Spatial Field Acquisition Using Sensors

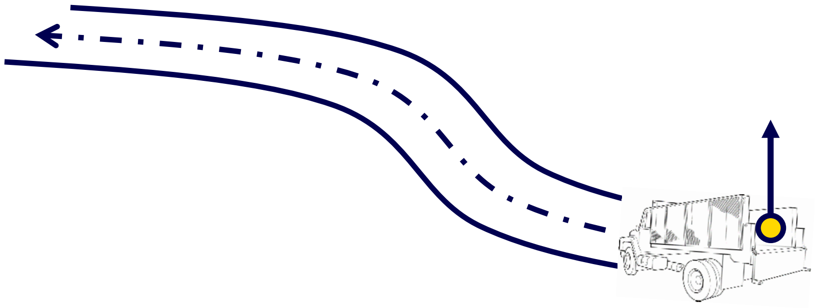
Prof. Animesh Kumar

Electrical Engineering

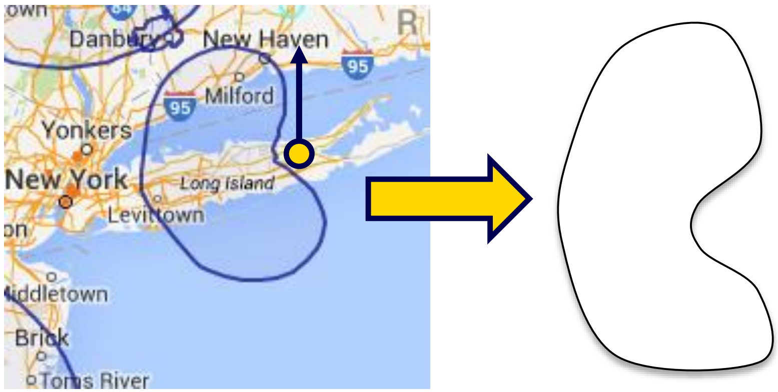
Indian Institute of Technology Bombay, India

