The role of 2-D Symmetries and Periodicities in Signal Extension Schemes for Nonseparable filter-banks

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***Abstract*— The symmetric extension method is used in multirate filter-banks to maintain critical sampling, i.e. total number of sub-band samples is equal to the number of input samples, when processing finite-length signals. This approach is widely used for subband image coding with linear-phase filter-banks. One of the important aspects in the study of symmetric extension schemes is an analysis of the effect of the subsampling (downsampling and upsampling) operations on the signal symmetry and periodicity. The filter-banks most commonly used are separable – so, essentially, a 1-D signal symmetry and periodicity analysis is done for the separable 2-D filter-banks. However, for non-separable sampling and filter-banks, a "true" 2-D analysis of the signal symmetry and periodicity is needed. 2-D signals can possess more variety of symmetries and periodicities than 1-D signals. Also, the non-separable subsampling operations change the "nature" of the periodicity and symmetry in the signal – i.e., for e.g., after downsampling a rectangularly periodic 2-D signal on a non-separable lattice, the downsampled signal does not in general remain rectangularly periodic, but its periodicity is along directions determined by the sampling matrix. For the separable case, the "nature" of the periodicity and symmetry remains the same. In this paper, we study the role played by the 2-D symmetry and periodicity in the development of signal extension schemes for 2-D nonseparable filter banks. We analyze the effect of non-separable subsampling operations on the periodicity and symmetry of a 2-D signal. Using this, we study the symmetric extension method for the case of the two-channel Quincunx filter-banks, and for the four-channel hexagonally sub-sampled filter-banks.**

I. INTRODUCTION

RECENTLY there has been a lot of interest in the use of non-separable systems for image processing and compression. The main motivation for this is the potential of non-separable systems to extract and represent directional information better than separable systems. Non-separable systems involve non-separable sampling and non-separable filters. In particular, the design of critically sampled non-separable filter-banks has attracted a lot of attention [1-5]. One of the main applications of 2-D critically sampled filter-banks is image compression.

The use of filter-banks on finite-extent images presents the problem of data-expansion. This is due to the implicit zero-padding of the input-signal that occurs with the convolution of finite extent sequences. The linear convolution of a finite extent input with a finite-extent filter impulse response (or, in other words, convolution of two finite extent signals), results in an output sequence which has, in general, more number of samples (i.e. a bigger region of support – ROS) than the input. This is highly undesirable in coding applications. A similar problem arises in the case of 1-D filter-banks, and one approach to address the problem in the 1-D case is by using various signal extension methods [6-8] to eliminate data-expansion. Note that when separable 2-D filter-banks are used for image coding, the signal extension is handled using the 1-D extension techniques, since separable 2-D processing operates on one dimension at a time. However, for the case of non-separable filter-banks, a "true" 2-D analysis of the signal symmetry and periodicity is needed. 2-D signals can possess more variety of symmetries and periodicities than 1-D signals. Also, the non-separable subsampling operations change the "nature" of the periodicity and symmetry in the signal – i.e., for e.g., after downsampling a rectangularly periodic 2-D signal on a non-separable lattice, the downsampled signal does not in general remain rectangularly periodic, but its periodicity is along directions determined by the sampling matrix. For the separable case, the "nature" of the periodicity and symmetry remains the same. In this paper, we study the role played by the 2-D symmetry and periodicity in the development of signal extension schemes for 2-D nonseparable filter banks.

The paper is organized as follows: In section II we review the characterization of 2-D signal symmetry and periodicity. In section III we analyze the effect of the nonseparable sub-sampling operations on the 2-D signal symmetry and periodicity. We apply the results of section III to analyze signal extension methods for the two-channel Quincunx filter-banks in section IV, and for the 4-channel hexagonally subsampled filter-banks in Section V. We conclude and point to some directions for further research in section VI.

A. *Notation and Background*

Notation: Boldfaced lower-case letters are used to represent vectors, and bold-faced upper case letters are used

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for matrices. $\det(\mathbf{A})$ denotes the determinant of the matrix $\mathbf{A}$.

