Admissibility condition on a wavelet

Continuous wavelet transform (CWT) of a function $x(t)$ w.r.t. $\psi(t)$ is given by:

$$\text{WT}_\psi\{x; b, a\} = C \int x(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt$$

where, $a$ and $b$ are dilation and translation parameters and $C$ is the normalizing factor such that,

$$C \int \left| \psi\left(\frac{t-b}{a}\right) \right|^2 dt \quad \text{has unit energy}$$

$$\Rightarrow \quad C = \frac{1}{\sqrt{a}}$$

Hence the CWT is expressed as:

$$W_\psi x(b, a) = \frac{1}{\sqrt{a}} \int x(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt$$

**Theorem:** The wavelet transform obeys a Parseval theorem like relationship.

$$\int \int W_\psi x(b, a) \overline{W_\psi y(b, a)} db \frac{da}{a^2} = C_\psi \int x(t) y(t) dt$$

**Proof:**

First let us consider the single integral of LHS with w.r.t. $b$ which can be transformed to w-Domain as:

$$\int W_\psi x(b, a) \overline{W_\psi y(b, a)} db = \frac{1}{2\pi} \int \hat{W_\psi x}(b, a) \overline{\hat{W_\psi y}(b, a)} d\omega$$

which is defined as

$$W_\psi x(b, a) = \frac{1}{\sqrt{a}} \int x(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt$$

ie. Transformation w.r.t $b$: 

$$= \frac{1}{\sqrt{a}} \int e^{-j\omega b} \left\{ \int x(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt \right\} db$$
Exchanging \(db\) and \(dt\):

\[
= \frac{1}{\sqrt{\alpha}} \int x(t) \left( \int \psi \left( \frac{t-b}{a} \right) e^{-jat} dt \right) dt \tag{2}
\]

Substitute \(\lambda = \frac{t-b}{a}\); \(t = a\lambda + b\); \(db = -ad\lambda\) in

\[
\int \psi \left( \frac{t-b}{a} \right) e^{-jat} dt
\]

to get

\[
= |a| \int \psi(\lambda) e^{-jat} e^{-jat} e^{-jat}\lambda dt
\]

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= |a| \int \psi(\lambda) e^{-jat} e^{-jat} e^{-jat}\lambda dt
\]

\[
= |a| \int \psi(\lambda) e^{-jat} e^{-jat} e^{-jat}\lambda dt
\]

\[
= |a| e^{-jat} \psi(a\omega) \tag{3}
\]

Substituting (3) back in (2)

\[
\therefore \mathcal{W}_y(x(b,a)) = \frac{1}{|a|^{\frac{1}{2}}} \int x(t) |a| e^{-jat} \psi(a\omega) dt
\]

\[
= |a|^{\frac{1}{2}} \left\{ \int x(t) e^{-jat} dt \right\} \psi(a\omega)
\]

\[
= |a|^{\frac{1}{2}} x(\omega) \psi(a\omega) \tag{4}
\]

Substituting (4) back in (1)

\[
\therefore \int \mathcal{W}_y(x(b,a)) \mathcal{W}_y(y(b,a)) db = \frac{1}{2\pi} \left[ |a|^{\frac{1}{2}} x(\omega) \psi(a\omega) \right] \left[ |a|^{\frac{1}{2}} y(\omega) \psi(a\omega) \right] d\omega
\]

\[
= \frac{1}{2\pi} \left[ \int a x(\omega) y(\omega) \psi(a\omega)^2 \right] d\omega \tag{5}
\]
\[ LHS = \int \int W_\psi x(b, a) \overline{W_\psi y(b, a)} db \frac{da}{a^2} = \frac{1}{2\pi} \int \int |x(\omega) \overline{y(\omega)}| \psi(\hat{a} \omega) \left( \frac{\hat{a}}{a} \right)^2 d\omega \frac{da}{a^2} \]

\[ = \frac{1}{2\pi} \int \int x(\omega) \overline{y(\omega)} \left| \psi(\hat{a} \omega) \right|^2 \frac{da}{|a|} d\omega \]

