Effect of Feedback on Linear Complementarity Systems

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***Abstract*— The wellposedness of a restricted class of Linear Complementarity Systems (LCS) has been recently studied [1] and [2]. One of the necessary conditions for the wellposedness of LCS is that the consistent space and jump space in every mode should should not intersect non-trivially. A given LCS with zero input may not satisfy this condition in every mode. By applying state feedback and port feedback one can change the orientations of consistent space and jump space and thereby possibly making them disjoint. In some applications it may be desirable to have the consistent space as large as possible and jump space as small as possible. This can also be achieved through feedback.**

***Index Terms*— Linear Complementarity Systems, Feedback, State Feedback, Port Feedback, Jump Space, Consistent Space**

I. INTRODUCTION

Linear Complementarity System(LCS) is a class of Hybrid Systems. A *hybrid system* is one that combines the features of continuous and discrete dynamics. The jump states of a hybrid systems are often called modes of the system. LCS commonly occur in (approximations to) circuit analysis and mechanical systems with unilateral constraints. *Complementarity problems* refers to a class of inequality problems. Two scalar variables are said to be *complementary* if they are both subject to an inequality constraint, and if at all times at least one of these constraints is satisfied with equality. Linear complementarity problem has a wide range of applications. In many applications complementarity conditions arise naturally. In many other applications the given set of equations and inequalities can be rewritten as complementarity form.

A. Notations

The symbols $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{C}$ denotes the set of real, natural and complex numbers respectively. $\mathbb{R}^n$ denotes a vector (column) of $n \in \mathbb{N}$ components in $\mathbb{R}$. $\mathbb{R}^{m\times n}$ denotes a matrix of order $m \times n$ in $\mathbb{R}$. The set $\{1, 2, \ldots, n\}$ where $n \in \mathbb{N}$ is denoted by $\bar{n}$. Let $M \in \mathbb{R}^{m\times n}$, $I \subseteq \bar{m}$ and $J \subseteq \bar{n}$ then M_{IJ} denotes the submatrix with components $\{M_{ij}\}, i \in I, j \in J$.

A vector $x \in \mathbb{R}^n$ is said to be non-negative, if $x_i \geq 0$ for all $i \in \bar{n}$ and is denoted by $x \geq 0$. A vector $x \in \mathbb{R}^n$ is said to be positive, if $x_i > 0$ for all $i \in \bar{n}$ and is denoted by $x > 0$. If two vectors $x, y \in \mathbb{R}^n$ are such that $x_i y_i = 0$ for all $i \in \bar{n}$ then it is denoted by $x \perp y$.

A function $u : [0, \infty) \to \mathbb{R}$ is said to be regular if it is infinitely differentiable on $[0, \infty)$. For a linear system an input is said to be admissible if output is regular.

B. Complementarity problems

A *Linear Complementarity Problem* (LCP) is described as follows[1], given $q \in \mathbb{R}^m$ and $M \in \mathbb{R}^{m\times m}$ find $u, y \in \mathbb{R}^m$ such that

$$y = q + Mu,$$
$$u, y \geq 0, \quad u^T y = 0,$$

where the $\geq$ sign is to be interpreted as component wise non-negativity.

A Linear Complementarity System (LCS) can be thought of as a dynamic LCP. A LCS is described as follows,

$$\dot{x}(t) = Jx(t) + Ku(t) + Lw(t), \tag{1a}$$
$$y(t) = Mx(t) + Nu(t), \tag{1b}$$
$$0 \leq u(t) \perp y(t) \geq 0, \tag{1c}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^p$, and matrices J, K, L, M, N are of compatible dimensions. The algebraic constraints (1c) are the LCP constraints, where u and y are the complementary variables. Here $w(t)$ is the input to the above LCS. When this system is viewed as hybrid system, the complementarity conditions (1c) gives rise to 2^m modes. Every index set $I \subseteq \bar{m}$ can be thought of as imposing constraints $y_I = 0$ and $u_{\bar{m}\setminus I} = 0$ and could be viewed as *mode* of operation of the system. With each mode

[1]See [3] for a detailed survey of LCP.

I the linear dynamics are given by

$$\begin{aligned}\dot{x}(t) &= Jx(t) + Ku(t) + Lw(t),\\ y(t) &= Mx(t) + Nu(t),\\ y_I(t) &= 0 \quad u_{\bar{m}\setminus I}(t) = 0,\\ y_{\bar{m}\setminus I}(t) &\geq 0, \quad u_I(t) \geq 0.\end{aligned}$$

Now consider the above LCS operating in mode $I \subseteq \bar{m}$, with input $w(t) = 0$, we have the following equations

$$\dot{x}(t) = Jx(t) + Ku(t), \tag{2a}$$
$$y_I(t) = 0 = M_{I\bullet}x(t) + N_{II}u_I(t), \tag{2b}$$
$$y_{\bar{m}\setminus I}(t) = M_{\bar{m}\setminus I}x(t) + N_{\bar{m}\setminus I}u(t), \tag{2c}$$
$$y_{\bar{m}\setminus I}(t) \geq 0, \quad u_I(t) \geq 0, \quad u_{\bar{m}\setminus I}(t) = 0. \tag{2d}$$

Equation (2b) is like output nulling equation.

The wellposedness of system (1) without input and with input has been studied in [2] and [1] respectively. It has been shown that the wellposedness of (1) depends on the consistent space and jump space of the linear system (2a) and (2b) in each mode $I \subseteq \bar{m}$. Associated with every linear system of type (2a) and (2b) one can define two subspaces of its state space, consistent space and jump space.

Definition 1.1: Consistent space (weakly unobservable space [4]): Set of all initial conditions $x_c(0) \in \mathbb{R}^n$ for which there exists an regular input $u_I(t)$ on $[0, \infty)$ such that the resulting output $y_I(t) = 0 \ \forall t \geq 0$. This set will be denoted by[2] $\mathscr{V}$ and it has been shown to be a vector space over $\mathbb{R}$ in [4].

Definition 1.2: Jump space (strongly reachable space [4]): Set of all states $x_i \in \mathbb{R}^n$ which are reachable from origin (i.e., $x(0+) = x_i$) by giving admissible impulsive input such that $y_I(t) = 0 \ \forall t \geq 0$. This set will be denoted[3] by $\mathscr{W}$ and it has been shown to be a vector space over $\mathbb{R}$ in [4].

One of the necessary conditions for the wellposedness of system (1) is that the consistent space and jump space should not intersect non-trivially in all modes. In [2] it is assumed that the linear system defined by (2a) and (2b) satisfies this condition.

A given system may not satisfy this condition. Now by applying state feedback and/or port feedback ($w(t) = Fx(t)$ is called state feedback and $w(t) = Gu(t)$ is port feedback) one may be able to satisfy this condition. In this paper we investigate how the consistent space and jump space can be oriented/changed by the application of state and port feedback.

