},0}(\mathbf{z})$ and $E_{\mathrm{k},1}(\mathbf{z})$ should have diagonal symmetry with centers of symmetry $\mathbf{d_k} = \mathbf{Q^{-1} c_k}$ and $\mathbf{d'_k}$ respectively. This can be summarized as follows.

Proposition 1: For the analysis filters $H_{\mathrm{k}}(\mathbf{z})$ to have quadrantal symmetry with center of symmetry $\mathbf{c_k}$, a sufficient condition is that each of the polyphase components has diagonal symmetry as in (7a) and (7b) with $\mathbf{d_k} = \mathbf{Q^{-1}c_k}$, and $\mathbf{d'_k} = \mathbf{d_k} - [1/2\ \ 1/2]^{\mathrm{T}}$ ■

We now impose additional constraints using the requirement that the determinant of the analysis polyphase matrix be a monomial. From the first equality relation of (7a) and (7b), we get the following:

$$\begin{bmatrix} E_{0,0}(\mathbf{z}) & E_{0,1}(\mathbf{z}) \\ E_{1,0}(\mathbf{z}) & E_{1,1}(\mathbf{z}) \end{bmatrix} = \begin{bmatrix} \gamma_{01} & 0 \\ 0 & \gamma_{11} \end{bmatrix} \times \begin{bmatrix} \mathbf{z}^{-\mathbf{A_3 d_0}} \mathrm{E}_{0,0}(\mathbf{z}^{\mathbf{T_3}}) & \mathbf{z}^{-\mathbf{A_3 d'_0}} \mathrm{E}_{0,1}(\mathbf{z}^{\mathbf{T_3}}) \\ \mathbf{z}^{-\mathbf{A_3 d_1}} \mathrm{E}_{1,0}(\mathbf{z}^{\mathbf{T_3}}) & \mathbf{z}^{-\mathbf{A_3 d'_1}} \mathrm{E}_{1,1}(\mathbf{z}^{\mathbf{T_3}}) \end{bmatrix}.$$

Taking determinants of both sides of above equation

$$\det(\mathbf{E}(\mathbf{z})) = \gamma_{01}.\gamma_{11} \mathbf{z}^{-\mathbf{A_3}\bar{\mathbf{d}}} \det\left(\mathbf{E}(\mathbf{z}^{\mathbf{T_3}})\right) \tag{8a}$$

where $\bar{\mathbf{d}} = \mathbf{d_0} + \mathbf{d_1} - [1/2\ \ 1/2]^{\mathrm{T}}$

Similarly, from the other equality relations in (7a) and (7b),

$$\det(\mathbf{E}(\mathbf{z})) = \gamma_{02}.\gamma_{12} \mathbf{z}^{-\mathbf{A_4}\bar{\mathbf{d}}} \det\left(\mathbf{E}(\mathbf{z}^{\mathbf{T_4}})\right) \tag{8b}$$

$$\det(\mathbf{E}(\mathbf{z})) = \gamma_{01}.\gamma_{11}.\gamma_{02}.\gamma_{12} \mathbf{z}^{-2\bar{\mathbf{d}}} \det\left(\mathbf{E}(\mathbf{z}^{-\mathbf{I}})\right). \tag{8c}$$

We require $\det(\mathbf{E}(\mathbf{z})) = \mathbf{z^r}$, where $\mathbf{r}$ is an arbitrary integer vector. Thus, we have $\det(\mathbf{E}(\mathbf{z}^{\mathbf{T_3}})) = \mathbf{z}^{\mathbf{T_3 r}}$, $\det(\mathbf{E}(\mathbf{z}^{\mathbf{T_4}})) = \mathbf{z}^{\mathbf{T_4 r}}$, and $\det(\mathbf{E}(\mathbf{z}^{-\mathbf{I}})) = \mathbf{z^{-r}}$

Using this in (8a), (b), (c), and also using the relation between $\mathbf{A_i}$ and $\mathbf{T_i}$ from (2b), we get the following conditions:

$$\mathbf{d_0} + \mathbf{d_1} = \mathbf{r} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{\mathrm{T}} \tag{9a}$$

$$\gamma_{01}.\gamma_{11} = 1, \quad \text{and} \quad \gamma_{02}.\gamma_{12} = 1. \tag{9b}$$

For (9b), we ignore the possibility $\gamma_{01} = \gamma_{11} = -1$ or $\gamma_{02} = \gamma_{12} = -1$ since in that case both the analysis filters have a zero at the same location, therefore perfect-reconstruction would not be possible. Thus, from (9b), we have $\gamma_{01} = \gamma_{11} = 1$ and $\gamma_{02} = \gamma_{12} = 1$.

