Distributed Computation of Minimum Time Consensus for Multi-Agent Systems

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Abstract—The problem of computing the minimum time to consensus of multiple identical double-integrator agents is considered. A distributed algorithm for computing the final consensus target state is proposed. Local feedback time optimal control laws are synthesized to drive each agent to the computed final consensus target state. Each agent is assumed to know the states of every other agent. As a part of the algorithm, every possible triplet of agents compute their mutual minimum time to consensus. The maximum value among these triplet minimum times is shown to be the required minimum time to consensus for the entire group. Since the required computation can be performed for each triple separately, it can be distributed evenly among the agents.

I. INTRODUCTION

Recent technological developments in unmanned aerial, ground and underwater vehicles have renewed interest in distributed control strategies for multi-agent systems that achieve consensus, i.e. the state of the various agents are controlled towards a common value (see [1], [2] and references therein). Most of the researchers have focussed on limitations imposed by partial information exchange allowed by incomplete communication among the various agents [3], [4], whereby various asymptotic [5], [6] and finite time [7], [8] consensus algorithms have been proposed. The speed at which consensus is achieved has been a cause of concern for various researchers, who have characterized the convergence rate using the algebraic connectivity of the communication graph [9], [10]. Still other results focus on finite time consensus [7], [8] to avoid exponential convergence rates. Recently there have been efforts for computing minimum time consensus for multi-agents communicating over directed trees [11], and a result on predicting the consensus value for discrete time multi-agent system with first-order agents [12].

The problem of computing the minimum time consensus point and the corresponding time optimal control for each agent can be viewed as a coupled minimum time problem for a 2N-dimensional multi-input system formed by appending the states of all agents. All final position states and all velocity states are constrained to be equal. While the Pontryagin’s Maximum Principle (PMP) does give necessary conditions for the optimal solution, there are no easy methods to compute the exact solution of such problems for arbitrary N. Known solution methods for this problem advocate solving either complicated boundary value problems or discretizing the infinite dimensional optimization problem to form a large finite dimensional non-linear program (see [13] and references therein). However, these methods are based on heuristics and do not guarantee a solution even after elaborate computations. In the context of multi-agent systems, it is reasonable to assume that computational power available onboard the agents is limited and no powerful central computer is unavailable, and hence the available methods are not appropriate in this scenario.

In this paper, we first develop a method for computing the minimum time to consensus and the corresponding target state for a three agent system. Then for a N agent system, we compute the minimum time to consensus for all possible triplets of agents separately. We show that the minimum time to consensus for the N-agent system is the maximum of these minimum time to consensus for all triplets of agents. Closed form expressions are derived for each quantity to be computed, thereby reducing the computational load substantially. Further, the minimum time computation for each triplet of agents is done separately. Hence apart from the comparison of these computed times, all other computations can be distributed among the agents and can be carried out in parallel, using on-board computers. Lastly, once the target state is identified and broad-casted to all the agents, a local feedback control law is devised for each agent that drives the states of each agent from its initial condition to the computed target state in the minimum time.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. Problem formulation

We consider a multi-agent system consisting of N agents with identical second order dynamics \( \dot{x}_i(t) = u_i(t) \). The state-space representation of each agent is given by

\[
\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, \ldots, N \tag{1}
\]

where \( x_i(t) = [r_i(t) \ v_i(t)]^T \), \( x_i(0) = x_{i0} = [r_{i0} \ v_{i0}^T] \), \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). The input \( u_i(t) \in \mathbb{R} \) is constrained within \( U_i = \{ u_i(t) : |u_i(t)| \leq 1 \} \). The states \( r_i(t) \), \( v_i(t) \in \mathbb{R} \) indicate the position and velocity of the \( i^{th} \) agent respectively while the input represents the acceleration imparted to each agent. System (1), is said to achieve consensus if for all \( i, j \in \{1, \ldots, n\} \), \( |x_i(t) - x_j(t)| \rightarrow 0 \) as \( t \rightarrow \infty \) and \( x_i(t) = x_j(t) \) for all \( t \geq \bar{t} \). The time \( \bar{t} \) is called the time to consensus and the point \( x_i(\bar{t}) = \bar{x} (i = 1, \ldots, N) \) is the consensus point. If \( \bar{t} < \infty \), we say that the consensus is achieved in finite time. In this case, the time to consensus \( \bar{t} \) is defined as the time at which the states of all agents are equal.

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paper, our objective is to identify a consensus point \( \bar{x} \in \mathbb{R}^2 \) such that the time to consensus \( \bar{t} \) is minimum.

**Problem 1.** Find \( \bar{x} \) and min \( \bar{t} \) such that for all \( i, j \in \{1, \ldots, N\} \), \( |x_i(t) - x_j(t)| \rightarrow 0 \) as \( t \rightarrow \bar{t} \), \( x_i(\bar{t}) = x_j(\bar{t}) = \bar{x} \) and \( x_i(t) = x_j(t) \) for \( t \geq \bar{t} \) and \( u_i(t) \in U_i \).

After the consensus point \( \bar{x} \) and time to consensus \( \bar{t} \) are identified, we design local control laws for the agents so that the consensus is achieved at point \( \bar{x} \) in time \( \bar{t} \).