II. STATE FEEDBACK

Consider an LCS

$$\begin{aligned}\dot{x}(t) &= Jx(t) + Ku^1(t) + Lw(t),\\ y^1(t) &= Mx(t) + Nu^1(t),\\ &0 \leq u^1(t) \perp y^1(t) \geq 0,\end{aligned} \tag{3}$$

where $x(t) \in \mathbb{R}^n$, $u^1(t) \in \mathbb{R}^m$, $y^1(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^p$ and matrices of appropriate dimensions. Here $w(t)$ is the input to the system. Suppose we apply state feedback i.e., $w(t) = Fx(t)$ to the above system we get the following

$$\begin{aligned}\dot{x}(t) &= (J + LF)x(t) + Ku^1(t),\\ y^1(t) &= Mx(t) + Nu^1(t),\\ &0 \leq u^1(t) \perp y^1(t) \geq 0.\end{aligned} \tag{4}$$

In a given mode $I \subseteq \bar{m}$ the dynamics of the system are as follows

$$\dot{x}(t) = (J + LF)x(t) + K_{\bullet I}u_I^1(t), \tag{5a}$$
$$y_I^1 = 0 = M_{I\bullet}x(t) + N_{II}u_I^1(t), \tag{5b}$$
$$y^1_{\bar{m}\setminus I}(t) = M_{\bar{m}\setminus I}x(t) + N_{\bar{m}\setminus I}u^1(t), \tag{5c}$$
$$y^1_{\bar{m}\setminus I}(t) \geq 0, \quad u^1_I(t) \geq 0, \quad u^1_{\bar{m}\setminus I}(t) = 0. \tag{5d}$$

The complementarity conditions (5d) are dealt in detail in [2]. Now consider (5a) and (5b), which forms the linear behaviour of the system in a given mode $I \subseteq \bar{m}$. Let $A = J$, $E = L$, $B = K_{\bullet I}$, $C = M_{I\bullet}$ and $D = N_{II}$. Also, let $y = y_I^1$ and $u = u_I^1$. We get the following linear equations

$$\begin{aligned}\dot{x}(t) &= (A + EF)x(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t).\end{aligned} \tag{6}$$

We now investigate the relation between consistent space and jump space of the above linear system with the system without feedback (with zero input $w = 0$ or $F = 0$). That is,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t).\end{aligned} \tag{7}$$

Algorithms for computing consistent space $\mathscr{V}$ and jump space $\mathscr{W}$ of the system (7) (without feedback) are given below [4]. $\mathscr{V}$ is given by the limit of sequence of subspaces as follows:

$$\begin{aligned}\mathscr{V}_0 &= \mathbb{R}^n\\ \mathscr{V}_{i+1} &= \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that}\\ &\qquad Ax + Bu \in \mathscr{V}_i,\ Cx + Du = 0\},\end{aligned} \tag{8}$$

and $\mathscr{W}$ is determined by the limit of the following sequence of subspaces:

$$\begin{aligned}\mathscr{W}_0 &= \{0\}\\ \mathscr{W}_{i+1} &= \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ and } \exists \bar{x} \in \mathscr{W}_i \text{ such that}\\ &\qquad x = A\bar{x} + Bu,\ C\bar{x} + Du = 0\}.\end{aligned} \tag{9}$$

Algorithms for computing consistent space $\mathscr{V}^F$ and jump space $\mathscr{W}^F$ of the system (6) (with state feedback) are given below. $\mathscr{V}^F$ is given by the limit of the following sequences of subspaces:

$$\begin{aligned}\mathscr{V}_0^F &= \mathbb{R}^n\\ \mathscr{V}_{i+1}^F &= \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that}\\ &\qquad (A + EF)x + Bu \in \mathscr{V}_i^F,\ Cx + Du = 0\},\end{aligned} \tag{10}$$

[2]this notation is used in [4]. It is denoted by V in [2].

[3]this notation is used in [4]. It is denoted by T in [2].

and $\mathscr{W}^F$ is given by the limit of the following sequences of subspaces:

$$\begin{aligned}\mathscr{W}_0^F &= \{0\}\\ \mathscr{W}_{i+1}^F &= \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ and } \exists \bar{x} \in \mathscr{W}_i^F \text{ such that}\\ &\quad x = (A + EF)\bar{x} + Bu,\ C\bar{x} + Du = 0\}.\end{aligned} \tag{11}$$

Definition 2.1: Let $\mathscr{P}_X$ be the projection of $\ker([C\ D])$ on the state space $\mathbb{R}^n$. $\mathscr{P}_X$ is the set of all states such that there exists an u which makes y (output) zero.

The following lemmas will be useful in deriving many results in this paper.

Lemma 2.1: If for a given $x \in \mathbb{R}^n$ we have $Cx+Du^1 = 0$ and $Cx + Du^2 = 0$ for some $u^1, u^2 \in \mathbb{R}^m$ then $u^2 - u^1 \in \ker D$.

The following lemmas are immediate from the algorithms above.

Lemma 2.2: Given systems (7) and (6), the following is true $\mathscr{V}_1 = \mathscr{V}_1^F = \mathscr{P}_X$ and for all F.

Lemma 2.3: Given system (6) the following is true $\mathscr{V}_{k+1}^F \subseteq \mathscr{V}_k^F$ for all $k \in \mathbb{N}$ and for all F.

Lemma 2.4: Given system (6) the following is true $\mathscr{W}_k^F \subseteq \mathscr{W}_{k+1}^F$ for all $k \in \mathbb{N}$ and for all F.

A. Effect on consistent space

In this subsection we will study how the state feedback affects the consistent space.

Lemma 2.5: Consider linear systems (6) and (7). If $EF(\mathscr{V}_{i+1}) + B(\ker D) \subseteq \mathscr{V}_i^F$ then $\mathscr{V} \subseteq \mathscr{V}^F$. Further, $\mathscr{V}_i \subseteq \mathscr{V}_i^F$ for all $i \in \mathbb{N}$.

Proof: We prove this result by induction on the number of steps in algorithms (8) and (10). For basis step we note that $\mathscr{V}_1 = \mathscr{V}_1^F = \mathscr{P}_X$, hence $\mathscr{V}_1 \subseteq \mathscr{V}_1^F$. Assume $\mathscr{V}_i \subseteq \mathscr{V}_i^F$ for all $i = 1, \ldots, k$ for some $1 \leq k \in \mathbb{N}$ as induction hypothesis. Now we have to show that $\mathscr{V}_{k+1} \subseteq \mathscr{V}_{k+1}^F$. Let $x \in \mathscr{V}_{k+1}$, therefore, $\exists \bar{u} \in \mathbb{R}^m$ such that $Cx + D\bar{u} = 0$ and $x^1 = Ax + B\bar{u} \in \mathscr{V}_k \subseteq \mathscr{V}_k^F$. Now, $\forall \tilde{u} \in \mathbb{R}^m$ such that $Cx + D\tilde{u} = 0$ consider $x^2 = Ax + B\tilde{u} + EFx$. Now, we have $x^2 - x^1 = B(\tilde{u} - \bar{u}) + EFx$ where $x \in \mathscr{V}_{k+1}$. Since, $(\tilde{u} - \bar{u}) \in \ker D$ from the lemma 2.1, therefore, we have $x^2 - x^1 \in \mathscr{V}_k^F$ from the main assumption. But, since, $x^1 \in \mathscr{V}_k \subseteq \mathscr{V}_k^F$ therefore, we have $x^2 \in \mathscr{V}_k^F$ which means $x \in \mathscr{V}_{k+1}^F$. Hence, we have $\mathscr{V} \subseteq \mathscr{V}^F$. ∎

We now give a partial converse for the above lemma. Let $\mathscr{V} \subseteq \mathscr{V}^F$. Consider $x \in \mathscr{V}$, therefore, $\exists \bar{u} \in \mathbb{R}^m$ such that $Cx + D\bar{u} = 0$ and $x^1 = Ax + B\bar{u} \in \mathscr{V} \subseteq \mathscr{V}^F$. Since, $x \in \mathscr{V}^F$ also we have, $\exists \tilde{u} \in \mathbb{R}^m$ such that $Cx + D\tilde{u} = 0$ and $x^2 = Ax + B\tilde{u} + EFx \in \mathscr{V}^F$. We now have $x^2 - x^1 = B(\tilde{u} - \bar{u}) + EFx$. Since, both $x^1, x^2 \in \mathscr{V}^F$ therefore, $B(\tilde{u}-\bar{u})+EFx \in \mathscr{V}^F$ where, $(\tilde{u}-\bar{u}) \in \ker D$ from lemma 2.1 and $x \in \mathscr{V}$. Thus we have the following proposition.