Thus, to summarize these results, the problem to construct quadrantally symmetric analysis filters can be formulated as: Construct the polyphase matrix $\mathbf{E}(\mathbf{z}) = \begin{bmatrix} E_{0,0}(\mathbf{z}) & E_{0,1}(\mathbf{z}) \\ E_{1,0}(\mathbf{z}) & E_{1,1}(\mathbf{z}) \end{bmatrix}$, such that $E_{0,0}(\mathbf{z})$, $E_{0,1}(\mathbf{z})$, $E_{1,0}(\mathbf{z})$, $E_{1,1}(\mathbf{z})$ have diagonal symmetry with symmetry parameters $\gamma' s = 1$ (i.e., with symmetry, and not anti-symmetry), and with centers of symmetry as $\mathbf{d_0}$, $\mathbf{d_0} - [1/2\ \ 1/2]^{\mathrm{T}}$, $\mathbf{d_1}$, and $\mathbf{d_1} - [1/2\ \ 1/2]^{\mathrm{T}}$ respectively, and satisfying the following constraints.

a) $\det(\mathbf{E}(\mathbf{z})) = \mathbf{z^r}$, where $\mathbf{r}$ is an arbitrary integer vector.

b) $\mathbf{d_0} + \mathbf{d_1} = \mathbf{r} - [1/2\ \ 1/2]^{\mathrm{T}}$. ■

Let us now discuss the properties of the synthesis filters. Here $\mathbf{E}(\mathbf{z})$ is the analysis polyphase matrix, and $\mathbf{R}(\mathbf{z})$ is the synthesis polyphase matrix. With $\det(\mathbf{E}(\mathbf{z})) = \mathbf{z^r}$, we have

$$\mathbf{R}(\mathbf{z}) = [\mathbf{E}(\mathbf{z})]^{-1} = \mathbf{z^{-r}} \begin{bmatrix} E_{1,1}(\mathbf{z}) & -E_{0,1}(\mathbf{z}) \\ -E_{1,0}(\mathbf{z}) & E_{0,0}(\mathbf{z}) \end{bmatrix}$$

Thus, from (1b), we have

$$\begin{aligned} \begin{bmatrix} \mathrm{G}_0(\mathbf{z}) \\ \mathrm{G}_1(\mathbf{z}) \end{bmatrix} &= \mathbf{R}^{\mathrm{T}}(\mathbf{z^Q}). \begin{bmatrix} \mathbf{z}^{-\mathbf{k_1}} \\ 1 \end{bmatrix} \\ &= \mathbf{z^{-Qr}} \begin{bmatrix} \mathbf{z}^{-\mathbf{k_1}} \mathrm{E}_{1,1}(\mathbf{z^Q}) - \mathrm{E}_{1,0}(\mathbf{z^Q}) \\ -\mathbf{z}^{-\mathbf{k_1}} \mathrm{E}_{0,1}(\mathbf{z^Q}) + \mathrm{E}_{0,0}(\mathbf{z^Q}) \end{bmatrix} \\ \Rightarrow \begin{bmatrix} G_0(\mathbf{z}) \\ G_1(\mathbf{z}) \end{bmatrix} &= \mathbf{z^{-Qr}} \begin{bmatrix} -H_1(-\mathbf{z}) \\ H_0(-\mathbf{z}) \end{bmatrix}. \end{aligned}$$

Thus, it can be seen that the synthesis filters also have the same symmetry as the analysis filters.

