Finite Rate of Innovation Signals: Quantization Analysis with Resistor-Capacitor Acquisition Filter

Srikanth Tenneti, California Institute of Technology, Pasadena CA
Animesh Kumar and Abhay Karandikar, Indian Institute of Technology Bombay, India

SampTA2013
10th International Conference on Sampling Theory and Applications
July 1st - July 5th, 2013
Jacobs University Bremen

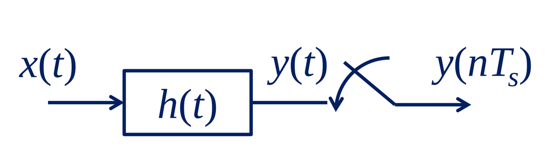
Introduction

◊ FRI signals were recently proposed by *Vetterli, Marziliano*, and *Blu*

$$x(t) = \sum_{k=0}^{K_0-1} c_k \delta(t - t_k), \quad c_k, t_k \in \mathbb{R}$$

◊ In the above signal, the parameters (c_k, t_k) for $k = 0, \ldots, K_0 - 1$ represent the signal's degrees of freedom

◊ The bandwidth of $x(t)$ is infinite; but, the following acquisition scheme can be used to reconstruct $x(t)$ from $(2K_0 + 1)$ samples



◊ The filter $h(t)$ can be lowpass with appropriate bandwidth or Gaussian for perfect reconstruction

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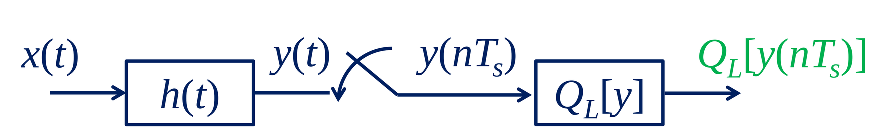
Quantization issues and contributions



◊ If the samples are quantized, the error analysis in determination of (c_k, t_k) is not analyzed, so far, in closed-form expressions

◊ Ideal lowpass or Gaussian acquisition filter reduces the problem of (c_k, t_k) retrieval to a non-linear power-sum series, which is complex to analyze under quantization (perturbation)

Contribution: In a first, we analyze the effect of (scalar) quantization error on FRI signals comprising of stream of Diracs or non-uniform splines. This analysis requires faster than minimum sampling rate

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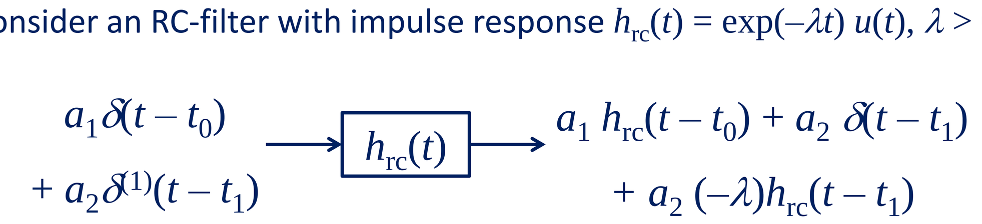
Signal model and insights into RC filtering

Non-uniform splines or neural signals can be reduced to stream of Diracs and their derivatives. This is the signal model for this work. Without loss of generality, the signal model with bounded $t_{k,i}$ and $c_{k,i}$ is

$$x(t) = \sum_{k=0}^{K_0-1} c_{k,0}\delta(t - t_{k,0}) + \sum_{k=0}^{K_1-1} c_{k,1}\delta^{(1)}(t - t_{k,1})$$

Consider an RC-filter with impulse response $h_{rc}(t) = \exp(-\lambda t)\, u(t)$, $\lambda > 0$



The output, except at $t = t_1$, is the response to $a_1\delta(t - t_0) + a_2(-\lambda)\delta(t - t_1)$!