Proposition 2.1: Consider linear systems (6) and (7). If $B(\ker D) = \{0\}$, then $\mathscr{V} \subseteq \mathscr{V}^F \implies EF(\mathscr{V}) \subseteq \mathscr{V}^F$.

Proof: Since, $B(\ker D) = \{0\}$ from the above partial converse we have $EFx \in \mathscr{V}^F$ for all $x \in \mathscr{V}$. ∎

Proposition 2.2: Consider linear systems (6) and (7). If $B(\ker D) = \{0\}$ and $\mathscr{V}_2 \subseteq \ker F$, then $\mathscr{V} \subseteq \mathscr{V}^F$. Further, $\mathscr{V}_i \subseteq \mathscr{V}_i^F$ for all $i \in \mathbb{N}$.

Proof: Let $\mathscr{V}_2 \subseteq \ker F$. Since, $\mathscr{V}_1 = \mathscr{P}_X$, we have $EF(\mathscr{P}_X) \subseteq \mathbb{R}^n = \mathscr{V}_0^F$. Now, Since, $F(\mathscr{V}_2) = \{0\}$ and $\mathscr{V}_{k+1} \subseteq \mathscr{V}_k$ for all $k \in \mathbb{N}$, we have $EF(\mathscr{V}_{i+1}) = \{0\} \subseteq \mathscr{V}_i^F$ for all $1 \leq i \in \mathbb{N}$. Thus, $EF(\mathscr{V}_{i+1}) \subseteq \mathscr{V}_i^F$ for all $i \in \mathbb{N}$. Since, $B(\ker D) = \{0\}$, therefore, from lemma 2.5 we have $\mathscr{V}_i \subseteq \mathscr{V}_i^F$ for all $i \in \mathbb{N}$ and hence $\mathscr{V} \subseteq \mathscr{V}^F$.

∎

Lemma 2.6: Consider linear systems (6) and (7). If $EF(\mathscr{P}_X) + B(\ker D) \subseteq \mathscr{V}$ then $\mathscr{V} = \mathscr{V}^F$. Further, $\mathscr{V}_i = \mathscr{V}_i^F$ for all $i \in \mathbb{N}$.

Proof: Let $EF(\mathscr{P}_X) + B(\ker D) \subseteq \mathscr{V}$. To show $\mathscr{V}_i = \mathscr{V}_i^F$ for all $i \in \mathbb{N}$. This is done in two steps, first we show $\mathscr{V}_i \subseteq \mathscr{V}_i^F$ and then $\mathscr{V}_i^F \subseteq \mathscr{V}_i$ for all $i \in \mathbb{N}$.

Case $\mathscr{V}_i \subseteq \mathscr{V}_i^F$ for all $i \in \mathbb{N}$: This will be shown by induction on the number of steps in the algorithms (8) and (10). For basis case observe that $\mathscr{V}_1 = \mathscr{V}_1^F = \mathscr{P}_X$. As induction hypothesis assume that $\mathscr{V}_j \subseteq \mathscr{V}_j^F$ for all $j = 1, 2, \ldots, k$ for some $1 \leq k \in \mathbb{N}$. Now we have to show that $\mathscr{V}_{k+1} \subseteq \mathscr{V}_{k+1}^F$. Let $x \in \mathscr{V}_{k+1}$, therefore, $\exists u^1 \in \mathbb{R}^m$ such that $Cx + Du^1 = 0$ and $x^1 = Ax + Bu^1 \in \mathscr{V}_k \subseteq \mathscr{V}_k^F$ by induction hypothesis. Now, $\forall u^2 \in \mathbb{R}^m$ such that $Cx + Du^2 = 0$ consider $x^2 = Ax + Bu^2 + EFx$. Now, we have $x^2 - x^1 = B(u^2 - u^1) + EFx$. Since, $(u^2 - u^1) \in \ker D$, and $x \in \mathscr{V}_{k+1} \subseteq \mathscr{P}_X$ therefore, we have $x^2 - x^1 \in \mathscr{V} \subseteq \mathscr{V}_k \subseteq \mathscr{V}_k^F$ from the main assumption. But, since, $x^1 \in \mathscr{V}_k \subseteq \mathscr{V}_k^F$ therefore, we have $x^2 \in \mathscr{V}_k^F$ which means $x \in \mathscr{V}_{k+1}^F$. Thus, we have $\mathscr{V}_{k+1} \subseteq \mathscr{V}_{k+1}^F$.

Case $\mathscr{V}_i^F \subseteq \mathscr{V}_i$ for all $i \in \mathbb{N}$: This will again be shown using induction on the number of steps in the algorithms (8) and (10). For basis case observe that $\mathscr{V}_1 = \mathscr{V}_1^F = \mathscr{P}_X$. For induction hypothesis we assume that $\mathscr{V}_j^F \subseteq \mathscr{V}_j$ for all $j = 1, 2, \ldots, k$ for some $1 \leq k \in \mathbb{N}$. Now, let $x \in \mathscr{V}_{k+1}^F$, therefore, $\exists u^1 \in \mathbb{R}^m$ such that $Cx + Du^1 = 0$ and $x^1 = Ax+Bu^1+EFx \in \mathscr{V}_k^F \subseteq \mathscr{V}_k$ by induction hypothesis. Now, $\forall u^2 \in \mathbb{R}^m$ such that $Cx+Du^2 = 0$ consider $x^2 = Ax+Bu^2$. We now consider $x^1 - x^2 = B(u^2 - u^1) + EFx$. Since, $(u^2 - u^1) \in \ker D$, and $x \in \mathscr{V}_{k+1}^F \subseteq \mathscr{P}_X$ therefore, we have $x^1 - x^2 \in \mathscr{V} \subseteq \mathscr{V}_k$ from the main assumption. But, since, $x^1 \in \mathscr{V}_k^F \subseteq \mathscr{V}_k$ therefore, we have $x^2 \in \mathscr{V}_k$ which means $x \in \mathscr{V}_{k+1}$. Thus, we have $\mathscr{V}_{k+1}^F \subseteq \mathscr{V}_{k+1}$.

Combining the above two cases we have $\mathscr{V}_i = \mathscr{V}_i^F$ for all $i \in \mathbb{N}$ and hence $\mathscr{V} = \mathscr{V}^F$.

∎

We also, give a partial converse for the above proposition. Let $\mathscr{V} = \mathscr{V}^F$. Consider $x \in \mathscr{V}$, therefore, $\exists u^1 \in \mathbb{R}^m$ such that $Cx+Du^1 = 0$ and $x^1 = Ax+Bu^1 \in \mathscr{V} = \mathscr{V}^F$. Since, $x \in \mathscr{V}^F$ also we have, $\exists u^2 \in \mathbb{R}^m$ such that $Cx + Du^2 = 0$ and $x^2 = Ax + Bu^2 + EFx \in \mathscr{V}^F = \mathscr{V}$. We now have $x^2 - x^1 = B(u^2 - u^1) + EFx$. Since, both $x^1, x^2 \in \mathscr{V}$ therefore, $B(u^2-u^1)+EFx \in \mathscr{V}$ where, $(u^2-u^1) \in \ker D$ and $x \in \mathscr{V}$. Thus, we have the following proposition.