Substituting \( aw = \lambda \); we get \( \frac{da}{a} = \frac{d\lambda}{\lambda} \) and

\[ \int \int W_\psi x(b, a) \overline{W_\psi y(b, a)} db \frac{da}{a^2} = \frac{1}{2\pi} \left\{ \int \int \left| x(\omega) \overline{y(\omega)} \right| d\omega \right\} \left\{ \int \left| \psi(\hat{a} \omega) \right|^2 \frac{d\lambda}{\lambda} \right\} \]

for the RHS of the above integral to converge \( \int_0^\infty |\hat{\psi}(\lambda)| \frac{d\lambda}{\lambda} \) must converge say to a positive finite constant \( C_\psi \)

\[ \Rightarrow LHS = C_\psi \left\{ \frac{1}{2\pi} \int \hat{x}(\omega) \overline{\hat{y}(\omega)} d\omega \right\} \]

\[ = C_\psi \int x(t) y(t) dt \quad \text{(RHS)} \quad \text{(Using Parseval’s Theorem)} \]

Hence Proved

**Example:** Checking admissibility condition for Haar MRA:

Expression for Haar wavelet: 
\[ |\hat{\psi}(\omega)| = \sin^2 \frac{\omega}{4} \]

Applying admissibility condition: 
\[ \int_0^\infty |\hat{\psi}(\omega)| \frac{d\omega}{\omega} = \int_0^\infty \frac{\sin^4 \left( \frac{\omega}{4} \right) d\omega}{\omega} \]

The integral can be expressed as: 
\[ \int_0^\infty = \int_0^\delta + \int_\delta^\infty \]
For integral
\[
\int_0^\infty \left| \frac{\sin^2 \frac{\lambda}{4}}{\frac{\lambda}{4}} \right|^2 \frac{d\lambda}{\lambda} \leq \int_0^\infty \left| \frac{\lambda}{4} \right|^2 \frac{d\lambda}{\lambda} = \int_0^\infty \frac{\lambda}{16} d\lambda (\because \sin \lambda \leq \lambda)
\]
which converges as \( \delta \to 0 \).

Also the integral
\[
\int_\delta^\infty \left| \frac{\sin^2 \frac{\lambda}{4}}{\frac{\lambda}{4}} \right|^2 \frac{d\lambda}{\lambda} < \int_\delta^\infty \frac{16}{\lambda^3} (\because \sin \lambda < 1)
\]
therefore as \( \delta \to \infty \) the integral converges.

Hence Haar wavelet satisfies admissibility condition.

Inverse Continuous Wavelet Transform (ICWT)

From Parseval theorem, to get inverse wavelet transform choose \( y(t) \) to be an unit area narrow pulse around \( t=t_0 \) (or as impulse)
\[
x(t_0) = K_\psi \int \int W_{\psi} x(b, a) \psi \left( \frac{t_0 - b}{a} \right) db da
\]
where again, \( K_\psi = \int |\hat{\psi}(\lambda)|^2 \frac{d\lambda}{\lambda} \); a constant

on the condition that \( \int |\hat{\psi}(\omega)|^2 \frac{d\omega}{\omega} \) converges ie. ADMISSIONSIBILITY CONDITION.

It is because of admissibility, that the CWT is invertible.

If spectrum of \( \psi(t) \) i.e. \( \hat{\psi}(\omega) \) does not \( \to 0 \) as \( \omega \to 0 \), then \( \int |\hat{\psi}(\omega)|^2 \frac{d\omega}{\omega} \) diverges.

Therefore \( \hat{\psi}(\omega) \) must decay atleast as fast as \( \omega \), which says if \( \psi(\cdot) \) is a band pass function then \( \int |\hat{\psi}(\omega)|^2 \frac{d\omega}{\omega} \) converges.

The transform we have seen so far are redundant in nature. So now we go for discretization of scale and translation parameter.
1.1 Discretization of scale parameter $a$:

Consider
\[
\int |\hat{\psi}(\lambda)|^2 \frac{d\lambda}{\lambda}
\]

Put $\lambda = e^{-\nu}$
\[
\int_{-\infty}^{\infty} \psi(-e^{-\nu})^2 d\nu
\]
This quantity is Fourier transform of some autocorrelation sequence.

For $\nu \rightarrow \nu + \nu_0$: $e^\nu \rightarrow e^\nu e^{\nu_0}$ i.e. for uniform movement on $\nu$ there is exponential movement in $\lambda$. This is logarithmic discretization. This means that the no. of points of discretization between $a = 1$ and $a = 10$ should be same as between $a = 10$ and $a = 100$ where ‘$a$’ is the scaling parameter.