**B. Preliminaries**

We briefly review some definitions related to the attainable and reachable sets of linear time optimal control from [14], [15]. Since all the agents are identical in (1), we drop the subscript \( i \) in this section. For any given point \( p \in \mathbb{R}^2 \), define the attainable set from \( p \) as follows:

**Definition 2.** [15] The set of all the states that any agent can reach from \( p \in \mathbb{R}^2 \) using admissible control \( u(t) \in U \) in time \( t > 0 \) is called the **attainable set** \( \mathcal{A}_p(t) \) from \( p \) at time \( t \). This set is given by,

\[
\mathcal{A}_p(t) = \left\{ x : x = e^{At}p + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \forall u(t) \in U \right\}
\]

**Definition 3.** [15] The set of points in \( \mathbb{R}^2 \) from which any agent can reach point \( p \) in time \( t \) is defined as the **reachable set** to point \( p \) at time \( t \): 

\[
\mathcal{R}_p(t) = \left\{ x : x = e^{At}p - \int_0^t e^{-At}Bu(\tau)d\tau, \forall u(t) \in U \right\}
\]

For a given \( p \) and \( t \), we write \( \mathcal{A}_p(t) = (e^{At} - I)p + e^{At}\mathcal{R}_p(t) \). The set \( \mathcal{R}_p(t) \) is a compact convex set with nonempty interior [15] and therefore \( \mathcal{A}_p(t) \) is also compact and convex with nonempty interior.

It is well known [16] that if there exists some input \( u(t) \in U \) which drives the states of any agent in (1) from initial state \( x_0 \) to final state \( x_f \) in some time \( t < \infty \), then there exists a unique time optimal input \( \bar{u}(t) \in U \) that drives the corresponding states from \( x_0 \) to \( x_f \) in minimum possible time. For an \( n \)th order system, the time optimal control is necessarily bang-bang switching between the extreme admissible values (\( \pm 1 \)) at most \( n - 1 \) times. We use these properties of the time optimal control to characterize the boundaries of the attainable set \( \mathcal{A}_p(t) \). The minimum time to reach the boundary \( \partial \mathcal{A}_p(t) \) from \( p \) is exactly \( t \). Any state transfer from \( p \) to \( \partial \mathcal{A}_p(t) \) will require bang-bang time optimal input \( \bar{u}(t) \in U \) with only one switch. Hence:

\[
\partial \mathcal{A}_p(t) = \left\{ x : x = e^{At}p + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \right\} \quad (2)
\]

The interior of \( \mathcal{A}_p(t) \) is denoted by \( \text{int}(\mathcal{A}_p(t)) \). Note that, for a given \( t \), the point \( p \) need not belong to \( \mathcal{A}_p(t) \). The structure of attainable set for one of the agents obeying (1) is shown in Figure 1.

**III. TIME OPTIMAL \( N \)-AGENT CONSENSUS**

All agents have identical dynamics as defined in (1). We associate with agent \( a_i \) an attainable set from the initial condition \( x_{i0} \) (of agent \( a_i \)) at time \( t \) and denote it by \( \mathcal{A}_i(t) \). Clearly, for consensus, we require that the intersection of the attainable sets for all the agents becomes non-empty at \( \bar{t} \), i.e., \( \bigcap_{1 \leq i \leq N} \mathcal{A}_i(\bar{t}) \neq \emptyset \). It seems that checking this condition directly would require computing solution of a large set of coupled polynomial equations and inequalities simultaneously (see Section IV-B). Moreover it is difficult to distribute this computation among the various agent’s on-board computers. However, since attainable sets \( \mathcal{A}_i(t) \) for all \( i = 1, \ldots, N \) are convex in \( \mathbb{R}^2 \), we use Helly’s theorem on intersection of convex sets, to parallelize this computation.

**Theorem 4.** (Helly’s Theorem)[17]Let \( F \) be a finite family of convex sets in \( \mathbb{R}^n \) with at least \( n+1 \) members. If every \( n+1 \) members of \( F \) have a point in common, then all members of \( F \) have a point in common.

Since, as \( t \to \infty \), \( \mathcal{A}_i(t) \to \mathbb{R}^2 \), we note that any two attainable sets \( \mathcal{A}_i(t) \) and \( \mathcal{A}_j(t) \) are guaranteed to intersect at some time \( 0 < t < \infty \). Now, consider any three agents \( \{a_i, a_j, a_k\} \) from (1), and let the minimum time to consensus for these triplet be denoted by \( \bar{t}_{ijk} \). Then, the minimum time to consensus \( \bar{t} \) for the \( N \)-agent system satisfies \( \bar{t} \geq \max_{1 \leq i,j,k \leq N} \bar{t}_{ijk} \). Now, by Helly’s theorem it follows that \( \bar{t} = \max_{1 \leq i,j,k \leq N} \bar{t}_{ijk} \) if and only if the attainable sets of all \( \binom{N}{3} \) triplets of agents have non-zero intersections at \( \bar{t} = \max_{1 \leq i,j,k \leq N} \bar{t}_{ijk} \). However, for some triplet of agents \( \{a_i, a_j, a_k\} \) it may so happen that \( \mathcal{A}_i(\bar{t}_{ijk}) \cap \mathcal{A}_j(\bar{t}_{ijk}) \cap \mathcal{A}_k(\bar{t}_{ijk}) = \emptyset \), but \( \mathcal{A}_i(t) \cap \mathcal{A}_j(t) \cap \mathcal{A}_k(t) = \emptyset \) for some \( t < \bar{t}_{ijk} \) in which case we have \( \bar{t} > \max_{1 \leq i,j,k \leq N} \bar{t}_{ijk} \). But, this situation is ruled out by using the following lemmas.

**Lemma 5.** For a pair of agents \( \{a_i, a_j\} \), if for some \( t' > 0 \), \( \mathcal{A}_i(t') \cap \mathcal{A}_j(t') = \emptyset \), then for all \( t > t' \), \( \mathcal{A}_i(t) \cap \mathcal{A}_j(t) = \emptyset \).