B. Polyphase Condition for Diagonally Symmetric Filters

We now require the analysis filters $H_k(\mathbf{z})$ to be diagonally symmetric as in (3a). Following a derivation along same lines as Section II-A, we arrive the following conditions on the polyphase components of $H_k(\mathbf{z})$.

$$\begin{aligned} \mathrm{E}_{k,0}(\mathbf{z}) &= \gamma_{k3} \mathbf{z}^{-2\mathbf{A_1 d_k}} \mathrm{E}_{k,0}(\mathbf{z}^{\mathbf{T_1}}) \\ &= \gamma_{k4} \mathrm{z}^{-2\mathbf{A_2 d_k}} \mathrm{E}_{k,0}(\mathbf{z}^{\mathbf{T_2}}) \\ &= \gamma_{k3}\gamma_{k4} \mathbf{z}^{-2\mathbf{d_k}} \mathrm{E}_{k,0}(\mathbf{z}^{-\mathbf{I}}) \end{aligned} \tag{10a}$$

and

$$\begin{aligned} \mathrm{E}_{k,1}(\mathbf{z}) &= \gamma_{k3} \mathbf{z}^{-2\mathbf{A_1 d'_k}} \mathrm{E}_{k,1}(\mathbf{z}^{\mathbf{T_1}}) \\ &= \gamma_{k4} \mathrm{z}^{-2\mathbf{A_2 d'_k}} \mathrm{E}_{k,1}(\mathbf{z}^{\mathbf{T_2}}) \\ &= \gamma_{k3}\gamma_{k4} \mathbf{z}^{-2\mathbf{d'_k}} \mathrm{E}_{k,1}(\mathbf{z}^{-\mathbf{I}}) \end{aligned} \tag{10b}$$

where $\mathbf{d_k} = \mathbf{Q^{-1}c_k}$ and $\mathbf{d'_k} = \mathbf{d_k} - [1/2\ \ 1/2]^{\mathrm{T}}$

Thus, (10a) and (10b) gives conditions on the polyphase components of the analysis filters, for the analysis filters to have diagonal symmetry. And, (10) says that $E_{k,0}(\mathbf{z})$ and $E_{k,1}(\mathbf{z})$ should have quadrantal symmetry with centers of symmetry $\mathbf{d_k} = \mathbf{Q^{-1}c_k}$ and $\mathbf{d'_k}$ respectively. This can be summarized as follows.

Proposition 2: For the analysis filters $H_k(\mathbf{z})$ to have diagonal symmetry with center of symmetry $\mathbf{c}_k$, a sufficient condition is that each of the polyphase components have quadrantal symmetry as in (10a) and (10b), with $\mathbf{d_k} = \mathbf{Q^{-1}c_k}$ and $\mathbf{d'_k} = \mathbf{d_k} - [1/2\ \ 1/2]^{\mathrm{T}}$ ■

We now require that the determinant of the analysis polyphase matrix be a monomial, i.e., $\det(\mathbf{E}(\mathbf{z})) = \mathbf{z^r}$, where $\mathbf{r}$ is an arbitrary integer vector, and following a similar derivation as in Section II-A, we arrive at the following constraints (which are similar to those obtained in (9a) and (9b).

For $\det(\mathbf{E}(\mathbf{z})) = \mathbf{z^r}$ with the elements of $\mathbf{E}(\mathbf{z})$ as in proposition 2, we require that

$$\mathbf{d_0} + \mathbf{d_1} = \mathbf{r} - \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{\mathrm{T}} \tag{11a}$$

$$\gamma_{03}.\gamma_{13} = 1, \quad \gamma_{04}.\gamma_{14} = 1. \tag{11b}$$

Again, as earlier, ignoring the possibility $\gamma_{03} = \gamma_{13} = -1$ or $\gamma_{04} = \gamma_{14} = -1$. Thus, we require $\gamma_{03} = \gamma_{13} = 1$ and $\gamma_{04} = \gamma_{14} = 1$.