Downsampling and upsampling operations on 2-D signals are defined using lattices [1-2]. The lattice generated by a sampling matrix $\mathbf{D}$ is the set of integer vectors $\mathbf{m}$ such that $\mathbf{m} = \mathbf{D}\mathbf{n}$, for some integer vector $\mathbf{n}$. We denote the lattice by $\mathrm{LAT}(\mathbf{D})$. The generating matrix $\mathbf{D}$ should be a non-singular integer matrix. The generating matrix for a given lattice is not unique. With this, downsampling a 2-D signal using a sampling matrix $\mathbf{D}$ is defined as $y[\mathbf{n}] = x[\mathbf{D}\mathbf{n}]$ and upsampling is defined as

$$\begin{aligned} y[\mathbf{n}] &= x[\mathbf{D}^{-1}\mathbf{n}], && \text{if } \mathbf{n} \in \mathrm{LAT}(\mathbf{D}) \\ y[\mathbf{n}] &= 0 && \text{otherwise} \end{aligned}$$

As an example, the quincunx lattice, generated by the matrix $\mathbf{D}_Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, is shown in Figure-1. And Figure-2 shows a 2-D maximally decimated 2-channel filter-bank employing Quincunx subsampling.

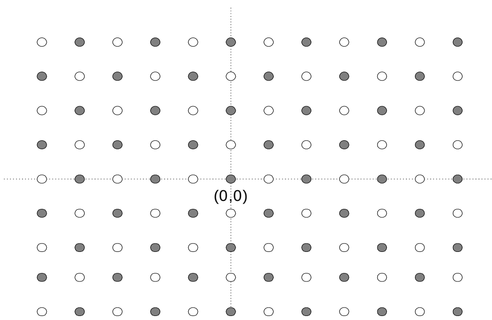

Figure 1 Quincunx sampling lattice

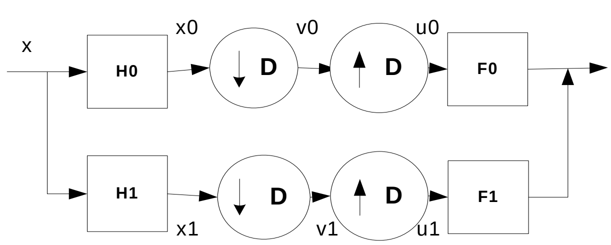

Figure 2 Quincunx filter bank

II. CHARACTERIZATION OF 2-D SIGNAL SYMMETRIES AND PERIODICITIES

A. *Characterization of 2-D symmetries*

We review the characterization of 2-D symmetries from [9] and [11]. A 2-D signal is said to be symmetric if $x[\mathbf{T}\mathbf{n} + \mathbf{b}] = x[\mathbf{n}]$ (identity-symmetry) or $x[\mathbf{T}\mathbf{n} + \mathbf{b}] = -x[\mathbf{n}]$ (anti-symmetry). Here $\mathbf{b}$ is a 2×1 vector, and $\mathbf{T}$ is a non-singular matrix. The most commonly used $\mathbf{T}$ matrices (and the ones we use in this paper) are :

$$\mathbf{T_1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \mathbf{T_2} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{T_3} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\mathbf{T_4} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \ \mathbf{T_5} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \mathbf{T_6} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that $\mathbf{T_1}\mathbf{T_2} = \mathbf{T_3}\mathbf{T_4} = \mathbf{T_5^2} = \mathbf{T_6^2} = -\mathbf{I}$, and $\mathbf{T_6} = \mathbf{T_5^3}$.

$\mathbf{T}$ is said to be an equiaffine transformation if $|\det(\mathbf{A})| = 1$. In this case, the corresponding regions in the transformed and original domains have the same area. $\mathbf{T}$ is said to be a congruent transformation if the Euclidean distance between any two points in the original region is equal to that between the corresponding (image) points in the transformed region. This will be so if and only if $\mathbf{T}$ is orthogonal. We note that all the matrices $\mathbf{T_1}\ldots\mathbf{T_6}$ given above are equiaffine and congruent. We say that a symmetry is k-fold if there are k "identical regions". In this paper we only consider 4-fold symmetries, i.e. quadrantal symmetry, diagonal symmetry, and 90^0 rotational symmetry. Below, we define these 4-fold symmetries, and discuss the various possibilities in each of these symmetry-types. For convenience of notation, we define constant matrices $\mathbf{A_i}$, i=1…6, as $\mathbf{A_i} = \frac{1}{2}(\mathbf{I} - \mathbf{T_i})$, for i=1,2, and $\mathbf{A_i} = \mathbf{I} - \mathbf{T_i}$, for i=3,4,5,6 , where $\mathbf{I}$ is the Identity matrix.