**Proof:** For some \( t' > 0 \), \( \mathcal{A}_i(t') = e^{At'}(x_{i0} + R_0(t')) \) where \( R_0(t) = \{ x : x = \int_0^t e^{-At}Bu(\tau)d\tau \} \). From [18], we have \( R_0(t_1) \subset R_0(t_2) \) if \( t_1 < t_2 \). Now, since \( \mathcal{A}_i(t') \cap \mathcal{A}_j(t') = \emptyset \) we have \( e^{At'}(x_{i0} + R_0(t')) \cap e^{At'}(x_{j0} + R_0(t')) = \emptyset \) Therefore, \( (x_{i0} + R_0(t')) \cap (x_{j0} + R_0(t')) = \emptyset \). Now for \( t > t' \), we have \( R_0(t') \subset R_0(t) \). Therefore, \( e^{At}(x_{i0} + R_0(t)) \cap e^{At}(x_{j0} + R_0(t)) = \emptyset \) and hence, \( \mathcal{A}_x_i(t) \cap \mathcal{A}_x_j(t) = \emptyset \).

From Lemma 5 we get the following.

**Lemma 6.** Consider three agents \( \{a_i, a_j, a_k\} \). If \( \mathcal{A}_i(t') \cap \mathcal{A}_j(t') \cap \mathcal{A}_k(t') = \emptyset \) for some \( t' \), then for all \( t > t' \), \( \mathcal{A}_i(t) \cap \mathcal{A}_j(t) \cap \mathcal{A}_k(t) = \emptyset \).
Fig. 2. Case 1- $\hat{x}_{ij} \in A_k(\bar{t}_{ij})$

Next theorem follows from lemma 6 and Helly’s theorem.

**Theorem 7.** For a multi-agent system described by (1), the minimum time to consensus is $\hat{t} = \max_{1 \leq i,j,k \leq N} \hat{t}_{ijk}$ and the consensus point $\hat{x} = \hat{x}_{ijk}$ is the minimum time consensus point of the triplet achieving the maximum.

**Remark 8.** From Helly’s theorem, it follows that the minimum time consensus point is unique for any two triplets $\{a_p, a_q, a_r\}$ and $\{a_e, a_f, a_g\}$ that satisfy $\hat{t}_{pqr} = \hat{t}_{efg} = \max_{i,j,k} \hat{t}_{ijk}$. Thus if $\hat{t}_{pqr} = \hat{t}_{efg}$ then $x_{pqr} = x_{efg}$.

**Remark 9.** Note that all the agents should reach the consensus point $\hat{x}$ exactly at time $\hat{t}$. Hence, to drive agents $a_i$ (for which $\hat{x} \in \partial A_i(\hat{t})$) to the computed consensus point, time optimal control $u_i(t)$ must be employed. On the other hand the input $u_j(t)$ used to drive agents $a_j$ (for which $\hat{x} \in \text{int}A_j(\hat{t})$) to the computed consensus point should be adjusted such that the state trajectory reaches $\hat{x}$ exactly at $\hat{t}$.

### IV. THREE AGENT CONSENSUS

Attainable sets of each of a triplet $\{a_i, a_j, a_k\}$ have a point in common if attainable set of every pair intersects. From Lemma 5, $\hat{t}_{ijk} \geq \max\{\hat{t}_{ij}, \hat{t}_{ik}, \hat{t}_{jk}\}$. Without loss of generality, assume $\hat{t}_{ij} = \max\{\hat{t}_{ij}, \hat{t}_{ik}, \hat{t}_{jk}\}$. We now have two cases: 1) $\hat{x}_{ij} \in A_k(\hat{t}_{ij})$ and 2) $\hat{x}_{ij} \notin A_k(\hat{t}_{ij})$. For each of these cases, we give a method to compute $t_{ijk}$.

**A. $\hat{x}_{ij} \in A_k(\hat{t}_{ij})$:**

This case is shown in Figure 2. There exists a control law $u_k(t) \in U$ which drives the states of $a_k$ from the initial condition $\bar{x}_{k0}$ to $\hat{x}_{ij}$. Hence, the consensus can be achieved by the 3-agent system at $\hat{t}_{ij}$ i.e. the equality $\hat{t}_{ijk} = \hat{t}_{ij} = \max\{\hat{t}_{ij}, \hat{t}_{ik}, \hat{t}_{jk}\}$ holds. The computation of $\hat{t}_{ij}$ for a pair of agents $\{a_i, a_j\}$ is done as follows:

1) **Time optimal consensus for two agent system:** Let the initial conditions of $a_i$ and $a_j$ be $x_{i0}$ and $x_{j0}$ respectively. For consensus we need, $|x_i(t) - x_j(t)| \to 0$ as $t \to \hat{t}_{ij}$ and $x_i(t) = x_j(t)$ for all $t > \hat{t}_{ij}$. At time $t$ the condition $x_i(t) = x_j(t)$ gives,

\[ e^{A_t}(x_{i0} - x_{j0}) + \int_0^t e^{A(t-\tau)}B(u_i(\tau) - u_j(\tau))d\tau = 0 \quad (3) \]

Consensus of two agent system is equivalent to driving the states of the difference system $\dot{z}(t) = A z(t) + B u_z(t)$ to origin where $z(t) = x_i(t) - x_j(t)$ and $u_z(t) = u_i(t) - u_j(t)$. As $|u_1(t)| \leq 1$ and $|u_j(t)| \leq 1$, $|u_z(t)| \leq 2$. The time optimal control is necessarily bang-bang switching between $\pm 2$ at most one switch. This implies, $u_i(t)$ and $u_j(t)$ should be bang-bang switching between $\pm 1$ with at most one switch i.e. both the agents should employ time optimal control for reaching consensus in minimum possible time.

Since, $u_i(t) = u_j(t) - u_j(t)$, it follows that $u_i(t) = -u_j(t)$. Moreover, the minimum time for consensus $(\hat{t}_{ij})$ of the two agents $\{a_i, a_j\}$ is also the minimum time required for driving the states of the difference system from $z(0)$ to origin and hence, is fixed for the given initial condition. Due to uniqueness of time optimal control, $u_i(t)$ and hence $u_i(t)$ and $u_j(t)$ are unique. Therefore, $x_i(\hat{t}_{ij})$ is a unique point.