Spatial sampling is everywhere

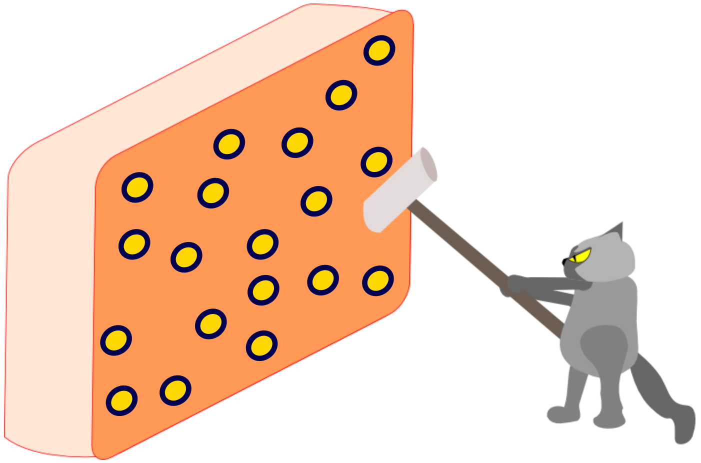
Emission monitoring with sensors



Sampling along a path with vehicle

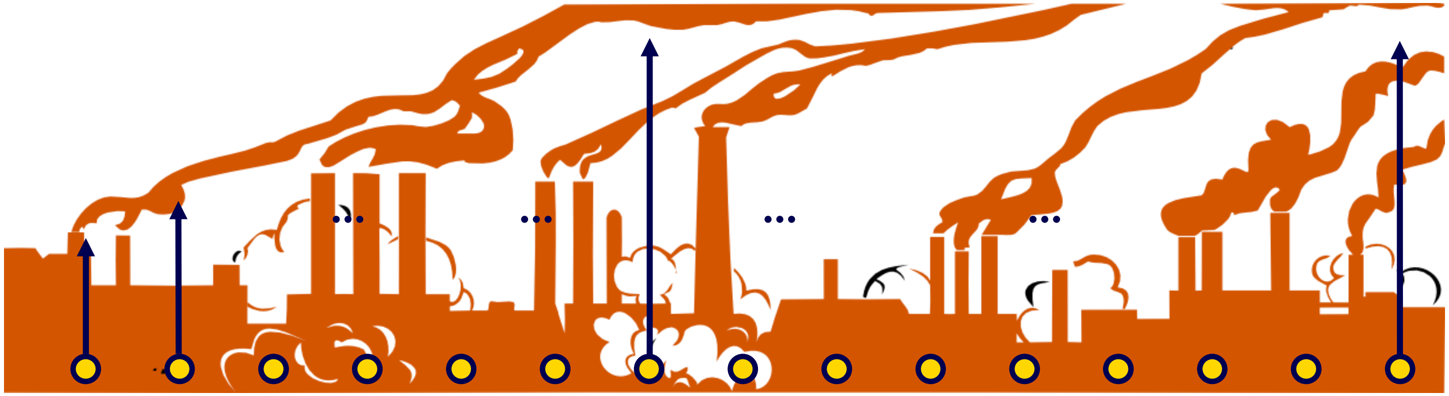


Coverage region for TV transmitters



Randomly sprayed smart-dust/paint

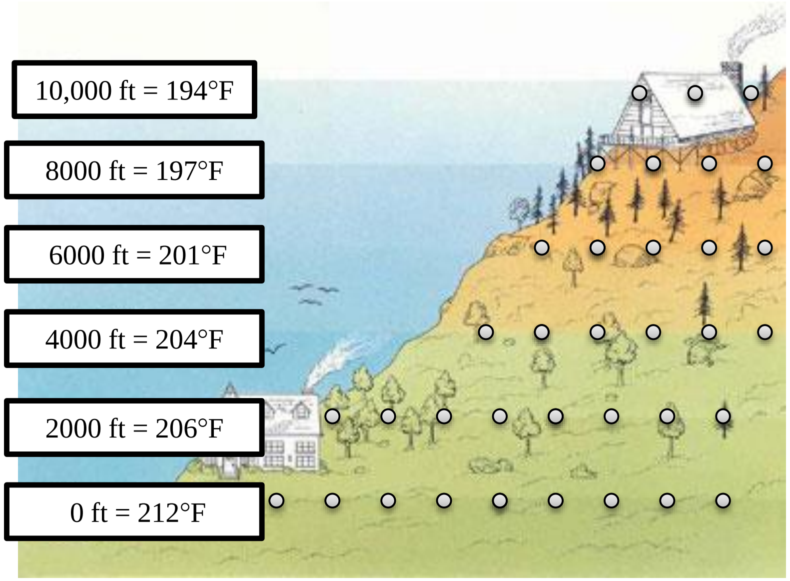
Field sampling with an array of (fixed) sensors

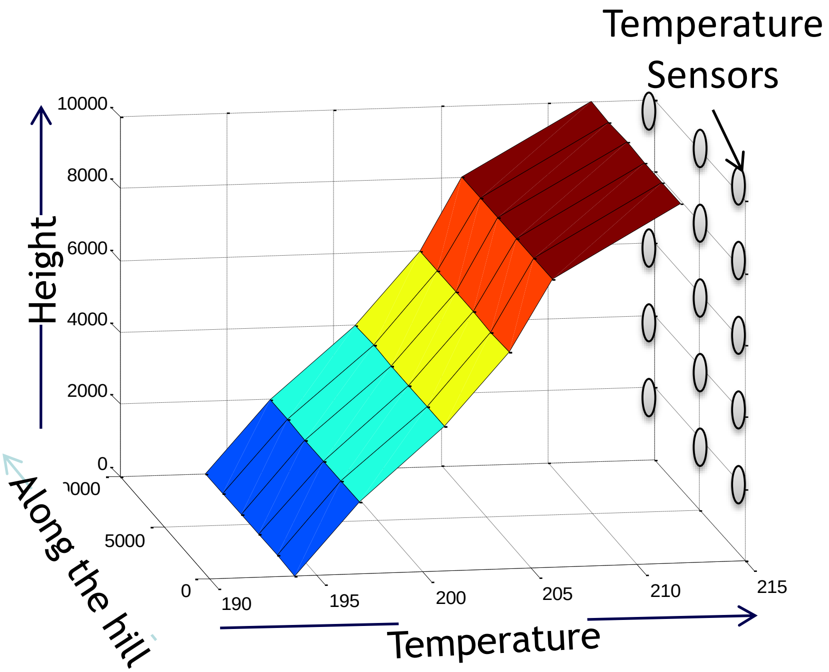


◊ Remote sensing of a physical (spatial) field using an array of wireless sensors