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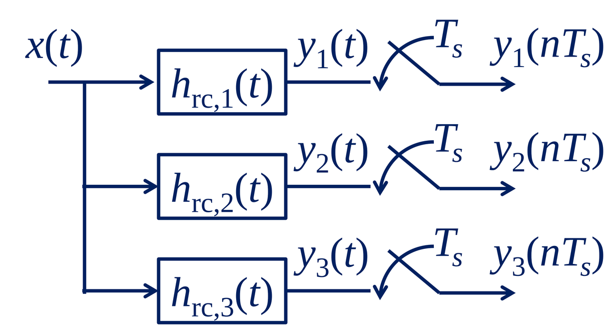
RC-filter bank for reconstruction



First note that

$$y_1(t) = e^{-\lambda_1 t} \sum_{k=0}^{K-1} c_k u(t - t_k)$$

where each $t_i = t_{li,ji}$ for some (l_i, j_i) pair, $K = K_0 K_1$, and $c_i = (-\lambda)^{ji} c_{li,ji}$

Under the (mild) assumption that $nT_s \neq t_k$, we get scaled output as piecewise constant, that is,

$$e^{\lambda_1 nT_s} y_1(nT_s) = \sum_{k=0}^{K-1} c_k e^{\lambda_1 t_k} u(nT_s - t_k)$$

Distinct three channels outputs can be used to solve for (c_k, t_k, j_k) in an ***easy*** fashion. One reading per Dirac requires $T_s < \min_k (t_k - t_{k-1})$

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Related work

◊ Two channel or multichannel RC filter banks for perfect reconstruction **[Dragotti, Vetterli, and Blu'2007] [Seelamantula and Unser'2008] [Olkkonen and Olkkonen'2008]**

◊ Accuracy in the presence of statistical noise **[Maravic and Vetterli'2005][Kusuma and Goyal'2009]**

◊Some quantization and oversampling results were studied by **[Jovanovic and Beferull-Lozano'2006]**

◊ Has been proposed for global positioning system and ultra wideband communications **[Kusuma-Maravic-Vetterli'2003]**

To the best of our knowledge, closed form error analysis to capture the effect of scalar quantization on FRI signal parameters is mostly unsolved

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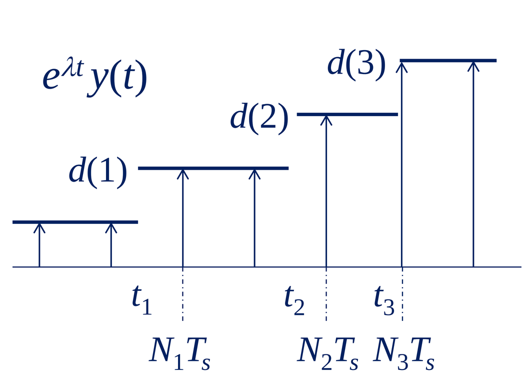
Perfect reconstruction method



Proposition: If $T_s < \min_k (t_k - t_{k-1})$ and that $nT_s \neq t_k$, then the parameters of our FRI signal are given by

$$t_{l_i,j_i} = \frac{1}{\lambda_1 + \lambda_3 - 2\lambda_2} \log\left[\frac{d_1(i) d_3(i)}{d_2^2(i)}\right],$$

$$j_i = \frac{1}{\log\left(\frac{\lambda_1}{\lambda_2}\right)} \left[\log\left[\frac{d_1(i)}{d_2(i)}\right] + (\lambda_2 - \lambda_1) t_{l_i,j_i}\right],$$

$$\text{and } c_i = (-\lambda_1)^{j_i} c_{l_i,j_i} = \frac{d_1(i)}{e^{\lambda_1 t_{l_i,j_i}}}.$$

◊ The RC filters have distinct $(\lambda_1, \lambda_2, \lambda_3)$ and they satisfy $\lambda_1 \lambda_3 = (\lambda_2)^2$

◊ Estimating j_i is easier since it is discrete (0 or 1). If L is large enough, this can be recovered exactly

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Reconstruction error with quantization