We denote this point by $\bar{x}_{ij}$.

Consider system (3). Let the time optimal input that drives the states $z(t) = x_i(t) - x_j(t)$ from $z(0) = x_{i0} - x_{j0}$ to origin, switch at $\hat{t}_s$. But, the initial value of input $u_2(0) = \pm 2$ is unknown. Therefore, if $u_2(t) = +2$ for $0 \leq t \leq t^*_+ \text{ and } u_2(t) = -2 \text{ for } t^*_+ < t \leq \hat{t}_{ij}$ we get, $e^{A t}(x_0-x_0) + 2(\int_0^{t^*_+} - \int_0^{t^*_2}) e^{A (t^*_2-t)} B d\tau = 0$. By this, we obtain $t^*_+ = \frac{(v_0-v_0) + 2t^*_2}{4}$ and $t^*_2 = \frac{-v_0-v_0 + \sqrt{2(v_0-v_0)^2 - 8(x_0-x_0)}}{4}$

where $x_0 = [r_0, v_0]^T$ and $x_0 = [r_0, v_0]^T$. Similarly, if $u_2(t) = -2$ for $0 \leq t \leq t^*_+$ and $u_2(t) = +2$ for $t^*_+ < t \leq \hat{t}_{ij}$, we compute the explicit formula for $t^*_+$ and $\hat{t}_{ij}$. The correct choice of $u_2(0)$ is obtained by checking the inequality $0 \leq t^*_+ \leq \hat{t}_{ij}$. If this inequality holds then $\bar{x}_{ij} = \bar{x}_{ij}^+ = \hat{x}_{ij}$ and $t_s = \hat{t}_{ij}$, otherwise $\bar{x}_{ij} = \bar{x}_{ij}^-$ and $t_s = t_s^*$. Now, by substituting the computed values, we write $\bar{x}_{ij} = \left[ \begin{array}{c} r_0 + \bar{x}_{ij}^+ v_0 + 2t_{\bar{x}_{ij}}^- v_0 + 2v_0 - \bar{x}_{ij} \end{array} \right]$ where $x_0 = [r_0, v_0]^T$ if $u_i(0) = 1$ and $x_0 = x_{j0}$ if $u_j(0) = 1$.

2) **Whether $\bar{x}_{ij} \in A_k(\hat{t}_{ij})$** : We compute $\partial A_k(\hat{t}_{ij})$ as follows. Recall the definition of $A_k(t)$ from definition 2. For any given initial condition $p$ and time $t$, any point $x(t) \in \partial A_k(t)$ satisfies (2) $x(t) = e^{A t} p + \int_0^t e^{A(t-\tau)} B d\tau + \int_{\tau^*}^t e^{A(t-\tau)} B d\tau$ for some $t^* \leq t$. Solving this expression for initial sequence $u(\tau) = 1$ for $0 \leq \tau \leq t_s$ and $u(\tau) = -1$ for $t_s < \tau \leq t$ we get, $t_s = t_s^* = \frac{(v_0-v_0) + 2t^*_2}{4}$ and $\hat{t}_{ij} = \left( \frac{(v_0-v_0)^2}{4} + 2t^*_2 + 2t(v_0+v_0) + 4p_1-t_1) \right) = 0 \ (4)$

Similarly, for input sequence $u(\tau) = -1$ for $0 \leq \tau \leq t_s$ and $u(\tau) = 1$ for $t_s < \tau \leq t$, we have $t_s = t_s^- = \frac{(p_2-v_0-t_1)}{2}$ and $\hat{t}_{ij} = \left( \frac{(v_0-v_0)^2}{4} - t^*_2 + 2t(v_0+v_0) + 4p_1-t_1) \right) = 0 \ (5)$

The states $x(t) = [r(t), v(t)]^T$ that satisfy either (4) or (5) form $\partial A_k(t)$ if corresponding value of $t_s$ satisfies $0 \leq t_s \leq t$. We define the following sets:

\[ \partial A_{k+}(t) := \{ x(t) : x(t) \text{ satisfies (4)} \text{ with } 0 \leq t_s^+ \leq t \} \]

\[ \partial A_{k-}(t) := \{ x(t) : x(t) \text{ satisfies (5)} \text{ with } 0 \leq t_s^- \leq t \} \]

Then, $\partial A_k(t) = \partial A_{k+}(t) \cup \partial A_{k-}(t)$. The attainable set $A_k(t)$ is enclosed by (4) and (5). The point $e^{A t} p \in$
Fig. 3. Case 2- $\bar{x}_{ij} \notin A_k(\bar{t}_{ij})$

Fig. 4. Minimum time consensus point for the case-2

int($A_p(t)$) for all $p$ and $t$ since this point is reached from $p$ using 0 input. At $e^{At}p$, LHS of Eq.(4) evaluates to $t^2 > 0$ and LHS of Eq.(5) evaluates to $-t^2 < 0$. Hence, for a point $x(t) = [x(t) \ v(t)]^T \in \text{int}(A_p(t))$, 

$$\text{poly}(\partial A_{p-}(t)) > 0 \ & \text{poly}(\partial A_{p-}(t)) < 0 \quad (6)$$

where $\text{poly}(\partial A_{p+}(t))$ are the polynomials on LHS of Eq.(4) and (5) respectively. For a point on $\partial A_{p+}(t)$, (4) is satisfied and $\text{poly}(\partial A_{p-}(t)) < 0$. Similarly, for a point on $\partial A_{p-}(t)$, (5) is satisfied and $\text{poly}(\partial A_{p+}(t)) > 0$. If both of the (4) and (5) are satisfied, then, the state trajectory can be driven from $p$ to $x(t)$ with constant input (either 1 or -1). Such a point $x(t) \in \partial A_{p+}(t)$ with $t^*_+ = 0$ and $x(t) \in \partial A_{p-}(t)$ with $t^*_- = t$ or vice versa. These conditions allow us to check if a point is contained in $A_p(t)$.