- ◊ **Key issues:** aliasing, quantization, noise
- ◊ **Signal structure:** smoothness, bandlimitedness, physics based evolution

Field sampling with an array of (fixed) sensors

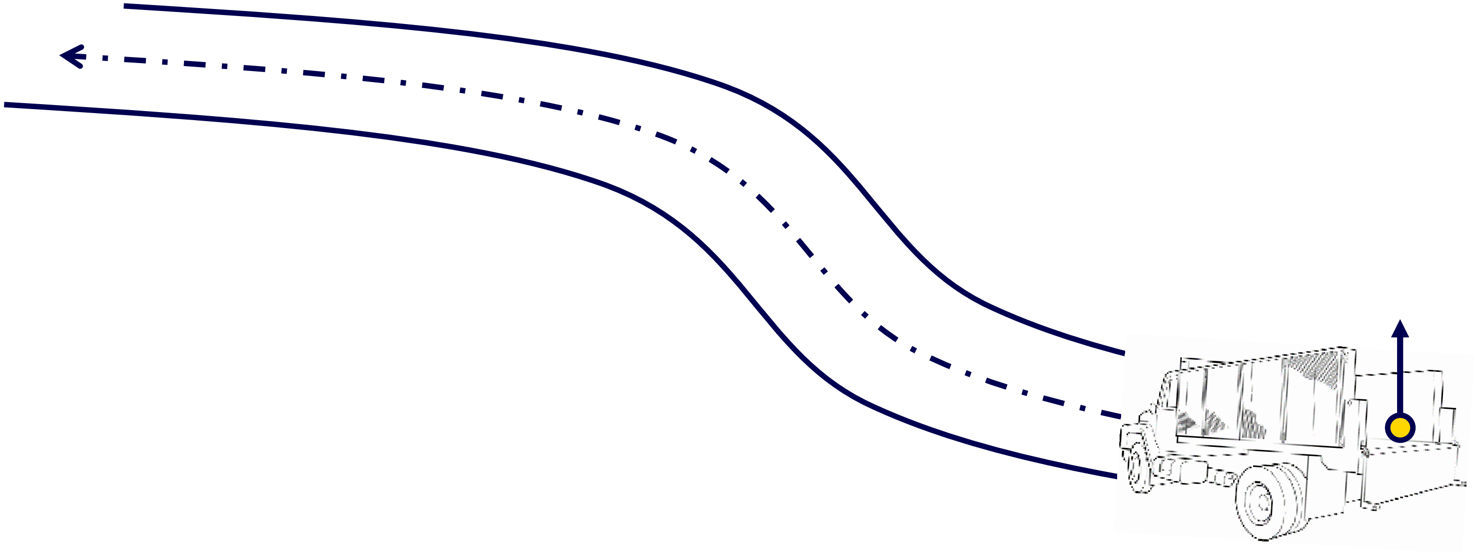




◊ We would like to address two-dimensional fields as well

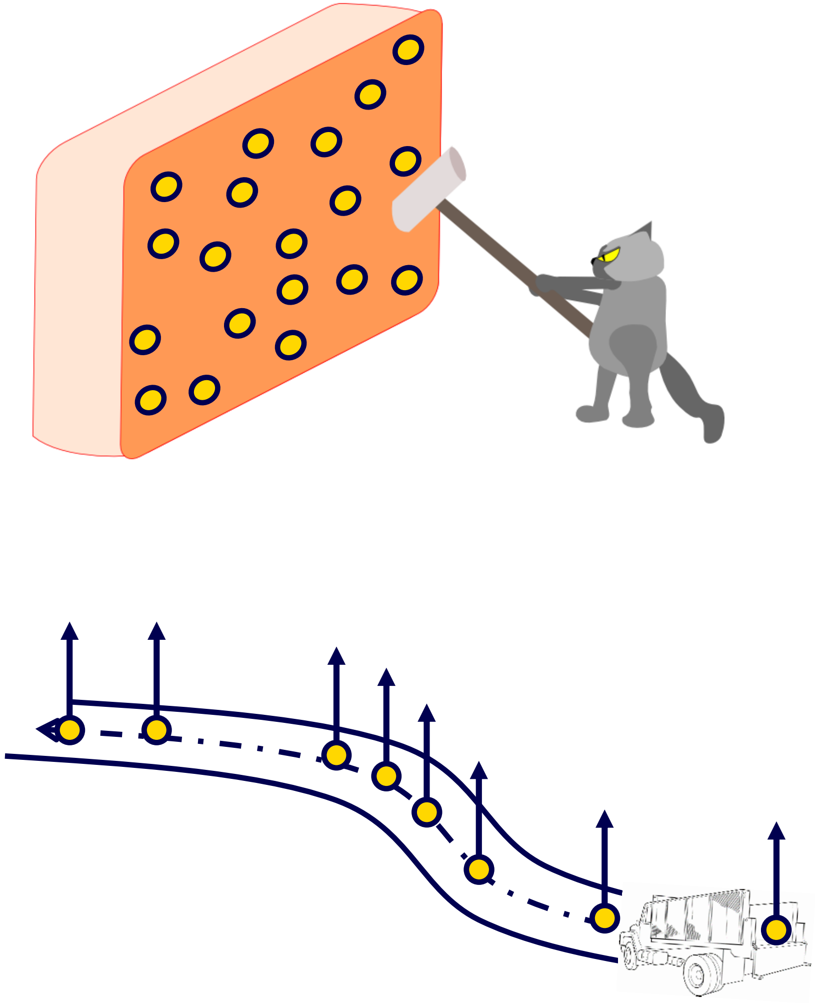
- ◊ **Key issues:** multidimensional aliasing, quantization, noise