Now choosing $a = a_0^m$ ($\infty > a_0 > 0$)

we get
\[
\int_{0}^{\infty} |\hat{\psi}(w)|^2 \frac{dw}{w} = \sum_{m = -\infty}^{+\infty} a_0^m \int |\hat{\psi}(w)|^2 \frac{dw}{w}
\]

now substituting $w = a_0^m \delta$; $\frac{dw}{w} = \frac{d\delta}{\delta}$ we get:
\[
= \sum_{m = -\infty}^{+\infty} a_0^m \int |\hat{\psi}(a_0^m \delta)|^2 \frac{d\delta}{\delta}
\]

taking the summation sign inside the integral we get:
\[
= \int \left\{ \sum_{m = -\infty}^{+\infty} |\hat{\psi}(a_0^m \delta)|^2 \right\} \frac{d\delta}{\delta}
\]
for this integral to converge;
\[ 0 < A \leq \sum_{m=-\infty}^{\infty} |\hat{\psi}(a^{-m}\delta)|^2 \leq B < \infty \]

ie. strictly between 2 positive constants.

The bounded quantity is called the **Sum of Dilated Spectra** given by:

\[ SDS(\psi, a_0)(\lambda) = \sum_{m=-\infty}^{\infty} |\hat{\psi}(a^{-m}\delta)|^2 \]

and the condition of bounding is referred as **Frame Property** (it is also a check for possibility of inversion).

Wavelet Transform on function \( x(t) \) is given by

\[
W_{\psi,x}(b, a) = \frac{1}{\sqrt{a}} \int x(t)\psi\left(\frac{t-b}{a}\right)dt
\]

Define \( g(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{-t}{a}\right) \)

\[
g(b-t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)
= \int x(t)g(b-t)dt
\]

This is convolution of \( x(t) \) with \( g(\cdot-t) \). So we can interpret wavelet transform as a linear filtering operation.

Frequency response of the LSI filter

\[
\hat{g}(w) = \int g(t)e^{-jwt} dt
\]

\[
= \frac{1}{\sqrt{a}} \int \psi\left(\frac{-t}{a}\right)e^{-jwt} dt \quad \text{Put } \frac{-t}{a} = \lambda
\]

\[
= \frac{1}{\sqrt{a}} \int \psi(\lambda)e^{-jw(-a\lambda)} - ad\lambda
\]

\[
= -\sqrt{a} \int \hat{\psi}(\lambda)e^{-j(aw)\lambda} d\lambda
\]

\[ \therefore \hat{g}(w) = -\sqrt{a} \hat{\psi}(aw) \]
That is we are filtering \( x(t) \).

\( \psi(t) \) is band-pass function and \( \sqrt{a} \hat{\psi}(aw) \) is a band-pass filter with center frequency \( W_\psi/a \).

In this way the signal \( x(t) \) is filtered by different filters with different scale parameter.

Similarly it can be shown that inverse wavelet transform is also a filtering operation.

Freq. domain output of \( m^{th} \) filter at analysis side is

\[
\hat{x}(w)\hat{\psi}(a_0^m w) \quad \text{where } a_0 > 1
\]

Output of \( m^{th} \) synthesis filter

\[
\hat{x}(w)\hat{\psi}(a_0^m w)\hat{\psi}(a_0^m w)
\]

\[
= \hat{x}(w) \left| \hat{\psi}(a_0^m w) \right|^2
\]

Output of the synthesis filter bank.

\[
= \sum_m \hat{x}(w) \left| \psi(a_0^m w) \right|^2
\]

\[
= \hat{x}(w) \sum_m \left| \psi(a_0^m w) \right|^2
\]

Therefore for perfect reconstruction (i.e. synthesis filter bank output = a multiple of the analysis filter bank input).
i.e. $\sum_{m} |\psi(a_0^m w)|^2 = C_1$ for all $w$.

The term $\sum_{m} |\psi(a_0^m w)|^2$, sum of dilated spectra $SDS_\psi$ should be a constant value. If $0 < A \leq \sum_{m} |\psi(a_0^m w)|^2 \leq B < \infty$ is true we can use different analysis and synthesis filter.

Also we can scale $\hat{\psi}(w)$ such that $SDS(\hat{\psi}, a_0)(w) = 1$ which is given by

$$\hat{\psi}(w) = \frac{\hat{\psi}(w)}{+\sqrt{SDS(\hat{\psi}, a_0)(w)}}$$

to get normalized bank of filters.