Proposition 2.3: Consider linear systems (6) and (7). If $B(\ker D) = \{0\}$ and $\mathscr{V} = \mathscr{V}^F$, then $EF(\mathscr{V}) \subseteq \mathscr{V}$.

Proof: Follows from the above partial converse. ∎

Proposition 2.4: Consider linear systems (6) and (7). If $\operatorname{im} E + B(\ker D) \subseteq \mathscr{V}$ then $\mathscr{V} = \mathscr{V}^F$. Further, $\mathscr{V}_i = \mathscr{V}_i^F$ for all $i \in \mathbb{N}$.

Proof: Follows from above lemma 2.6. ■

Corollary 2.1: Consider linear systems (6) and (7). If $B(\ker D) = \{0\}$ and $\mathscr{V}_i = \mathscr{V}_i^F$ for all $i \in \mathbb{N}$, then $EF(\mathscr{V}_{i+1}) \subseteq \mathscr{V}_i$ for all $i \in \mathbb{N}$.

Proof: Let $\mathscr{V}_i = \mathscr{V}_i^F$ for all $i \in \mathbb{N}$. Let $x \in \mathscr{V}_{k+1}$, therefore, $\exists u^1 \in \mathbb{R}^m$ such that $Cx + Du^1 = 0$ and $x^1 = Ax + Bu^1 \in \mathscr{V}_k = \mathscr{V}_k^F$ for $k \in \mathbb{N}$. Now, since, $x \in \mathscr{V}_{k+1}^F$ also we have, $\exists u^2 \in \mathbb{R}^m$ such that $Cx + Du^2 = 0$ and $x^1 = Ax + Bu^2 + EFx \in \mathscr{V}_k^F = \mathscr{V}_k$ for all $k \in \mathbb{N}$. Now, $x^2 - x^1 = B(u^2 - u^1) + EFx$. Since, $(u^2 - u^1) \in \ker D$, and $x \in \mathscr{V}_{k+1}$, we have $EFx \in \mathscr{V}_k = \mathscr{V}_k^F$. Since this holds for all $k \in \mathbb{N}$ therefore, $EF(\mathscr{V}_{k+1}) \subseteq \mathscr{V}_k$. ■

Corollary 2.2: Consider linear systems (6) and (7). If $B(\ker D) = \{0\}$ then $\operatorname{im} E \subseteq \mathscr{V} \iff \mathscr{V} = \mathscr{V}^F$ for all F. Further, $\mathscr{V}_i = \mathscr{V}_i^F$, for all $i \in \mathbb{N}$.

Proof: ($\Rightarrow$): Let $\operatorname{im} E \subseteq \mathscr{V}$. Then we have $EFx \in \mathscr{V}$ for all $x \in \mathbb{R}^n$. In particular $EF(\mathscr{P}_X) \subseteq \mathscr{V}$. Since, $B(\ker D) = \{0\}$, therefore, from lemma 2.6 we have $\mathscr{V}_i = \mathscr{V}_i^F$ for all $i \in \mathbb{N}$.

($\Leftarrow$): Let $\mathscr{V} = \mathscr{V}^F$. Consider $x \in \mathscr{V}$, therefore, $\exists u^1 \in \mathbb{R}^m$ such that $Cx + Du^1 = 0$ and $x^1 = Ax + Bu^1 \in \mathscr{V} = \mathscr{V}^F$. Since, $x \in \mathscr{V}^F$ also we have, $\exists u^2 \in \mathbb{R}^m$ such that $Cx + Du^2 = 0$ and $x^2 = Ax + Bu^2 + EFx \in \mathscr{V}^F = \mathscr{V}$. We now have $x^2 - x^1 = B(u^2 - u^1) + EFx$. Since, both $x^1, x^2 \in \mathscr{V}$ therefore, $B(u^2 - u^1) + EFx \in \mathscr{V}$ where, $(u^2 - u^1) \in \ker D$ and $x \in \mathscr{V}$. But, since $B(\ker D) = \{0\}$, we have $EFx \in \mathscr{V}$ for all $x \in \mathscr{V}$. Since, F is arbitrary we have $\operatorname{im} E \subseteq \mathscr{V}$. ■

Corollary 2.3: Consider linear systems (6) and (7). If $B(\ker D) = \{0\}$ then if $\mathscr{P}_X \subseteq \ker F$ then $\mathscr{V} = \mathscr{V}^F$. Further, $\mathscr{V}_i = \mathscr{V}_i^F$, for all $i \in \mathbb{N}$.

Proof: Let $\mathscr{P}_X \subseteq \ker F$. Then we have $F(\mathscr{P}_X) = \{0\}$. Therefore, $EF(\mathscr{P}_X) = \{0\}$. Hence $EF(\mathscr{P}_X) \subseteq \mathscr{V}$. Since, $B(\ker D) = \{0\}$, therefore, from lemma 2.6 the result follows. ■

We know that $\mathscr{V}^F \subseteq \mathscr{P}_X$ for all F's. Therefore, one could ask the question when is $\mathscr{V}^F = \mathscr{P}_X$.

Proposition 2.5: Consider linear systems (6) and (7) and algorithms (8) and (10), then $[A + EF|B](\ker[C|D]) \subseteq \mathscr{P}_X \iff \mathscr{V}^F = \mathscr{P}_X$.

Proof: ($\Rightarrow$): Let $[A+EF|B](\ker[C|D]) \subseteq \mathscr{P}_X$. Since, $\mathscr{V}_1^F = \mathscr{P}_X$ from the algorithm (8) it is clear that it is enough to show $\mathscr{V}_2^F = \mathscr{V}_1^F$. Since, $\mathscr{V}_{k+1}^F \subseteq \mathscr{V}_k^F$ for all $k \in \mathbb{N}$, therefore, it is enough to $\mathscr{V}_1^F \subseteq \mathscr{V}_2^F$. Let $x \in \mathscr{V}_1^F = \mathscr{P}_X$, then $\exists u$ such that $Cx + Du = 0$. Consider, $x^1 = Ax + Bu + EFx$. Now, because of the main assumption of this proposition we have $x^1 \in \mathscr{P}_X = \mathscr{V}_1^F$. Which means $x \in \mathscr{V}_2^F$ from the algorithm (10). Therefore, $\mathscr{V}_1 = \mathscr{V}_2 = \mathscr{V}^F = \mathscr{P}_X$.

($\Leftarrow$): Let $\mathscr{V}^F = \mathscr{P}_X$. Since, $\mathscr{V}_i^F = \mathscr{P}_X$ for all $i \in \mathbb{N}$, therefore, $\forall x \in \mathscr{V}_2^F = \mathscr{P}_X$, $\exists u \in \mathbb{R}^m$ such that $Cx + Du = 0$ and $Ax + Bu + EFx \in \mathscr{V}_1^F = \mathscr{P}_X$. Which means $[A + EF|B](\ker[C|D]) \subseteq \mathscr{P}_X$. ■

Thus, from the above discussion it is clear that the consistent space has a tendency to increase with state feedback. The largest consistent space that can be obtained is $\mathscr{P}_X$ through state feedback.