1) *Quadrantal Symmetry*

Quadrantal symmetry, with center of symmetry $\mathbf{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T$, can be defined as follows:

$$x[\mathbf{n}] = x[\mathbf{T_1}\mathbf{n} + 2\mathbf{A_1}\mathbf{c}] = x[\mathbf{T_2}\mathbf{n} + 2\mathbf{A_2}\mathbf{c}] = x[\mathbf{T_1}\mathbf{T_2}\mathbf{n} + 2\mathbf{c}]$$

There are four different types for the location of the center of symmetry, with c_1 and c_2 each independently taking the value of a "full integer (F)" i.e. Z (where Z denotes the set of integers) or a "half integer (H)" i.e. $\frac{1}{2}Z_{odd}$ (where Z_{odd} denotes the set of odd integers). We abbreviate these four cases as FF (c_1 and c_2 are both F), FH (c_1 is F, c_2 is H), HF (c_1 is H, c_2 is F), and HH (c_1, c_2 are both H). Figure-3 shows these four cases of quadrantal symmetry.

The above symmetries use identity-symmetry. When considering anti-symmetry, we can have anti-symmetry independently for the $\mathbf{T_1}$ and $\mathbf{T_2}$ operations i.e. we can have

$$x[\mathbf{n}] = \gamma_1 x[\mathbf{T_1}\mathbf{n} + 2\mathbf{A_1}\mathbf{c}] = \gamma_2 x[\mathbf{T_2}\mathbf{n} + 2\mathbf{A_2}\mathbf{c}] = \gamma_1\gamma_2 x[\mathbf{T_1}\mathbf{T_2}\mathbf{n} + 2\mathbf{c}] \tag{1a}$$

In z-domain,

$$X(\mathbf{z}) = \gamma_1 \mathbf{z}^{-2\mathbf{A_1}\mathbf{c}} X(\mathbf{z}^{\mathbf{T_1}}) = \gamma_2 z^{-2\mathbf{A_2}\mathbf{c}} X(\mathbf{z}^{\mathbf{T_2}}) = \gamma_1\gamma_2 \mathbf{z}^{-2\mathbf{c}} X(\mathbf{z}^{\mathbf{T_1}\mathbf{T_2}}) \tag{1b}$$

For notation, we have used the same subscript for γ as its associated $\mathbf{T}$-operation. γ_1 and γ_2 can each independently be +1 (symmetry) or –1 (anti-symmetry), thus giving four possibilities : SS, SA, AS, and AA. So, there are a total of

16 types of quadrantal symmetry

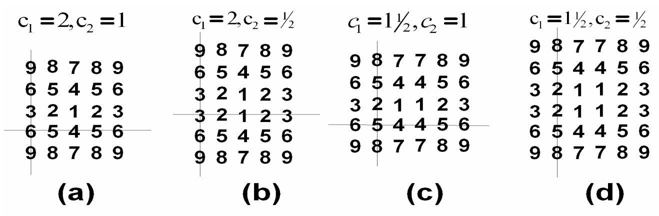

Figure 3 Examples of quadrantally symmetric signals with types of center of symmetry (a) FF, (b) FH, (c) HF, (d) HH

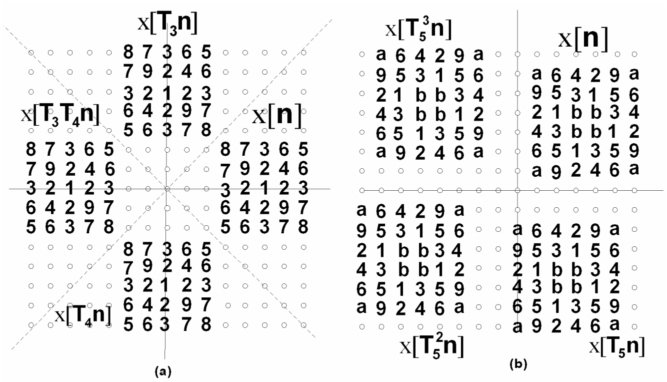

Figure 4 Illustration of characterization of (a) Diagonal symmetry, (b) 90˚ rotational symmetry

2) Diagonal Symmetry

Diagonal symmetry, with center of symmetry $\mathbf{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T$, can be characterized as follows :

$$x[\mathbf{n}] = x[\mathbf{T_3 n} + \mathbf{A_3 c}] = x[\mathbf{T_4 n} + \mathbf{A_4 c}] = x[\mathbf{T_3 T_4 n} + \mathbf{2c}]$$