Theorem: If $T_s < \min_k (t_k - t_{k-1})$ and $nT_s \neq t_k$, then the parameters of our FRI signal can be reconstructed in the following way

$$\widehat{t}_{l_i,j_i} = \frac{1}{\lambda_1 + \lambda_3 - 2\lambda_2} \log\left[\frac{\widehat{d}_1(i)\widehat{d}_3(i)}{\widehat{d}_2^2(i)}\right],$$

$$\widehat{j}_i = \frac{1}{\log\left(\frac{\lambda_1}{\lambda_2}\right)} \left[\log\left(\frac{\widehat{d}_1(i)}{\widehat{d}_2(i)}\right) + (\lambda_2 - \lambda_1)\widehat{t}_{l_i,j_i}\right]$$

$$\text{and } \widehat{c}_{l_i} = (-\lambda_1)^{\widehat{j}_i} \widehat{c}_{l_i,j_i} = \frac{\widehat{d}_1(i)}{\exp(\lambda_1 \widehat{t}_{l_i,j_i})}.$$

◊ $(\lambda_1, \lambda_2, \lambda_3)$ are distinct and $\lambda_1 \lambda_3 = (\lambda_2)^2$

◊Denote $\Delta = \lambda_1 - 2\lambda_2 + \lambda_3$

◊ With $c_i(m) = c_{li,ji}(-\lambda_m)^{ji}$, our accuracy results are as follows:

Accuracy of the time-locations

$$|\widehat{t}_i - t_i| \leq -\frac{4}{\Delta} \min_m \log\left[1 - \frac{2^{-L}(1 + e^{\lambda_m T_s})}{|c_i(m)|}\right]$$

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Reconstruction with quantization (contd.)

$$\left|\frac{\widehat{c}_i(m) - c_i(m)}{c_i(m)}\right| \leq \underbrace{\left[1 + \frac{2^{-L}(1 + e^{\lambda_m T_s})}{|c_i(m)|}\right]\left[1 - \frac{2^{-L}(1 + e^{\lambda_{m^*} T_s})}{|c_i(m^*)|}\right]^{\frac{-4\lambda_m}{\Delta}}} - 1$$

Accuracy of the scale parameters (proportionality constants)

Comment: Since $\log(1-x) \approx -x$ for small x, therefore, for large L the error in locations t_i is proportional to 2^{-L}. This is the best one can expect with scalar quantizers

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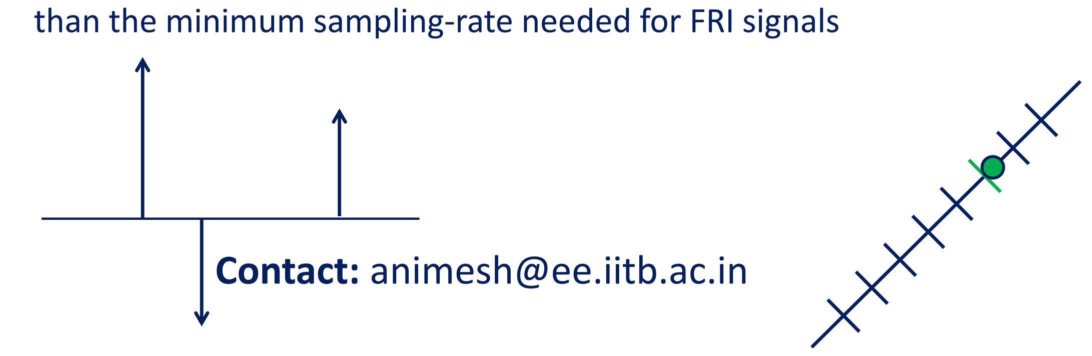
Conclusions

◊ In the presence of scalar quantizers, for sampling and reconstruction error analysis of stream of Diracs and non-uniform splines, a new three-channel RC filter bank was introduced

◊ Worst case bounds on FRI signal parameter error were established in terms of the quantizer precision L (in bits). The error scales eventually as 2^{-L}

◊ The sampling-interval required is less than $\min_k (t_k - t_{k-1})$, which is smaller than the minimum sampling-rate needed for FRI signals



Contact: animesh@ee.iitb.ac.in

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