B. Effect on jump space

We will now consider the effect of state feedback on the jump space.

Lemma 2.7: Consider linear systems (6) and (7). Then $\mathscr{W}^F \subseteq \mathscr{W} \iff EF(\mathscr{W}^F) \subseteq \mathscr{W}$.

Proof: ($\Rightarrow$): Let $\mathscr{W}^F \subseteq \mathscr{W}$. Let, $x \in \mathscr{W}^F$ then $\forall u^1 \in \mathbb{R}^m$ such that $C\bar{x} + Du^1 = 0$ we have $x^1 = Ax + Bu^1 + EFu^1 \in \mathscr{W}^F \subseteq \mathscr{W}$. Since, $x \in \mathscr{W}$ also, we have $\forall u^2 \in \mathbb{R}^m$ such that $C\tilde{x} + Du^2 = 0$, $x^2 = Ax + Bu^2 \in \mathscr{W}$. Therefore, $x^1 - x^2 = B(u^1 - u^2) + EFx \in \mathscr{W}$. Now, since, $B(u^1 - u^2) \in B(\ker D)$ from lemma 2.1 and $B(\ker D) \subseteq \mathscr{W}$ from algorithm (9) we have $EFx \in \mathscr{W}$ for all $\forall x \in \mathscr{W}^F$. Therefore, $EF(\mathscr{W}^F) \subseteq \mathscr{W}$.

($\Leftarrow$): Let $EF(\mathscr{W}^F) \subseteq \mathscr{W}$. We will show that $\mathscr{W}_i^F \subseteq \mathscr{W}$ for all $i \in \mathbb{N}$ using induction on number of steps of algorithm (11). For the basis step we have $\mathscr{W}_1^F = \mathscr{W}_1 = B(\ker D) \subseteq \mathscr{W}$. Now, assume that $\mathscr{W}_i^F \subseteq \mathscr{W}$ for all $i = 1, 2, \ldots, k$ for some $k \in \mathbb{N}$. We have to show that $\mathscr{W}_{k+1}^F \subseteq \mathscr{W}$. Let $x \in \mathscr{W}_{k+1}^F$, then $\exists \bar{x} \in \mathscr{W}_k^F \subset \mathscr{W}$ and $\exists u^1 \in \mathbb{R}^m$ such that $C\bar{x} + Du^1 = 0$ and $x = A\bar{x} + Bu^1 + EF\bar{x}$. Since, $\mathscr{W}_i^F \subset \mathscr{W}^F$ for all $i \in \mathbb{N}$, we have $\bar{x} \in \mathscr{W}^F$. Therefore, $EF\bar{x} \in \mathscr{W}$. But, since $\bar{x} \in \mathscr{W}_k$, and $C\bar{x} + Du^1 = 0$, we have $A\bar{x} + Bu^1 \in \mathscr{W}$. Therefore, $x = A\bar{x} + Bu^1 + EF\bar{x} \in \mathscr{W}$. Therefore, $\mathscr{W}_{k+1}^F \subseteq \mathscr{W}$. Since, this is true for all $k \in \mathbb{N}$, we have $\mathscr{W}^F \subseteq \mathscr{W}$ ■

Proposition 2.6: Consider linear systems (6) and (7). If $\operatorname{im} E \subseteq \mathscr{W}$ then $\mathscr{W}^F \subseteq \mathscr{W}$.

Proof: This follows from ($\Leftarrow$) part of lemma 2.7. ■

Proposition 2.7: Consider linear systems (6) and (7). If $\mathscr{W}^F = \mathscr{W}$ then $EF(\mathscr{W}) \subseteq \mathscr{W}$.

Proof: Let $\mathscr{W}^F = \mathscr{W}$. Let $x \in \mathscr{W}$, then $\forall u^1 \in \mathbb{R}^n$ such that $Cx + Du^1 = 0$, we have $x^1 = Ax + Bu^1 \in \mathscr{W}$. Now, since, $x \in \mathscr{W}^F$ also, we have $\forall u^2 \in \mathbb{R}^n$ such that $Cx + Du^2 = 0$, we have $x^2 = Ax + Bu^2 + EFx \in \mathscr{W}^F = \mathscr{W}$. Therefore, we have $x^2 - x^1 = B(u^2 - u^1) + EFx \in \mathscr{W}$. But, since, $(u^2 - u^1) \in \ker(D)$ and $B(\ker(D)) = \mathscr{W}_1 \subseteq \mathscr{W}$ we have $B(u^2 - u^1) \in \mathscr{W}$. Therefore $EFx \in \mathscr{W}$ for all $x \in \mathscr{W}$. Therefore, $EF(\mathscr{W}) \subseteq \mathscr{W}$. ■

Proposition 2.8: Consider linear systems (6) and (7). If $\mathscr{W} \subseteq \ker(F)$ then $\mathscr{W}^F = \mathscr{W}$.

Proof: Let $\mathscr{W} \subseteq \ker(F)$. We will show that $\mathscr{W}_i = \mathscr{W}_i^F$ for all $i \in \mathbb{N}$. This will be shown in two steps, first we will show $\mathscr{W}_i \subseteq \mathscr{W}_i^F$ and then $\mathscr{W}_i^F \subseteq \mathscr{W}_i$.

Case $\mathscr{W}_i \subseteq \mathscr{W}_i^F$ for all $i \in \mathbb{N}$: We will show this by induction on number of steps in algorithm (11). For the basis step we have $\mathscr{W}_1 = \mathscr{W}_1^F = B(\ker D)$. As induction hypothesis assume that $\mathscr{W}_j \subseteq \mathscr{W}_j^F$ for all $j = 1, 2 \ldots k$ for some $1 \leq k \in \mathbb{N}$. We have to show that $\mathscr{W}_{k+1} \subseteq \mathscr{W}_{k+1}^F$. Let $x \in \mathscr{W}_{k+1}$, then $\exists \bar{x} \in \mathscr{W}_k \subseteq \mathscr{W}_k^F$ and $\exists u^1 \in \mathbb{R}^n$ such that $C\bar{x} + Du^1 = 0$, and $x = A\bar{x} + Bu^1$. Since, $\bar{x} \in \mathscr{W}_k \subseteq \mathscr{W}$ and $\mathscr{W} \subseteq \ker(F)$ we have $EF\bar{x} = 0$. Thus, we have $x = (A+EF)\bar{x} + Bu^1$ and $C\bar{x} + Du^1 = 0$. Which means $x \in \mathscr{W}_{k+1}^F$. Therefore, $\mathscr{W}_{k+1} \subseteq \mathscr{W}_{k+1}^F$.

Case $\mathscr{W}_i^F \subseteq \mathscr{W}_i$ for all $i \in \mathbb{N}$: We will again show this by induction on number of steps in algorithm (11). For the basis step we have $\mathscr{W}_1^F = \mathscr{W}_1 = B(\ker D)$. As induction hypothesis assume that $\mathscr{W}_j^F \subseteq \mathscr{W}_j$ for all $j = 1, 2 \ldots k$ for some $1 \leq k \in \mathbb{N}$. We have to show that $\mathscr{W}_{k+1}^F \subseteq \mathscr{W}_{k+1}$. Let $x \in \mathscr{W}_{k+1}^F$, then $\exists \bar{x} \in \mathscr{W}_k^F \subseteq \mathscr{W}_k$ and $\exists u^1 \in \mathbb{R}^n$ such that $C\bar{x} + Du^1 = 0$, and $x = (A+EF)\bar{x} + Bu^1$. Since, $\bar{x} \in \mathscr{W}_k^F \subseteq \mathscr{W}_k \subseteq \mathscr{W}$ and $\mathscr{W} \subseteq \ker(F)$ we have $EF\bar{x} = 0$. Thus, we have $x = A\bar{x} + Bu^1$ and $C\bar{x} + Du^1 = 0$. Which means $x \in \mathscr{W}_{k+1}$. Therefore, $\mathscr{W}_{k+1}^F \subseteq \mathscr{W}_{k+1}$. ∎

The following proposition shows that if $\mathscr{W} \neq \emptyset$ then state feedback alone cannot reduce $\mathscr{W}^F$ to zero. This is because $\mathscr{W}_1 \subseteq \mathscr{W}$ is independent of state feedback.