Figure 4-a shows $x[\mathbf{n}]$ (which is diagonally symmetric), and $x[\mathbf{T_3 n}]$, $x[\mathbf{T_4 n}]$, and $x[\mathbf{T_3 T_4 n}]$, to illustrate that the above characterization holds. In the case of diagonal symmetry, due to the nature of the symmetry, the center of symmetry can only be of type FF or HH. The center of symmetry in the signal in Figure 4-a is of type FF. And similar to the case of quadrantal symmetry, a symmetry or antisymmetry can be associated independently with $\mathbf{T_3}$ and $\mathbf{T_4}$ operations. This can be written as:

$$x[\mathbf{n}] = \gamma_3 x[\mathbf{T_3 n} + \mathbf{A_3 c}] = \gamma_4 x[\mathbf{T_4 n} + \mathbf{A_4 c}] = \gamma_3 \gamma_4 x[\mathbf{T_3 T_4 n} + \mathbf{2c}] \quad (2a)$$

In z-domain:

$$X(\mathbf{z}) = \gamma_3 \mathbf{z}^{-\mathbf{A_3 c}} X(\mathbf{z}^{\mathbf{T_3}}) = \gamma_4 z^{-\mathbf{A_3 c}} X(\mathbf{z}^{\mathbf{T_4}}) = \gamma_3 \gamma_4 \mathbf{z}^{-\mathbf{2c}} X(\mathbf{z}^{\mathbf{T_3 T_4}}) \quad (2b)$$

γ_3 and γ_4 can each independently be +1 or –1, thus giving four possibilities of symmetry/antisymmetry (just like the quadrantal case). Combining this with the two types of the center of symmetry, there are 8 types of diagonal symmetries.

3) 90˚ rotational Symmetry

90˚ rotational symmetry, with center of symmetry $\mathbf{c} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T$, can be characterized as follows :

$$x[\mathbf{n}] = x[\mathbf{T_5 n} + \mathbf{A_5 c}] = x[\mathbf{T_5^2 n} + \mathbf{2c}] = x[\mathbf{T_5^3 n} + \mathbf{A_6 c}]$$

Figure 4-b shows $x[\mathbf{n}]$ (which is 90˚ rotationally symmetric), and $x[\mathbf{T_5 n}]$, $x[\mathbf{T_5^2 n}]$, and $x[\mathbf{T_5^3 n}]$, to illustrate that the above characterization holds. In the case of 90˚ rotational symmetry, the center of symmetry can only be of type FF or HH. The center of symmetry in the signal in Figure 4-b is of type HH. And we can have an identity-symmetry or antisymmetry associated with the $\mathbf{T_5}$-operation. This can be written as:

$$x[\mathbf{n}] = \gamma_5 x[\mathbf{T_5 n} + \mathbf{A_5 c}] = x[\mathbf{T_5^2 n} + \mathbf{2c}] = \gamma_5 x[\mathbf{T_5^3 n} + \mathbf{A_6 c}] \quad (3a)$$

Or in z-domain:

$$X(\mathbf{z}) = \gamma_5 \mathbf{z}^{-\mathbf{A_5 c}} X(\mathbf{z}^{\mathbf{T_5}}) = \mathbf{z}^{-\mathbf{2c}} X(\mathbf{z}^{\mathbf{T}_5^2}) = \gamma_5 z^{-\mathbf{A_6 c}} X(\mathbf{z}^{\mathbf{T}_5^3}) \quad (3b)$$

γ_5 can be +1 or –1 thus giving two possibilities of symmetry. Combining this with the two types of the center of symmetry, there are 8 different types of 90˚ rotational symmetries.

4) Centro rotational Symmetry

Centro-symmetry with center of symmetry $\mathbf{c}$ can be characterized as :

$$x[\mathbf{n}] = \gamma x[\mathbf{-In} + \mathbf{2c}]$$

In z-transforms: $X(\mathbf{z}) = \gamma \mathbf{z}^{-\mathbf{2c}} X(\mathbf{z}^{\mathbf{-I}})$

In the 2-D case, centro symmetry in the 2-D impulse response of a filter implies linear-phase filter. Also, the filter is “zero-phase” when the filter impulse response is centro-symmetric with the center of symmetry at the origin i.e. $\mathbf{c} = \mathbf{0}$

B. Characterization of 2-D periodicity

2-D signals can have periodicities along different directions. A general 2-D periodic signal can be described as [10]:

$$x[\mathbf{n} + \mathbf{p}_0] = x[\mathbf{n}], \text{ and } x[\mathbf{n} + \mathbf{p}_1] = x[\mathbf{n}]$$

where $\mathbf{p}_0$ and $\mathbf{p}_1$ are constant integer vectors $\mathbf{p}_0 = \begin{bmatrix} p_{00} \\ p_{10} \end{bmatrix}$ and $\mathbf{p}_1 = \begin{bmatrix} p_{01} \\ p_{11} \end{bmatrix}$ with $(p_{00} p_{11} - p_{01} p_{10}) \neq 0$.