Thus, the problem to construct diagonally symmetric analysis filters can be formulated as follows:

Construct the polyphase matrix $\mathbf{E}(\mathbf{z}) = \begin{bmatrix} E_{0,0}(\mathbf{z}) & E_{0,1}(\mathbf{z}) \\ E_{1,0}(\mathbf{z}) & E_{1,1}(\mathbf{z}) \end{bmatrix}$, such that $E_{0,0}(\mathbf{z})$, $E_{0,1}(\mathbf{z})$, $E_{1,0}(\mathbf{z})$, $E_{1,1}(\mathbf{z})$ have quadrantal symmetry with symmetry parameters $\gamma's = 1$ (i.e., with symmetry, and not anti-symmetry), and with centers of symmetry as $\mathbf{d_0}, \mathbf{d_0} - [1/2\ \ 1/2]^{\mathrm{T}}$, $\mathbf{d_1}, \mathbf{d_1} - [1/2\ \ 1/2]^{\mathrm{T}}$ respectively, and satisfying the following constraints.

a) $\det(\mathbf{E}(\mathbf{z})) = \mathbf{z}^{\mathbf{r}}$, where $\mathbf{r}$ is an arbitrary integer vector.

b) $\mathbf{d_0} + \mathbf{d_1} = \mathbf{r} - [1/2\ \ 1/2]^{\mathrm{T}}$. ■

III. Structures for Constructing Symmetric Filters

Using the above conditions, we now construct two examples of matrix structures which can be used to construct quincunx filter banks with symmetric filters. The structure in Section III-A satisfies the conditions in Section II-A, and thus yields filters with quadrantal symmetry. The structure in Section III-B satisfies *both* the conditions in Sections II-A and II-B, and thus yields filters having *both* quadrantal and diagonal symmetry. We note that this is also referred to as octagonal symmetry.

A. A Polyphase Matrix Structure for Quadrantally Symmetric Quincunx Filter Banks

Consider the polyphase matrix $\mathbf{E}(\mathbf{z}) = \begin{bmatrix} E_{0,0}(\mathbf{z}) & E_{0,1}(\mathbf{z}) \\ E_{1,0}(\mathbf{z}) & E_{1,1}(\mathbf{z}) \end{bmatrix}$, with the elements $E_{0,0}(\mathbf{z})$, $E_{0,1}(\mathbf{z})$, $E_{1,0}(\mathbf{z})$, $E_{1,1}(\mathbf{z})$ as follows:

$$E_{0,0}(\mathbf{z}) = a + h\left(z_1^{-1}z_2 + z_1 z_2^{-1}\right) + g\left(z_1 z_2 + z_1^{-1}z_2^{-1}\right) \quad (12a)$$

$$E_{0,1}(\mathbf{z}) = c(z_1 + z_2) + b(1 + z_1 z_2) \quad (12b)$$

$$E_{1,0}(\mathbf{z}) = d\left(1 + z_1^{-1}z_2^{-1}\right) + e\left(z_1^{-1} + z_2^{-1}\right) \quad (12c)$$

$$E_{1,1}(\mathbf{z}) = f. \quad (12d)$$

Here a, b, c, d, e, f, g, h are scalar parameters. It can be verified that with $cd = -be$, $fh = ce$, and $fg = bd$, we have $\det(\mathbf{E}(\mathbf{z})) = -2bd - 2ce + af$ i.e., $\mathbf{r} = [0\,0]^{\mathrm{T}}$. We also require that $\det(\mathbf{E}(\mathbf{z})) = -2bd - 2ce + af \neq 0$.

From the above constraints on the parameters, the dependent parameters can be expressed in terms of the others as

$$\mathrm{c} = -\,\mathrm{be/d}, \quad \mathrm{h} = -\mathrm{be}^2/\mathrm{df}, \quad \text{and } \mathrm{g} = \mathrm{bd/f}, \quad \text{with } d \neq 0, \\ f \neq 0 \quad \text{and } -2bd - 2ce + af \neq 0.$$

Thus, the set of functions in (12a–(d) satisfy all the constraints of Proposition-1 and (9a) and (b). This leads to a quincunx filter bank with the analysis and synthesis filters having quadrantal symmetry.