Proposition 2.9: Consider linear systems (6) and (7) and algorithms (9) and (11), then $\mathscr{W} = \{0\} \iff \ker(D) \subseteq \ker(B)$.

Proof: ($\Rightarrow$): Let $\mathscr{W} = \{0\}$. Then from algorithm (9) it is clear that $\mathscr{W}_1 = B\ker(D) = \{0\}$. Thus, $\ker(D) \subseteq \ker(B)$. ($\Leftarrow$): Let $\ker(D) \subseteq \ker(B)$. Then $\mathscr{W}_1 = B\ker(D) = \{0\}$. Thus, $\mathscr{W} = \{0\}$. ∎

Hence, the above discussion makes it clear that the jump space has a tendency to decrease under state feedback. The minimum possible jump space is $B\ker(D)$ with state feedback.

III. PORT FEEDBACK

We now apply port feedback $w(t) = Gu(t)$ to the LCS (3) in each mode $I \subset \bar{m}$. After renaming the variables we consider the following linear equations after port feedback.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + (B+EG)u(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{12}$$

An algorithm for computing consistent space with port feedback is given below

$$\begin{aligned} \mathscr{V}_0^G &= \mathbb{R}^n \\ \mathscr{V}_{i+1}^G &= \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that} \\ &\qquad Ax + (B+EG)u \in \mathscr{V}_i^G,\ Cx + Du = 0\}, \end{aligned} \tag{13}$$

For every $\mathscr{V}_i$ space where $i \in \mathbb{N}$ from the algorithm (8) there is an associated $\mathscr{V}_i^U$ which is defined as follows:

$$\begin{aligned} \mathscr{V}_0^U &= \mathbb{R}^m \\ \mathscr{V}_{i+1}^U &= \{u \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \text{ such that} \\ &\qquad Ax + Bu \in \mathscr{V}_i,\ Cx + Du = 0\}, \end{aligned} \tag{14}$$

The following is an algorithm for computing the jump space after port feedback.

$$\begin{aligned} \mathscr{W}_0^G &= 0 \\ \mathscr{W}_{i+1}^G &= \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ and } \exists \bar{x} \in \mathscr{W}_i^G \text{ such that} \\ &\qquad x = A\bar{x} + (B+EG)u,\ C\bar{x} + Du = 0\}. \end{aligned} \tag{15}$$

Let $\mathscr{W}_U = \{u \in \mathbb{R}^m \mid \exists x \in \mathscr{W} \text{ and } Cx + Du = 0\}$. $\mathscr{W}_U^G$ is defined in a similar way.

Definition 3.1: Let $\mathscr{P}_U$ be the projection of $\ker([C\ D])$ on the port space $\mathbb{R}^m$. $\mathscr{P}_U$ is the set of all u's such that there exists an state x which makes y (output) zero.

The following lemmas are immediate from the algorithms above.

Lemma 3.1: Given systems (7) and (12), the following is true $\mathscr{V}_1 = \mathscr{V}_1^G = \mathscr{P}_X$ and for all G.

Lemma 3.2: Given system (12) the following is true $\mathscr{V}_{k+1}^G \subseteq \mathscr{V}_k^G$ for all $k \in \mathbb{N}$ and for all G.

Lemma 3.3: Given system (12) the following is true $\mathscr{W}_k^G \subseteq \mathscr{W}_{k+1}^G$ for all $k \in \mathbb{N}$ and for all G.

We will now investigate the relation between consistent space and jump space of the above linear system with the system without feedback (with zero input $w = 0$ or $G = 0$) i.e., system (7).

A. Effect on consistent space

In this section we will consider the effect of port feedback on consistent space.

Lemma 3.4: Consider linear systems (12) and (7). If $EG(\mathscr{V}_{i+1}^U) + B(\ker D) \subseteq \mathscr{V}_i^G$ then $\mathscr{V} \subseteq \mathscr{V}^G$. Further, $\mathscr{V}_i \subseteq \mathscr{V}_i^G$ for all $i \in \mathbb{N}$.

Proof: The proof is similar to one in lemma 2.5 with state feedback. ∎

We now give a partial converse for the above lemma. Let $\mathscr{V} \subseteq \mathscr{V}^G$. Consider $x \in \mathscr{V}$, therefore, $\exists u^1 \in \mathbb{R}^m$ such that $Cx + Du^1 = 0$ and $x^1 = Ax + Bu^1 \in \mathscr{V} \subseteq \mathscr{V}^G$. Since, $x \in \mathscr{V}^G$ also we have, $\exists u^2 \in \mathbb{R}^m$ such that $Cx + Du^2 = 0$ and $x^2 = Ax + Bu^2 + EGu^2 \in \mathscr{V}^G$. We now have $x^2 - x^1 = B(u^2 - u^1) + EGu^2$. Since, both $x^1, x^2 \in \mathscr{V}^G$ therefore, $B(u^2 - u^1) + EGu^2 \in \mathscr{V}^G$ where, $(u^2 - u^1) \in \ker D$ from lemma 2.1 and $u^2 \in \mathscr{P}_U$.

Proposition 3.1: Consider linear systems (12) and (7). If $B(\ker D) = \{0\}$ and $\mathscr{V}_2^U \subseteq \ker G$ then $\mathscr{V} \subseteq \mathscr{V}^G$. Further, $\mathscr{V}_i \subseteq \mathscr{V}_i^G$ for all $i \in \mathbb{N}$.

Proof: The proof is similar to one in proposition 2.2. ∎

Lemma 3.5: Consider linear systems (12) and (7). If $EG(\mathscr{P}_U) + B(\ker D) \subseteq \mathscr{V}$ then $\mathscr{V} = \mathscr{V}^G$. Further $\mathscr{V}_i = \mathscr{V}_i^G$ for all $i \in \mathbb{N}$.

Proof: The proof is similar to the one in lemma 2.6. ∎

We also, give a partial converse for the above proposition. Let $\mathscr{V} = \mathscr{V}^G$. Consider $x \in \mathscr{V}$, therefore, $\exists u^1 \in \mathbb{R}^m$ such that $Cx + Du^1 = 0$ and $x^1 = Ax + Bu^1 \in \mathscr{V} = \mathscr{V}^G$. Since, $x \in \mathscr{V}^G$ also we have, $\exists u^2 \in \mathbb{R}^m$ such that $Cx + Du^2 = 0$ and $x^2 = Ax + Bu^2 + EGu^2 \in \mathscr{V}^G = \mathscr{V}$. We now have $x^2 - x^1 = B(u^2 - u^1) + EGu^2$. Since, both $x^1, x^2 \in \mathscr{V}$ therefore, $B(u^2 - u^1) + EGu^2 \in \mathscr{V}$ where, $(u^2 - u^1) \in \ker D$ and $u^2 \in \mathscr{P}_U$.