The vectors $\mathbf{p}_0$ and $\mathbf{p}_1$ represent the displacement from any sample to the corresponding sample of two other periods. One period of the periodic signal is contained in the parallelogram-shaped region whose two adjacent sides are formed by the vectors $\mathbf{p}_0$ and $\mathbf{p}_1$. These vectors are called periodicity-vectors, and they can be arranged to form columns of a 2x2 nonsingular integer matrix $\mathbf{P}$ called the periodicity matrix. The number of samples in one period is equal to $|\det(\mathbf{P})|$. In the special case where $\mathbf{P}$ is diagonal, the signal is said to be rectangularly-periodic.

III. Effect of Subsampling and Filtering Operations on Signal Periodicity and Symmetry

The question that we now address is as follows: In a non-separable filter-bank, assuming the input signal has certain symmetry and periodicity, what symmetries or periodicities do the subband signals exhibit. This requires us to analyze the effect of the nonseparable filtering and downsampling operation on the signal periodicity and symmetry.

A. Effect of nonseparable filtering and subsampling on signal periodicity

Consider one branch of a non-separable filter-bank (like Fig-1) with sampling matrix $\mathbf{D}_1$. Let the input signal $x[\mathbf{n}]$ be periodic with periodicity matrix $\mathbf{P}$, i.e. $x[\mathbf{n}+\mathbf{Pr}] = x[\mathbf{n}]$, where $\mathbf{r}$ is any integer vector (see section II). With the periodic signal $x[\mathbf{n}]$ as input to the analysis filter H (a shift-invariant system), the output is also periodic with the same periodicity matrix as $x[\mathbf{n}]$ i.e. $x_0[\mathbf{n}+\mathbf{Pr}] = x_0[\mathbf{n}]$. Consider the sub-band signal v_0, which is obtained by downsampling x_0 using the sampling matrix $\mathbf{D}_1$ i.e. $v_0[\mathbf{n}] = x_0[\mathbf{D_1 n}]$.

But $x_0[\mathbf{n}+\mathbf{Pr}] = x_0[\mathbf{n}]$

$$\Rightarrow x_0[\mathbf{D_1 n}] = x_0[\mathbf{D_1 n}+\mathbf{Pr}] = x_0[\mathbf{D}_1(\mathbf{n}+\mathbf{D}_1^{-1}\mathbf{Pr})]$$
$$= v_0[\mathbf{n}+\mathbf{D}_1^{-1}\mathbf{Pr}]$$

Thus, $v_0[\mathbf{n}] = v_0[\mathbf{n}+\mathbf{D}_1^{-1}\mathbf{Pr}]$, i.e. v_0 is periodic with periodicity matrix $\mathbf{D}_1^{-1}\mathbf{P}$. Note that, in general, this is not rectangularly periodic. Also, since we are dealing with discrete-time signals, we need $\mathbf{D}_1^{-1}\mathbf{P}$ to be an integer matrix. In the context of a subband image coding scheme, this usually implies a "proper" choice of the input image size. For e.g., for Quincunx filter-banks, if we assume that one period of the input signal is supported on a "square region" $N \times N$, with even N, then $\mathbf{P} = N\mathbf{I}$, and $\mathbf{D}_Q^{-1}\mathbf{P}$ is an integer matrix for even N.

Since it is common to have a cascaded tree structure of filter-banks, the signal v_0 will be further downsampled (in general, the second stage of the filter-bank can be a $|\mathbf{D}_2|$ – band filter-bank, using a sampling matrix $\mathbf{D}_2$ - which need not be the same as in Stage-1). Then, following an analysis similar to above, the sub-band signal of the second stage with sampling matrix $\mathbf{D}_2$ is periodic with periodicity matrix $\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{P}$ i.e. $v[\mathbf{n}] = v[\mathbf{n}+\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{Pr}]$. Again, we need $\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{P}$ to be an integer matrix. We summarize this as follows:

Proposition 1: Consider a tree-structured filter-bank, with $\mathbf{D}_m$ denoting the sampling matrix of stage-m. Let the input to the (first-stage of) tree-structured filter-bank be a periodic signal with a periodicity matrix $\mathbf{P}$ i.e. $x[\mathbf{n}+\mathbf{Pr}] = x[\mathbf{n}]$. Then,

a) The sub-band signals of the first stage (m=1) are periodic with periodicity matrix $\mathbf{D}_1^{-1}\mathbf{P}$ i.e. $v[\mathbf{n}] = v[\mathbf{n}+\mathbf{D}_1^{-1}\mathbf{Pr}]$. Here, $\mathbf{D}_1^{-1}\mathbf{P}$ should be an integer matrix