- ◊ **Signal structure:** smooth, bandlimited, physics based

Spatial sampling using vehicles



- ◊ Sampling with moving vehicles can be used as well!
  - ◊ **Key issues:** nonuniform vehicle speeds, imprecise location, noise quantization, temporal variation
  - ◊ **Signal structure:** bandlimited, smooth

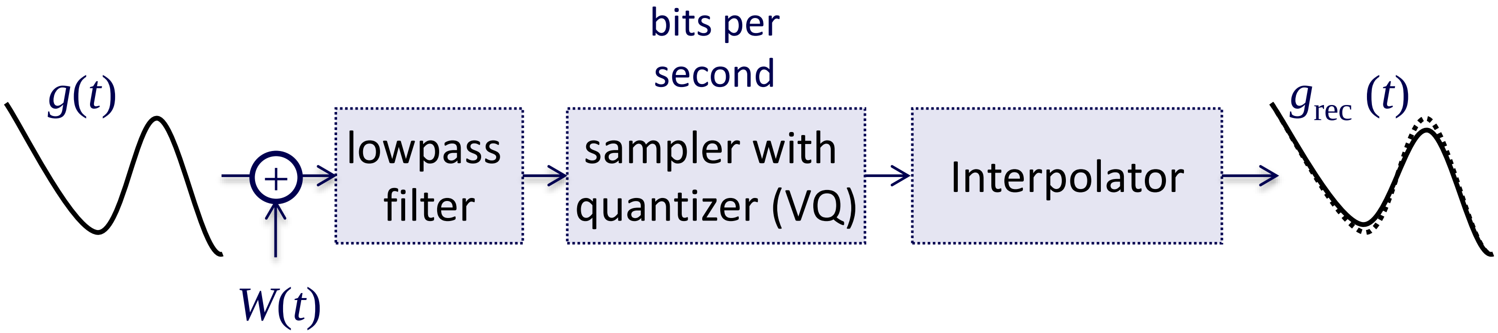
Spatial sampling with “unknown” location