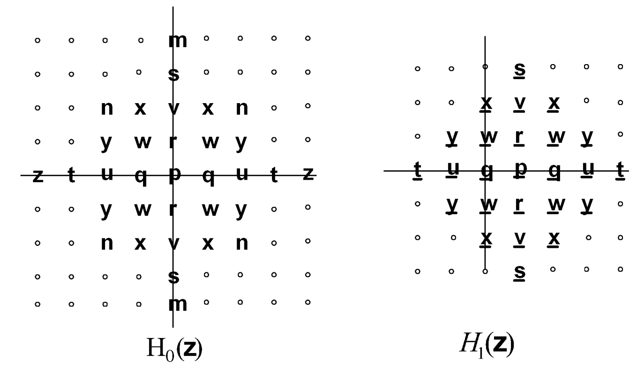

Fig. 3. Quadrantally symmetric analysis filters with two stages as in (12).

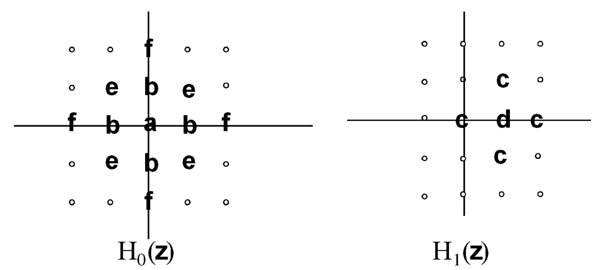

Fig. 4. Quadrantally and diagonally symmetric analysis filters with polyphase components as in (14).

Cascades of the Above Structure: In order to get filters of higher order, we now consider cascades of the matrix constructed in (12). Consider two matrices $\mathbf{E}(\mathbf{z})$ and $\mathbf{E}'(\mathbf{z})$, with elements of the form of (12) with a different set of parameters. Now consider the cascade $\mathbf{E}(\mathbf{z})\mathbf{E}'(\mathbf{z})$, which can be written as in (13) at the bottom of the page.

We now state the following facts.

a) The product of two diagonal symmetric transfer functions is also diagonally symmetric, with the center of symmetry equal to the sum of the centers of symmetry of its factors.

b) The sum of two diagonal symmetric transfer functions having the same center of symmetry is also diagonally symmetric with the same center of symmetry

Using the above facts, and analyzing the centers of symmetry of the elements of $\mathbf{E}(\mathbf{z})\mathbf{E}'(\mathbf{z})$ in (13), it can be seen that $\mathbf{E}(\mathbf{z})\mathbf{E}'(\mathbf{z})$ also satisfies all the conditions of proposition 1 and (9). Thus, filters of higher order can be constructed using such cascades.

Fig. 3 shows the supports of the analysis filters for a two-stage cascade, with each stage as in (12). The coefficients of the filters in Fig. 3 can be expressed in terms of the independent set of parameters of stage-1 and stage-2 of the cascade.

$$\mathbf{E}(\mathbf{z})\mathbf{E}'(\mathbf{z}) = \begin{bmatrix} \left(E_{0,0}(\mathbf{z}).E'_{0,0}(\mathbf{z}) + E_{0,1}(\mathbf{z}).E'_{1,0}(\mathbf{z})\right) & \left(E_{0,0}(\mathbf{z}).E'_{0,1}(\mathbf{z}) + E_{0,1}(\mathbf{z}).E'_{1,1}(\mathbf{z})\right) \\ \left(E_{1,0}(\mathbf{z}).E'_{0,0}(\mathbf{z}) + E_{1,1}(\mathbf{z}).E'_{1,0}(\mathbf{z})\right) & \left(E_{1,0}(\mathbf{z}).E'_{0,1}(\mathbf{z}) + E_{1,1}(\mathbf{z}).E'_{1,1}(\mathbf{z})\right) \end{bmatrix} \quad (13)$$