b) The sub-band signals of the second stage (m=2) are periodic with periodicity matrix $\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{P}$ i.e. $v[\mathbf{n}] = v[\mathbf{n}+\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{Pr}]$. Here, $\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{P}$ should be an integer matrix

c) In general, the sub-band signals of the m'th stage are periodic with periodicity matrix $\mathbf{D}_m^{-1}\ldots\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{P}$ i.e. $v[\mathbf{n}] = v[\mathbf{n}+\mathbf{D}_m^{-1}\ldots\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{Pr}]$. Here, $\mathbf{D}_m^{-1}\ldots\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{P}$ should be an integer matrix

Note: It follows that if the same sampling matrix $\mathbf{D}$ is used is used for all the stages, then sub-band signals of the m'th stage are periodic with periodicity matrix $\mathbf{D}^{-m}\mathbf{P}$ i.e. $v[\mathbf{n}] = v[\mathbf{n}+\mathbf{D}^{-m}\mathbf{Pr}]$. Again, we need $\mathbf{D}^{-m}\mathbf{P}$ to be an integer matrix.

B. Effect of nonseparable filtering and subsampling on signal symmetry

We now consider signal symmetries. Again, consider one branch of a non-separable filter-bank with sampling matrix $\mathbf{D}_1$. Let the input signal $x[\mathbf{n}]$ be symmetric as follows: $x[\mathbf{n}] = x[\mathbf{S_1 n}] = x[\mathbf{S_2 n}] = x[\mathbf{S_3 n}]$ (for e.g., this could be one of quadrantal, or diagonal, or 4-fold rotational symmetries, as described in section II). If the filter impulse-response has the same symmetry as the input, then the output of the filter will also have the same symmetry.

Consider the sub-band signal v_0, which is obtained by downsampling x_0 using the sampling matrix $\mathbf{D}_1$

$$v_0[\mathbf{n}] = x_0[\mathbf{D_1 n}]$$

$\Rightarrow v_0[\mathbf{D}_1^{-1}\mathbf{S}_1\mathbf{D}_1\mathbf{n}] = x_0[\mathbf{D_1 n}] = v_0[\mathbf{n}]$ (due to symmetry in $x[\mathbf{n}]$)

Similarly, we can show that $v_0[\mathbf{D}_1^{-1}\mathbf{S}_2\mathbf{D}_1\mathbf{n}] = v_0[\mathbf{n}]$, and $v_0[\mathbf{D}_1^{-1}\mathbf{S}_3\mathbf{D}_1\mathbf{n}] = v_0[\mathbf{n}]$

Thus v_0 has the following symmetry: $v_0[\mathbf{n}] = v_0[\mathbf{D}_1^{-1}\mathbf{S}_1\mathbf{D}_1\mathbf{n}] = v_0[\mathbf{D}_1^{-1}\mathbf{S}_2\mathbf{D}_1\mathbf{n}] = v_0[\mathbf{D}_1^{-1}\mathbf{S}_3\mathbf{D}_1\mathbf{n}]$. Here, for the symmetry to be meaningful, we need that the matrices $\mathbf{D}_1^{-1}\mathbf{S}_1\mathbf{D}_1$, $\mathbf{D}_1^{-1}\mathbf{S}_2\mathbf{D}_1$, and $\mathbf{D}_1^{-1}\mathbf{S}_3\mathbf{D}_1$ are integer, orthogonal, and have determinant = ± 1. This makes the transformations equiaffine and congruent.

We can repeat this analysis for the next stages in the treestructures filter-bank. We summarize this as :

Proposition 2: Consider a tree-structured filter-bank, with $\mathbf{D}_m$ denoting the sampling matrix of stage-m. Let the input to the (first-stage of) tree-structured filter-bank be a symmetric signal, as $x[\mathbf{n}] = x[\mathbf{S_1 n}] = x[\mathbf{S_2 n}] = x[\mathbf{S_3 n}]$. Then,

a) The symmetries in the sub-band signals of stage-1 (m=1) are as follows:

$$v[\mathbf{n}] = v[\mathbf{D}_1^{-1}\mathbf{S}_1\mathbf{D}_1\mathbf{n}] = v[\mathbf{D}_1^{-1}\mathbf{S}_2\mathbf{D}_1\mathbf{n}] = v[\mathbf{D}_1^{-1}\mathbf{S}_3\mathbf{D}_1\mathbf{n}]$$

b) The symmetries in the sub-band signals of stage-2 (m=2) are as follows:

$$v[\mathbf{n}] = v[\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{S}_1\mathbf{D}_1\mathbf{D}_2\mathbf{n}] = v[\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{S}_2\mathbf{D}_1\mathbf{D}_2\mathbf{n}]$$
$$= v[\mathbf{D}_2^{-1}\mathbf{D}_1^{-1}\mathbf{S}_3\mathbf{D}_1\mathbf{D}_2\mathbf{n}]$$

c) In general, the symmetries in the sub-band signals of stage-m are as follows:

$$v[\mathbf{n}] = v[\mathbf{D}_m^{-1}\ldots\mathbf{D}_1^{-1}\mathbf{S}_1\mathbf{D}_1\ldots\mathbf{D}_m n]$$
$$= v[\mathbf{D}_m^{-1}\ldots\mathbf{D}_1^{-1}\mathbf{S}_2\mathbf{D}_1\ldots\mathbf{D}_m\mathbf{n}]$$
$$= v[\mathbf{D}_m^{-1}\ldots\mathbf{D}_1^{-1}\mathbf{S}_3\mathbf{D}_1\ldots\mathbf{D}_m\mathbf{n}]$$

In all cases, we need the matrices describing the transformations to be integer, orthogonal, and having determinant = ± 1 (thus making the transformations equiaffine and congruent).

Note: It follows that if the same sampling matrix D is used for all stages, then the symmetries in the sub-band signals of stage-m are as follows:

$$v[\mathbf{n}] = v[\mathbf{D}^{-m}\mathbf{S}_1\mathbf{D}^m\mathbf{n}] = v[\mathbf{D}^{-m}\mathbf{S}_2\mathbf{D}^m\mathbf{n}]$$
$$= v[\mathbf{D}^{-m}\mathbf{S}_3\mathbf{D}^m\mathbf{n}]$$

Again, we need the matrices describing the transformations to be integer, orthogonal, and having determinant = ± 1.

IV. Analysis of Symmetric Signal Extension Method for Quincunx Filter Banks

We now use the results from section III to analyze the symmetric extension method for Quincunx filter-banks shown in Fig-1.

Assume that the input 2-D signal has square Region Of Support (ROS) of $(0 \cdots N-1, 0 \cdots N-1)$, where assume that N is even. Symmetric extension of the input gives us a quadrantally symmetric and rectangularly periodic signal. Consider a rectagularly-periodic input signal with periodicity matrix $M\mathbf{I}$, where $M = 2N - 2$ (obtained by mirroring the quadrantal-symmetrically extended signal).

We now analyze how rectangular-periodicity and quadrantal-symmetry is affected by the Quincunx down-sampling operation. From Proposition-1, with a rectangularly-symmetric signal with periodicity matrix $M\mathbf{I}$ as input, the sub-band signal v_0 of Stage-1 is periodic with periodicity matrix $M\mathbf{D}_Q^{-1}$, i.e. $v_0[\mathbf{n}] = v_0[\mathbf{n} + M\mathbf{D}_Q^{-1}\mathbf{r}]$ (Note that for the Quincunx sampling matrix $\mathbf{D}_Q$, $\mathbf{D}_Q^{-1} = \frac{1}{2}\mathbf{D}_Q^T$. Thus $M\mathbf{D}_Q^{-1} = \frac{M}{2}\mathbf{D}_Q^T$). Since, for the Quincunx sampling matrix, $\mathbf{D}_Q^2 = 2\mathbf{I}$, the subband signal of stage-2 is periodic with periodicity matrix $\frac{M}{2}\mathbf{I}$ (i.e. it is rectangularly periodic).

We now analyze the symmetries of the sub-band signals, when the input signal is quadrantally symmetric. From proposition-2, and since, for the Quincunx sampling matrix $\mathbf{D}_Q$ we have $\mathbf{D}_Q^{-1}\mathbf{T}_1\mathbf{D}_Q = \mathbf{T}_3$, $\mathbf{D}_Q^{-1}\mathbf{T}_2\mathbf{D}_Q = \mathbf{T}_4$, and $\mathbf{D}_Q^{-1}\mathbf{T}_1\mathbf{T}_2\mathbf{D}_Q = -\mathbf{I} = \mathbf{T}_3\mathbf{T}_4$, it follows that the sub-band signal of stage-1 has the following symmetry: $v_0[\mathbf{n}] = v_0[\mathbf{T}_3\mathbf{n}] = v_0[\mathbf{T}_4\mathbf{n}] = v_0[\mathbf{T}_3\mathbf{T}_4\mathbf{n}]$. This is diagonal symmetry. And, (again using $\mathbf{D}_Q^2 = 2\mathbf{I}$) it follows that the sub-band signal of stage-2 is quadrantal-symmetric.