Proposition 3.2: Consider linear systems (12) and (7). If $\operatorname{im} E + B(\ker D) \subseteq \mathscr{V}$ then $\mathscr{V} = \mathscr{V}^G$. Further, $\mathscr{V}_i = \mathscr{V}_i^G$ for all $i \in \mathbb{N}$.

Proof: The proof follows from above lemma 3.5. ∎

Corollary 3.1: Consider linear systems (12) and (7). If $B(\ker D) = \{0\}$ then if $\mathscr{V}_i = \mathscr{V}_i^G$ for all $i \in \mathbb{N}$ then $EG(\mathscr{V}_{i+1}^U) \subseteq \mathscr{V}_i$ for all $i \in \mathbb{N}$.

Proof: This proof is similar to the one in corollary 2.1. ∎

Corollary 3.2: Consider linear systems (12) and (7). If $B(\ker D) = \{0\}$ then $\operatorname{im} E \subseteq \mathscr{V} \iff \mathscr{V} = \mathscr{V}^G$ for all E. Further, $\mathscr{V}_i = \mathscr{V}_i^G$, for all $i \in \mathbb{N}$.

Proof: ($\Rightarrow$): Let $\operatorname{im} E \subseteq \mathscr{V}$. Then we have $EGu \in \mathscr{V}$ for all $u \in \mathbb{R}^m$. In particular $EG(\mathscr{P}_U) \subseteq \mathscr{V}$. Since, $B(\ker D) = \{0\}$, thus, from lemma 3.5 we have $\mathscr{V}_i = \mathscr{V}_i^G$ for all $i \in \mathbb{N}$. Therefore, $\mathscr{V} = \mathscr{V}^G$.

($\Leftarrow$): Let $\mathscr{V} = \mathscr{V}^G$. Let $x \in \mathscr{V}$ then, $\exists u^1 \in \mathbb{R}^m$ such that $Cx + Du^1 = 0$ and $x^1 = Ax + Bu^1 \in \mathscr{V} = \mathscr{V}^G$. Since, $x \in \mathscr{V}^G$ also then, $\exists u^2 \in \mathbb{R}^m$ such that $Cx + Du^2 = 0$ and $Ax + Bu^2 + EGu^2 \in \mathscr{V}^G = \mathscr{V}$. Therefore $x^2 = B(u^2 - u^1) + EGu^2 \in \mathscr{V} = \mathscr{V}^G$. But, since $(u^2 - u^1) \in \ker(D)$ we have $EGu^2 \in \mathscr{V} = \mathscr{V}^G$. Since G is arbitrary we this means $\operatorname{im}(E) \subseteq \mathscr{V}$ for all $i \in \mathbb{N}$. ∎

Corollary 3.3: Consider linear systems (12) and (7). If $B(\ker D) = \{0\}$ then if $\mathscr{P}_U \subseteq \ker G$ then $\mathscr{V} = \mathscr{V}^G$ and moreover $\mathscr{V}_i = \mathscr{V}_i^G$, for all $i \in \mathbb{N}$.

Proof: Let $\mathscr{P}_U \subseteq \ker G$. Then we have $G(\mathscr{P}_U) = \{0\}$. Therefore, $EG(\mathscr{P}_U) = \{0\}$. Hence $EG(\mathscr{P}_U) \subseteq \mathscr{V}$. Since, $B(\ker D) = \{0\}$, therefore, from lemma 3.5 the result follows. ∎

The following proposition characterises the class of feedback which maximises the consistent space.

Proposition 3.3: Consider linear systems (12) and (7). Then $[A|B + EG](\ker[C|D]) \subseteq \mathscr{P}_X \iff \mathscr{V}^G = \mathscr{P}_X$.

Proof: This proof is similar to one in proposition 2.5. ∎

The above results show that even under port feedback the consistent space has a tendency to increase. The largest possible consistent is $\mathscr{P}_X$ which is the same for state feedback also.

B. Effect on jump space

In this section we will see how the jump space can be changed with port feedback.

Lemma 3.6: Consider linear systems (12) and (7). Let $(B + EG)\ker(D) \subseteq \mathscr{W}$ then $\mathscr{W}^G \subseteq \mathscr{W} \iff EG(\mathscr{W}_U^G) \subseteq \mathscr{W}$.

Proof: ($\Rightarrow$): The proof of this proposition is similar ($\Rightarrow$) part of proof of proposition 2.7.

($\Leftarrow$):Let $EG(\mathscr{W}_U^G) \subseteq \mathscr{W}$. We will show this result by induction on the number of steps in algorithm (15). For basic step we have $\mathscr{W}_1^G = (B + EG)\ker(D) \subseteq \mathscr{W}$. Assume $\mathscr{W}_i^G \subseteq \mathscr{W}$ for all $i = 1, 2, \ldots, k$ for some $1 \leq k \in \mathbb{N}$. We have to show that $\mathscr{W}_{k+1}^G \subseteq \mathscr{W}$. Let $x \in \mathscr{W}_{k+1}^G$. Then, $\exists u^1 \in \mathbb{R}^m$ and $\bar{x} \in \mathscr{W}_k^G \subseteq \mathscr{W}$ such that $C\bar{x} + Du^1 = 0$ and $x = A\bar{x} + Bu^1 + EGu^1$. Now, since, $\bar{x} \in \mathscr{W}$, and $C\bar{x} + Du^1 = 0$ we have $A\bar{x} + Bu^1 \in \mathscr{W}$. Since, $EGu^1 \in \mathscr{W}$ from the main assumption, we have $x = A\bar{x} + Bu^1 + EGu^1 \in \mathscr{W}$. Which means $\mathscr{W}_{k+1}^G \subseteq \mathscr{W}$. Therefore, $\mathscr{W}^G \subseteq \mathscr{W}$. ∎

Proposition 3.4: Consider linear systems (12) and (7). If $(B + EG)\ker(D) \subseteq \mathscr{W}$ then if $\operatorname{im}(E) \subseteq \mathscr{W}$ then $\mathscr{W}^G \subseteq \mathscr{W}$

Proof: Let $\operatorname{im}(E) \subseteq \mathscr{W}$. Then, $EG(\mathscr{W}_U^G) \subseteq \mathscr{W}$. Therefore, from lemma 3.6 $\mathscr{W}^G \subseteq \mathscr{W}$. ∎

Proposition 3.5: Consider linear systems (12) and (7). If $\mathscr{W}^G = \mathscr{W}$ then $EG(\mathscr{W}_U) \subseteq \mathscr{W}$.

Proof: The proof is similar to the one in proposition 2.7. ∎

Proposition 3.6: Consider linear systems (12) and (7). If $\mathscr{P}_U \subseteq \ker(G)$ then $\mathscr{W}^G = \mathscr{W}$.

Proof: From the algorithm (15) it is clear that the term $EGu^1 = 0$ in $\mathscr{W}_i^G$ for all $i = 1, 2, \ldots$. Thus, $\mathscr{W}_i^G = \mathscr{W}_i$ for all $i \in \mathbb{N}$. Hence, $\mathscr{W}^G = \mathscr{W}$. ∎

The following proposition characterises all the port feedback which will reduce the jump space to zero.

Proposition 3.7: Consider linear systems (12) and (7). Then $\mathscr{W}^G = \{0\} \iff \ker(D) \subseteq \ker(B + EG)$.