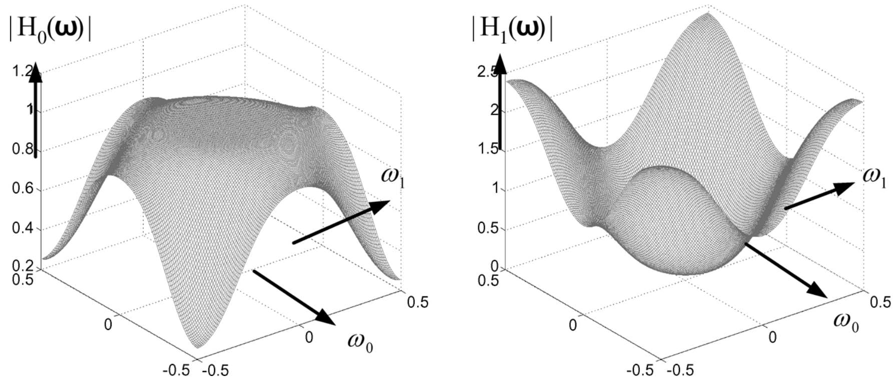

Fig. 5. Frequency plots of analysis filters for two-stage cascade, with each stage as in equation (14).

B. Polyphase Matrix Structure for Quadrantally and Diagonally Symmetric Quincunx Filter Banks

Consider the following set of symmetric 2-D functions:

$$\begin{aligned} E_{0,0}(\mathbf{z}) &= a + e\left(z_1 + z_1^{-1} + z_2 + z_2^{-1}\right) \\ &\quad + f\left(z_1^{-1}z_2 + z_1^{-1}z_2^{-1} + z_1 z_2 + z_1 z_2^{-1}\right) \qquad (14a) \\ E_{0,1}(\mathbf{z}) &= b(1 + z_1 + z_2 + z_1 z_2), \qquad (14b) \\ E_{1,0}(\mathbf{z}) &= c\left(1 + z_1^{-1} + z_2^{-1} + z_1^{-1}z_2^{-1}\right) \qquad (14c) \\ E_{1,1}(\mathbf{z}) &= d. \qquad (14d) \end{aligned}$$

Here, a, b, c, d, e, f are scalar parameters. It can be verified that with $de = 2cb$, and $df = cb$, we have $\det(\mathbf{E}(\mathbf{z})) = ad - 4cb$, i.e., $\mathbf{r} = [0\ 0]^{\mathrm{T}}$. We also require that $\det(\mathbf{E}(\mathbf{z})) = ad - 4cb \neq 0$.

From the above constraints, we can express e and f as: $\mathrm{e} = 2\mathrm{cb}/\mathrm{d}$, and $\mathrm{f} = \mathrm{cb}/\mathrm{d}(= \mathrm{e}/2)$, with $d \neq 0$, and $ad \neq 4cb$.

Thus, the set of functions in (14) satisfy all the constraints of Proposition-1 and (9) as well as the constraints of Proposition-2 and (11), and thus leads to quincunx filter banks with the analysis and synthesis filters having both quadrantal and diagonal symmetry. The analysis filters obtained from (14) are shown in Fig. 4.

Again, as in Section III-A, we can verify that cascades of the above structure also satisfies the constraints of Proposition-1 and (9) as well as the constraints of Proposition-2 and (11). Thus, we can generate filters of higher order using such cascades.

This brief only intends to present polyphase conditions and example cascade structures for synthesizing symmetric quincunx filter banks. However, to illustrate the frequency response of the filters, we used the matlab optimization toolbox to choose the parameters of a two-stage cascade structure with each stage as in (14). For the objective function to be minimized, we used the stopband errors of the analysis filters, with a diamond shaped passband of the low-pass analysis filter. The frequency responses of the analysis filters we obtained are shown in Fig. 5. We would like to note that a detailed formulation of optimization methods for the cascade structures is outside the scope of this brief, and is a subject for future work.

IV. Conclusion

In this brief, we derived conditions on the polyphase components for the filters in quincunx filter bank to have quadrantal or diagonal symmetry. In particular, we showed that: 1) for quadrantally symmetric filters, the polyphase components should be diagonally symmetric; 2) for diagonally symmetric filters, the polyphase components should be quadrantally symmetric. Using the requirement that the determinant of the analysis polyphase matrix be a monomial, we derived further constraints on the polyphase matrix. With this, the synthesis filters are also FIR, and have the same symmetry as the analysis filters. We constructed example of small polyphase matrices satisfying the above derived constraints, and it was also shown that cascades of the constructed matrix can be used to construct filters of higher order. Further issues to be addressed for future work are the issues of minimal structures, and design procedures for optimization of the parameters to obtain good frequency response characteristics.

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