By substituting the values of $\bar{x}_{ij}$, $x_{k0}$ and $\bar{t}_{ij}$ in (6), we check whether $\bar{x}_{ij} \in A_k(\bar{t}_{ij})$. If $\bar{x}_{ij} \in A_k(\bar{t}_{ij})$, then $\bar{t}_{ij} = \bar{t}_{ij}$. Otherwise we have $\bar{x}_{ij} \notin A_k(\bar{t}_{ij})$, which is treated next.

B. $\bar{x}_{ij} \notin A_k(\bar{t}_{ij})$:

This case is shown in Figure 3. In this case, for each pair of agents, the corresponding attainable sets intersect at time $\bar{t}_{ij} = \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$. However, $A_i(\bar{t}_{ij}) \cap A_j(\bar{t}_{ij}) \cap A_k(\bar{t}_{ij}) = \phi$. We need to find the minimum time $\bar{t}_{ijk}$ such that $A_i(\bar{t}_{ijk}) \cap A_j(\bar{t}_{ijk}) \cap A_k(\bar{t}_{ijk}) \neq \phi$. This scenario is shown in Figure 4. In such a situation, we compute $\bar{t}_{ijk}$ as follows below:

1) Time optimal consensus point for 3-agent system: Consider a 3-agent system composed of $\{a_i, a_j, a_k\}$. Let $x_{a0}$, $x_{j0}$ and $x_{k0}$ be the respective initial conditions. Let $\bar{t}_{ijk} > \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$ be the minimum time to consensus for this 3-agent system. Note that, at $\bar{t}_{ij} = \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$, $A_i(\bar{t}_{ij}) \cap A_j(\bar{t}_{ij}) = \bar{x}_{ij}$. For time $t > \bar{t}_{ij}$, the boundary of the set $A_i(t) \cap A_j(t)$ i.e. $\partial (A_i(t) \cap A_j(t))$ is composed of three disjoint parts namely $\partial A_i(t) \cap \text{int}(A_j(t))$, $\partial A_j(t) \cap \text{int}(A_i(t))$ and $\partial A_i(t) \cap \partial A_j(t)$. Now, $\bar{t}_{ijk}$ is the first instant $t$ at which $A_k(t)$ and $A_i(t) \cap A_j(t)$ intersect. As both the sets are convex with boundaries determined by quadratic curves in $\mathbb{R}^2$ (see (4) and (5)), at time $\bar{t}_{ijk}$, these sets meet each other at a unique point (see Figure 4).

Claim 10. $A_i(\bar{t}_{ijk}) \cap A_j(\bar{t}_{ijk}) \cap A_k(\bar{t}_{ijk}) = \partial A_i(\bar{t}_{ijk}) \cap \partial A_j(\bar{t}_{ijk}) \cap \partial A_k(\bar{t}_{ijk})$.

Proof: $\partial (A_i(t) \cap A_j(t))$ is composed of three disjoint parts namely $\partial A_i(t) \cap \text{int}(A_j(t))$, $\partial A_j(t) \cap \text{int}(A_i(t))$ and $\partial A_i(t) \cap \partial A_j(t)$ for $t > \bar{t}_{ij}$. At $\bar{t}_{ijk}$, $\partial A_k(\bar{t}_{ijk})$ meets $\partial (A_i(\bar{t}_{ijk}) \cap A_j(\bar{t}_{ijk}))$ at a unique point say $\bar{x}_{ijk}$. Hence exactly one of the following holds:

1) $\partial A_k(\bar{t}_{ijk}) \cap \partial A_i(\bar{t}_{ijk}) \cap \text{int}(A_j(\bar{t}_{ijk})) = \{\bar{x}_{ijk}\}$
2) $\partial A_k(\bar{t}_{ijk}) \cap \partial A_j(\bar{t}_{ijk}) \cap \text{int}(A_i(\bar{t}_{ijk})) = \{\bar{x}_{ijk}\}$
3) $\partial A_k(\bar{t}_{ijk}) \cap \partial A_i(\bar{t}_{ijk}) \cap \partial A_j(\bar{t}_{ijk}) = \{\bar{x}_{ijk}\}$

Suppose (1) is true. Then $\partial A_i(\bar{t}_{ijk}) \cap \partial A_j(\bar{t}_{ijk}) \cap \text{int}(A_k(\bar{t}_{ijk})) = \{\bar{x}_{ijk}\}$. This implies that intersection of $\partial A_k(\bar{t}_{ijk})$ and $\partial A_j(\bar{t}_{ijk})$ is a singleton set containing $\bar{x}_{ijk}$, otherwise due to continuity of $\partial A_k(\bar{t}_{ijk})$ and convexity of $A_i(\bar{t}_{ijk}) \cap A_j(\bar{t}_{ijk})$, there exists another point on $\partial A_k(\bar{t}_{ijk})$ in the neighbourhood of $\bar{x}_{ijk}$ which lies in the interior of $A_i(\bar{t}_{ijk}) \cap A_j(\bar{t}_{ijk})$ which contradicts to the fact that $\bar{x}_{ijk}$ is unique point of intersection of $A_k(\bar{t}_{ijk})$ and $A_i(\bar{t}_{ijk})$, $\partial A_k(\bar{t}_{ijk}) \cup \partial A_i(\bar{t}_{ijk}) = \{\bar{x}_{ijk}\}$, the point $\bar{x}_{ijk}$ is the first intersection point of $A_i(\bar{t}_{ijk})$ and $A_k(\bar{t}_{ijk})$. That is $\bar{t}_{ijk}$ is the minimum time to consensus for $a_i$ and $a_k$ agent. As $\bar{t}_{ijk} > \bar{t}_{ij} = \max\{\bar{t}_{ij}, \bar{t}_{jk}, \bar{t}_{ik}\}$ gives a contradiction. So (1) cannot be true. Similarly it can be shown that (2) is not true. Which implies (3) has to be true, justifying the claim.