Proof: ($\Rightarrow$): Let $\mathscr{W}^G = \{0\}$. Then from algorithm (15) it is clear that $\mathscr{W}_1^G = (B + EG)\ker(D) = \{0\}$. Thus, $\ker(D) \subseteq \ker(B + EG)$.

($\Leftarrow$): Let $\ker(D) \subseteq \ker(B + EG)$. Then $\mathscr{W}_1^G = (B + EG)\ker(D) = \{0\}$. Thus, $\mathscr{W}^G = \{0\}$. ∎

Therefore, it is clear that jump space shows a tendency to decrease in size under port feedback. Here one can reduce the jump space to zero by a proper choice of port feedback matrix G. Whereas in the case of state feedback the minimum possible jump space was $B(\ker D)$.

IV. COMBINED FEEDBACK

We will now apply a combined feedback $w(t) = Fx(t) + Gu(t)$ to the LCS (3) in each mode $I \subset \bar{m}$. After renaming the variables we consider the following linear equations after feedback.

$$\begin{aligned} \dot{x}(t) &= (A + EF)x(t) + (B + EG)u(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{16}$$

A similar algorithm like (8) can be given for computing consistent space with combined feedback. We call the limit of subspace sequences $\mathscr{V}_i^{FG}$ where $i \in \mathbb{N}$ as $\mathscr{V}^{FG}$ the consistent space of (16). Similarly, the limit of subspace sequences $\mathscr{W}_i^{FG}$ will be called as $\mathscr{W}^{FG}$ the jump space of (16). This can be computed using an algorithm similar to (9).

The following results are immediate from the algorithm above $\mathscr{V}_1 = \mathscr{V}_1^{FG} = \mathscr{P}_X$, $\mathscr{V}_{k+1}^{FG} \subseteq \mathscr{V}_k^{FG}$ and $\mathscr{W}_k^{FG} \subseteq \mathscr{W}_{k+1}^{FG}$ for all F and G.

We will need the following spaces in this section.

$$\begin{aligned} \mathscr{W}^{XU} &= \{(x, u) \mid Cx + Du = 0, Ax + Bu \in \mathscr{W}\}, \\ \mathscr{W}^{FGXU} &= \{(x, u) \mid Cx + Du = 0, \\ &\qquad (A + EF)x + (B + EG)u \in \mathscr{W}^{FG}\}. \end{aligned} \tag{17}$$

$$\begin{aligned} \mathscr{V}_0 &= \mathbb{R}^{m+n} \\ \mathscr{V}_{i+1}^{XU} &= \{(x, u) \mid Cx + Du = 0, Ax + Bu \in \mathscr{V}_i\}, \end{aligned} \tag{18}$$

$$\mathscr{V}^{XU} = \{(x, u) \mid Cx + Du = 0, Ax + Bu \in \mathscr{V}\}. \tag{19}$$

We will now investigate the relation between consistent space and jump space of the above linear system with the system without feedback (with zero input $w = 0$ or $F = 0$ and $G = 0$) i.e., (7).

A. *Effect on consistent space*

In this section we will derive the effect of combined feedback on consistent space.

From lemmas 2.5 and 3.4 we have $E(F|G)\mathscr{V}_{i+1}^{XU} + B(\ker D) \subseteq \mathscr{V}_i^{FG} \implies \mathscr{V} \subseteq \mathscr{V}^{FG}$. Further, $\mathscr{V}_i \subseteq \mathscr{V}_i^{FG}$ for all $i \in \mathbb{N}$. Propositions 2.2 and 3.1 gives that if $B(\ker D) = \{0\}$, then $\mathscr{V}_2^{XU} \subseteq \ker(F|G) \implies \mathscr{V} \subseteq \mathscr{V}^{FG}$. Further, $\mathscr{V}_i \subseteq \mathscr{V}_i^{FG}$ for all $i \in \mathbb{N}$. We can derive $E(F|G)\ker(C|D) + B(\ker D) \subseteq \mathscr{V} \implies \mathscr{V} = \mathscr{V}^{FG}$, further, $\mathscr{V}_i = \mathscr{V}_i^{FG}$ from lemmas 2.6 and 3.5. Also $\operatorname{im} E + B(\ker D) \subseteq \mathscr{V} \implies \mathscr{V} = \mathscr{V}^{FG}$, further, $\mathscr{V}_i = \mathscr{V}_i^{FG}$ for all $i \in \mathbb{N}$ from propositions 2.4 and 3.2. From corollaries 2.1 and 3.1 it follows that if $B(\ker D) = \{0\}$ then $\mathscr{V}_i = \mathscr{V}_i^{FG}$ for all $i \in \mathbb{N} \implies E(F|G)(\mathscr{V}_{i+1}^{XU}) \subseteq \mathscr{V}_i$ for all $i \in \mathbb{N}$. Then, corollaries 2.2 and 3.2 imply that if $B(\ker D) = \{0\}$ then $\operatorname{im} E \subseteq \mathscr{V} \iff \mathscr{V} = \mathscr{V}^{FG}$ for all F and G. Further, $\mathscr{V}_i = \mathscr{V}_i^{FG}$, for all $i \in \mathbb{N}$. The results of corollaries 2.3 and 3.3 imply that if $B(\ker D) = \{0\}$ then $\ker(C|D) \subseteq \ker(F|G) \implies \mathscr{V}_i = \mathscr{V}_i^{FG}$, for all $i \in \mathbb{N}$ and hence $\mathscr{V} = \mathscr{V}^{FG}$. The condition for maximisation of consistent space with combined feedback is $[A + EF|B + EG](\ker[C|D]) \subseteq \mathscr{P}_X \iff \mathscr{V}^{FG} = \mathscr{P}_X$. This result directly follows from propositions 2.5 and 3.3.

Thus, the consistent space has a tendency to increase with combined feedback. Also the maximum possible consistent space is $\mathscr{P}_X$.

B. *Effect on jump space*

In this section the effect of combined feedback on jump space will be considered. The results in this section is analogous to the combined results of state feedback and port feedback.

Let $(B + EG)\ker(D) \subseteq \mathscr{W}$ then $E(F|G)(\mathscr{W}^{FGXU}) \subseteq \mathscr{W} \iff \mathscr{W}^{FG} \subseteq \mathscr{W}$, this follows from $(\Leftarrow)$ part of lemmas 2.7 and 3.6. Thus, we have $\operatorname{im}(E) \subseteq \mathscr{W} \implies \mathscr{W}^{FG} \subseteq \mathscr{W}$. From propositions 2.7 and 3.5 one gets $\mathscr{W}^{FG} = \mathscr{W} \implies E(F|G)(\mathscr{W}^{XU}) \subseteq \mathscr{W}$. The result $\ker(C|D) \subseteq \ker(F|G) \implies \mathscr{W}^{FG} = \mathscr{W}$ can be obtained from $(\Leftarrow)$ part of propositions 2.8 and 3.6. The characterisation for the set of all the feedbacks which will reduce the jump space to zero is given by $\mathscr{W}^{FG} = \{0\} \iff \ker(D) \subseteq \ker(B + EG)$.

Hence, it is clear that the jump space has a tendency to decrease with combined feedback. It is also possible to reduce the jump space to zero with combined feedback.

V. CONCLUSION

It is clear from the above results that consistent space and jump space behave dually with both state and port feedback. While the consistent space tends to increase with feedback, jump space tends to decrease with feedback. In general a different feedback matrix may be required at each mode $I \subseteq \bar{m}$.

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