So, for Quincunx filter-banks, the sub-band signals of odd-stages have diagonal symmetries, and the sub-band signals of even-stages have quadrantal symmetries.

A. Symmetry requirements on the filters

For the symmetric extension method, the filters in the Quincunx filter-bank need to have the same symmetries as the input signal i.e. the filters need to be quadrantally-symmetric for Stage-1 (all odd numbered stages), and diagonally-symmetric for Stage-2 (all even numbered stages) of the tree-structured filter-bank, so that the filtering operation maintains the symmetries. The method of transformations of variables as proposed in [12] can be used to design filters with the required symmetries. In fact, the transformation functions $M(\mathbf{z})$ used in [12] exhibit quadrantal as well as diagonal symmetry. Thus, the filters designed have quadrantal as well as diagonal symmetries. Thus these filter-banks can be used for all the stages (even-numbered as well as odd-numbered stages) with the symmetric signal extension method for the Quincunx filter-bank.

V. Analysis of Symmetric Signal Extension Method for 4-Channel Filter-Banks with Hexagonal Subsampling

We now analyze the case of non-separable 4-channel filter-banks with hexagonal subsampling. Assume that the input signal has square ROS of $(0 \cdots N-1, 0 \cdots N-1)$, where assume that N is a multiple of 4. Symmetric extension of the input gives us a quadrantally symmetric and rectangularly periodic signal. Consider a rectagularly-periodic input signal with periodicity matrix $M\mathbf{I}$, where $M = 2N$ (Note that we do not use “mirroring” in this case, since we need M to be a multiple of 4). For the filter-bank, we let the first-stage have the sampling matrix $\mathbf{D_{H1}} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$, and the second-stage have the sampling matrix $\mathbf{D_{H2}} = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}$. Here, since $\mathbf{D_{H1}} \mathbf{D_{H2}} = \mathbf{D_{H2}} \mathbf{D_{H1}} = 4\mathbf{I}$, the sub-band signal at the output of the second-stage is again rectangularly periodic as with periodicity matrix $\frac{M}{4}\mathbf{I}$.

We now see what happens to the symmetry (the input is quadrantally symmetric). Consider the sub-band signals of the first stage. Observing that

$$\mathbf{D_1^{-1} T_1 D_1} = \mathbf{T_3}, \quad \text{and } \mathbf{D_1^{-1} T_2 D_1} = \mathbf{T_4}, \text{ and}$$

$$\mathbf{D_1^{-1} T_1 T_2 D_1} = -\mathbf{I} = \mathbf{T_3 T_4}$$

we see that the sub-band signals of the first stage have diagonal symmetry. And, for the second stage, again observing that

$$\mathbf{D_2^{-1} T_3 D_2} = \mathbf{T_1}, \quad \text{and } \mathbf{D_2^{-1} T_4 D_2} = \mathbf{T_2}, \quad \text{and}$$

$$\mathbf{D_2^{-1} T_3 T_4 D_2} = -\mathbf{I} = \mathbf{T_1 T_2}$$

the sub-band signals of the second stage have quadrantal symmetry.

A. *Symmetry requirements on the filters*

For the above analysis, we also need the filters to have the same symmetries as the input signal. In the case of the above example, we require that the filters for stage-1 with sampling matrix $\mathbf{D_{H1}}$ should have quadrantal symmetry, and the filters for stage-2 with sampling matrix $\mathbf{D_{H2}}$ should have diagonal symmetry. A 4-channel filter-bank with filters having symmetries as above can be done using the approach described in [13].

VI. CONCLUSION AND FUTURE WORK

We analyzed the effect of the non-separable subsampling operation on the periodicity and symmetry of a 2-D signal. We observed that the nature of signal symmetry and periodicity changes after non-separable subsampling operations – something that does not occur in the 1-D case (or in 2-D separable subsampling). We analyzed the symmetric extension method for the case of Quincunx filter-banks, and for hexagonally subsampled 4-channel filter-banks.

A subject for further work is to address problem of signal extension methods for a general non-separable filter bank with a given sampling matrix $\mathbf{M}$. Design of symmetric Quincunx filter banks with the filters having various symmetries has been addressed in the literature [11-12]. The design of symmetric $\mathbf{M}$-channel filter banks, where the subsampling matrix $\mathbf{M}$ has a specific form, has been addressed in [13]. However, the design of non-separable filter banks with the filters having various symmetries is not addressed for the case of general $\mathbf{M}$-channel filter banks.

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