2) Computation of $\bar{t}_{ijk}$: Now, $\bar{x}_{ijk} = \partial A_i(\bar{t}_{ijk}) \cap \partial A_j(\bar{t}_{ijk}) \cap \partial A_k(\bar{t}_{ijk})$. Hence all the three agents must employ time optimal control to reach $\bar{x}_{ijk}$ at time $\bar{t}_{ijk}$. For $\alpha = i, j, k$, either $\bar{x}_{ijk} \in \partial A_{\alpha+}(\bar{t}_{ijk})$ or $\bar{x}_{ijk} \in \partial A_{\alpha-}(\bar{t}_{ijk})$ depending on whether the initial value of time optimal input $u_{\alpha}(t)$ is +1 or -1. Note that, $u_{\alpha}(0)$, $\alpha = i, j, k$ cannot have the same value, as $\partial A_{\alpha+}(\bar{t}_{ijk}) \cap \partial A_{\alpha+}(\bar{t}_{ijk}) \cap \partial A_{\alpha-}(\bar{t}_{ijk})$ or $\partial A_{\alpha-}(\bar{t}_{ijk}) \cap \partial A_{\alpha-}(\bar{t}_{ijk}) \cap \partial A_{\alpha-}(\bar{t}_{ijk})$ does not satisfy $0 \leq t_{\alpha} < \bar{t}_{ijk}$ (where $t_{\alpha}$ is the switching instant for agent $\alpha$) for any $\alpha$ and $\bar{t}_{ijk}$. Hence, depending on the values of $u_{\alpha}(0)$, $\bar{x}_{ijk}$ can be one of the following:

1) $\partial A_{+}(\bar{t}_{ijk}) \cap \partial A_{-}(\bar{t}_{ijk}) \cap \partial A_{+}(\bar{t}_{ijk})$
2) $\partial A_{+}(\bar{t}_{ijk}) \cap \partial A_{+}(\bar{t}_{ijk}) \cap \partial A_{-}(\bar{t}_{ijk})$
3) $\partial A_{-}(\bar{t}_{ijk}) \cap \partial A_{+}(\bar{t}_{ijk}) \cap \partial A_{+}(\bar{t}_{ijk})$
4) $\partial A_{-}(\bar{t}_{ijk}) \cap \partial A_{+}(\bar{t}_{ijk}) \cap \partial A_{-}(\bar{t}_{ijk})$
5) $\partial A_{-}(\bar{t}_{ijk}) \cap \partial A_{-}(\bar{t}_{ijk}) \cap \partial A_{+}(\bar{t}_{ijk})$
6) $\partial A_{-}(\bar{t}_{ijk}) \cap \partial A_{-}(\bar{t}_{ijk}) \cap \partial A_{-}(\bar{t}_{ijk})$

Figure 4 shows the case where $\bar{x}_{ijk} = \partial A_{+}(\bar{t}_{ijk}) \cap \partial A_{-}(\bar{t}_{ijk}) \cap \partial A_{+}(\bar{t}_{ijk})$. For $\alpha = i, j, k$, the expression for $\partial A_{+}(\bar{t}_{ijk})$ and $\partial A_{-}(\bar{t}_{ijk})$ can be obtained by plugging in the value of $p = x_{a0}$ in (4) and (5) respectively. For each of the six cases mentioned above, we have three equations, one
corresponding to each agent of \( \{a_i, a_j, a_k\} \). Solving these three equations for each case, we obtain six closed form formulas for \( t \) in terms of \( x_{i0}, x_{j0} \) and \( x_{k0} \). We denote these values of \( t \) in terms of \( x_{i0}, x_{j0} \) and \( x_{k0} \) by \( t_1, \cdots, t_6 \) respectively. For \( t_m, m = 1, \cdots, 6 \), corresponding values of \( x(t_m) \) are also obtained by solving the three equations related to that case. The formula for \( x(t_1) \) is:

\[
\begin{bmatrix}
x(t_1) \\
v(t_1)
\end{bmatrix} = \begin{bmatrix}
-\frac{v(t_1)^2 - 2(v(t_1) + t_1)(v(t_1) - v_{k0})}{v(t_1) - v_{k0} - 4v_{k0}} \\
-t_1 + \frac{v(t_1)^2 - v_{k0}^2 - 4v_{k0}v_{k0}}{2(v(t_1) - v_{k0})}
\end{bmatrix}
\]

Similarly, formulas for \( x(t_2), \cdots, x(t_6) \) are obtained.

From \( t_m \) and \( x(t_m) \), the expression for the switching instant, say \( t_{sa} \), \( \alpha = i, j, k \) is obtained as follows:

For \( \alpha = i, j, k \), if \( \partial A_{\alpha_+}(t) \) is used, then the switching instant of agent \( \alpha_+ \) for \( m^{th} \) case \( t_{sa, m} = \frac{v(t_m) - v_{a_0} + v_{t_m}}{2} \) and if \( \partial A_{\alpha_-}(t) \) is used, then \( t_{sa, m} = \frac{v(t_m) - v_{a_0} - v_{t_m}}{2} \). Now, the \( m \) for which \( 0 \leq t_{sa,m} \leq t_m \) is satisfied for all \( \alpha = i, j, k \) gives \( \bar{t}_{ij} = t_m \). The corresponding value of \( x(t_m) \) is \( \bar{x}_{ij} \).

Remark 11. If the consensus is to be achieved at a stationary point i.e. on the set \( X_0 = \{x \mid r = 0 \} \), the intersection of attainable sets of every pair of agents should intersect with \( X_0 \). Computation of minimum time to consensus at stationary point can be done identically as described above.

Computation of \( \bar{t}_{ij} \) and \( \bar{x}_{ij} \) for each triplet \( \{a_i, a_j, a_k\} \) is independent. Also since the communication graph is complete, each agent can compute these values for \( \frac{1}{N} \) such systems. After performing computations of \( \bar{t}_{ij} \) and \( \bar{x}_{ij} \) for all the triplets, the agents broadcast the computed data. Then they choose the maximum value of \( \bar{t}_{ij} \) over all the triplets as the time to consensus.

Remark 12. For each triplet, we require a total of 691 floating point operations to compute \( \bar{t}_{ij} \) and \( \bar{x}_{ij} \). It is easy to verify that modern embedded processors, with typically gigaflops performance, can perform \( \frac{1}{N} \times 691 \) floating point operations in milliseconds for up to 100 agent groups.

V. COMPUTATION OF FEEDBACK CONTROL LAWS

As discussed in Section III, once the minimum time consensus point \( \bar{x} = [\bar{x} \ \bar{v}]^T \) and corresponding time to consensus \( \bar{t} \) are computed, for each agent \( a_i \) we need to find a feedback control strategy \( u_i(t) \) which will drive the states from \( x_{i0} \) to \( \bar{x} \) exactly at \( \bar{t} \). If \( \bar{x} \in \partial A_i(\bar{t}) \), then \( \bar{t} \) is the minimum time required for transferring states from \( x_{i0} \) to \( \bar{x} \) using admissible control i.e. \( u_i(t) \in U \), then \( u_i(t) \) has to be time optimal control. On the other hand \( \bar{x} \in \text{int}(A_i(\bar{t})) \), then \( u_i(t) \) cannot be time optimal control. We can check whether \( \bar{x} \) is in the interior of the attainable set or is on the boundary as explained in Section II-B.

A. Feedback control for state-transfer in time \( \bar{t} \)

We use following feedback strategy to drive the state of an agent \( a_i \), for which \( \bar{x} \in \text{int}(A_i(\bar{t})) \), to \( \bar{x} \) in time \( \bar{t} \). We adjust the input bounds to a lower value say \( w \) and employ time optimal control with these revised input constraints. For given initial state \( x_{i0} \), final state \( \bar{x} \) and the input constraint \( U_i = \{u_i(t) : |u_i(t)| \leq w \} \), there exists a unique time optimal input that drives the states from \( x_{i0} \) to \( \bar{x} \) in minimum possible time. Given \( \bar{t} \), we compute the revised input bound \( w < 1 \) as follows. From the solution of (1) and the condition that the input is bang-bang with at most one switch we get, \( \bar{x} = e^{\hat{A}\bar{t}}(x_{i0} + w(\int_{t_0}^{\hat{t}} e^{\hat{A}\tau} B\bar{d}\tau + \int_{\hat{t}}^{t_0} e^{\hat{A}\tau} B\bar{d}\tau)) \).

Solving the above equation for input sequence \( u_i(t) = w \) in \( 0 \leq t < t_s \) and \( u_i(t) = -w \) in \( t_s < t \leq \bar{t} \), we get

\[
t_s = \frac{w - y_1}{2y_0} \quad \text{and} \quad w = \frac{-w + 2y_0 + \sqrt{(y_0 + 2y_1)^2 + (y_0 + y_1)^2}}{2y_0}.
\]

Similarly for input sequence \( u_i(t) = -w \) in \( 0 \leq t < t_s \) and \( u_i(t) = w \) in \( t_s < t \leq \bar{t} \), we get

\[
t_s = \frac{w + y_1}{2y_0} \quad \text{and} \quad w = \frac{-w + 2y_0 + \sqrt{(y_0 + 2y_1)^2 + (y_0 + y_1)^2}}{2y_0}.
\]

For given \( x_{i0}, \bar{x} \) and \( \bar{t} \), we compute the two values of \( w \) (say \( w^+ \) and \( w^- \)) with corresponding values of \( t_s \) (say \( t_{s}^+ \) and \( t_{s}^- \)). The value \( (w_+ \text{ or } w_-) \) that satisfies \( 0 < w \leq 1 \) and \( 0 \leq t_s \leq \bar{t} \) is chosen as the new bound on the admissible control. Once this value is determined, use time optimal control with revised bounds to drive the states from \( x_{i0} \) to \( \bar{x} \).

VI. MINIMUM TIME CONSENSUS FOR SINGLE INTEGRATORS

Consider multi-agent systems with agent dynamics \( \dot{x}_i(t) = u_i(t) \), \( |u_i(t)| \leq 1 \) for \( i = 1, \cdots, N \). For each agent we have \( x_i(t) = x_{i0} + \int_0^t u(\tau)\,d\tau \). The time optimal input for each agent is a constant with value either 1 or -1 depending on the initial and final condition i.e., \( x_i(t) = x_{i0} \pm t \). Then, the minimum time to consensus between two agents \( a_i \) and \( a_j \) is \( \bar{t}_{ij} = \frac{1}{2}|x_{i0} - x_{j0}| \).

Here, the attainable set from a point \( p \in \mathbb{R} \) at time \( t \), \( A_p(t) \) is a closed interval \( [p - t, p + t] \subset \mathbb{R} \). By Helly's theorem for a finite collection of intervals in \( \mathbb{R} \), if every pair of the intervals intersect, then all intervals intersect. Hence, the minimum time consensus point for all agents is the consensus point of those agents say \( a \) and \( b \) for which \( \bar{t}_{ab} = \max_{ij} \bar{t}_{ij} = \max_{ij} \frac{|x_{a0} - x_{b0}|}{2} \). Without loss of generality, we can assume \( x_{i0} > x_{a0} \). For all agents, the input \( u_i(t) = \text{sign}(\bar{x} - x_i(t)) \) drives the agent state to \( \bar{x} \) at time \( t \leq \bar{t}_{ab} \). The state-trajectory of agents \( a \) and \( b \) reaches \( \bar{x} \) at time \( t = \bar{t}_{ab} \). For remaining agents the state-trajectory reaches \( \bar{x} \) at time \( t < \bar{t}_{ab} \) and stays at \( \bar{x} \) for all future time. However, for agents having first order dynamics, prior computation of the minimum time consensus point \( \bar{x} \) and corresponding time to consensus \( \bar{t} \) is not required.

Given a multi-agent system with agents communicating over any connected undirected graph, a feedback control law is required that ensures \( u_{a_0}(t) = 1 \) and \( u_{a_1}(t) = -1 \) for time \( 0 \leq t \leq \bar{t} \). At time \( \bar{t} \), we need \( x_{a_0}(t) = x_{b}(t) = \bar{x} \) for agents \( a \) and \( b \) as mentioned in above paragraph. For remaining agents \( \alpha, u_{a_1}(t) \) must be such that, \( x_{a_1}(t) = \bar{x} \). For these remaining agents one has a freedom to choose non-optimal control laws.

We may as well use \( u_i(t) = \text{sign}(\sum_{j \in N_i}(x_j - x_i)) \), where
\( N_i \) is the set of neighbours of agent \( a_i \) in the communication graph. Now, the agent \( a \) does not have any neighbour such that \( x_{i0} < x_{a0} \). Similarly, the agent \( b \) does not have any neighbour such that \( x_{i0} > x_{a0} \). Hence \( u_a(0) = -u_b(0) = 1 \).
The inputs \( u_a(t) \) and \( u_b(t) \) do not become 0 or change sign unless for all their neighbours \( x_i(t) = x_a(t) \) and \( x_i(t) = x_b(t) \) respectively. If for some neighbour \( a_i \) of \( a \), \( x_i(t') = x_a(t') \) for some time \( t' \) and \( a \) is the only neighbour of \( a_i \), then \( u_i(t) = u_a(t) \) for \( t > t' \). If \( a_i \) has some other neighbours other than \( a \), then \( u_i(t) = 1 \) for \( u_a(t) \) because for such a neighbour \( a_j \) (other than \( a \)), we have \( x_i(t) > x_i(t) \). Similar arguments are valid for neighbours of \( b \). Since, the graph is connected, these arguments are valid, until, for all \( i \), \( x_i(t) \) are equal and consensus is achieved. Let us denote by \( L \) the Laplacian matrix of the communication graph. Then a combined system for all interactions is written as \( \dot{x}(t) = -\text{sign}(Lx(t)) \) where \( x(t) = [x_1(t), x_2(t), \ldots, x_N(t)]^T \).

This system achieves consensus in minimum possible time \( \bar{t} \) at \( \bar{x} = \max_{ij} \frac{(x_{i0}+x_{j0})}{2} \).

**VII. Example**

**Example 13.** Consider a multiagent system with 6 agents \( \{a_1, \ldots, a_6\} \). The initial conditions of these agents are:

\[
\begin{align*}
{x_{10}} &= \begin{bmatrix} -2.08 & 1.08 \end{bmatrix}^T, \\
{x_{20}} &= \begin{bmatrix} -1.1 & 0 \end{bmatrix}^T, \\
{x_{30}} &= \begin{bmatrix} -1 & -1 \end{bmatrix}^T, \\
{x_{40}} &= \begin{bmatrix} -1 & 0.25 \end{bmatrix}^T, \\
{x_{50}} &= \begin{bmatrix} -1.5 & -0.5 \end{bmatrix}^T \\
{x_{60}} &= \begin{bmatrix} -2.5 & 1 \end{bmatrix}^T
\end{align*}
\]

There are \( \binom{6}{3} = 20 \) possible triplets of the agents.

Out of these 20 triplets, the minimum time to consensus is highest for \( \{a_3, a_4, a_6\} \) i.e. \( t_{346} = \max_{1 \leq i,j,k \leq 6} t_{ijk} = 1.5625 \). Hence, the minimum time to consensus for the multiagent system \( \bar{t} = t_{346} = 1.5625 \) and the minimum time consensus point \( \bar{x} = \frac{1}{3} [x_{30} + x_{40} + x_{60}] \).

The minimum time consensus point \( \bar{x} \in \text{int}A(\bar{t}) \) for \( i = 1, 2, 5 \), while for \( i = 3, 4, 6 \), \( \bar{x} \in \partial A(\bar{t}) \). The agents \( a_3, a_4 \) and \( a_6 \) have to use time optimal control for reaching \( \bar{x} \) at \( \bar{t} \). For remaining agents, non-time optimal controls are required for reaching \( \bar{x} \) at \( \bar{t} \).

**VIII. Conclusion**

Assuming complete communication graph, a method to compute the minimum time to consensus and the corresponding consensus point for a 3-agent system with second order agents is given. It is shown that the maximum of these times over all triplets of the agents in a multiagent system is the minimum time to consensus for that system. Further the consensus point of the corresponding triplet is the consensus point of the multiagent system. Once the minimum time to consensus and corresponding consensus point are computed, feedback strategies are proposed to drive each agent to the consensus point at the minimum time to consensus. Equivalent formulation for multiagent system with single integrator agents is discussed and it is shown that for this case, assumption of complete graph as well as explicit computation of minimum time to consensus are not required.

**REFERENCES**