◊ Sampling with fixed array of randomly deployed sensors:

- ◊ **Key issues:** nonuniform unknown locations, quantization, noise, temporal variation
- ◊ **Signal structure:** bandlimited, smooth

Sampling versus spatial (distributed) sampling



$g(t)$

$W(t)$

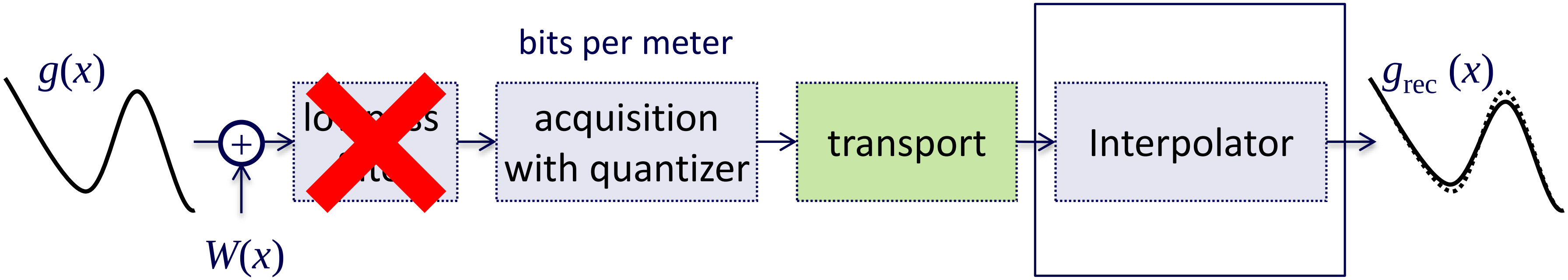
lowpass filter

bits per second

sampler with quantizer (VQ)

Interpolator

$g_{\text{rec}}(t)$



$g(x)$

$W(x)$

~~lowpass filter~~

bits per meter

acquisition with quantizer

transport

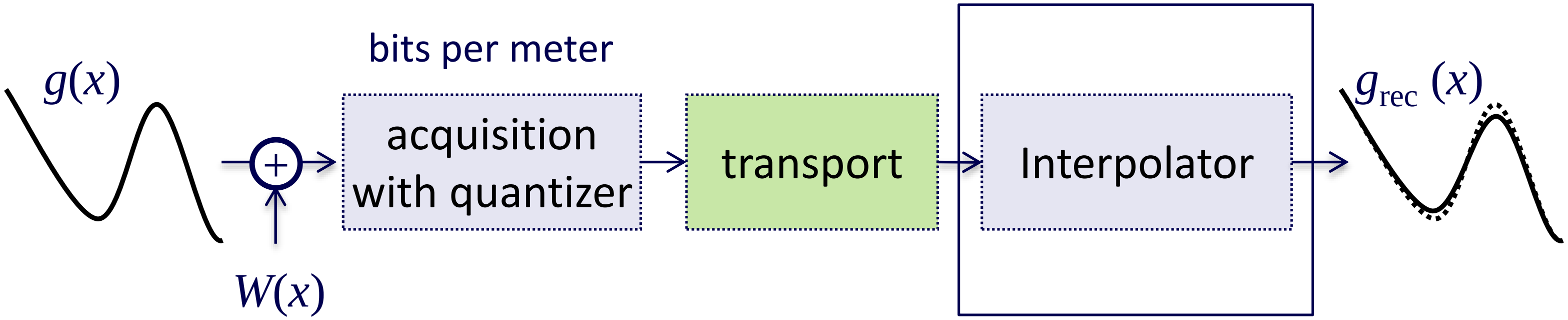
Interpolator

$g_{\text{rec}}(x)$

◊ Spatial field cannot be prefiltered due to distributed nature of sampling

◊ Field samples have to be directly acquired (and scalar quantized)

Acquiring high-precision field snapshots



◊ Bandlimited or smooth nonbandlimited fields will be considered

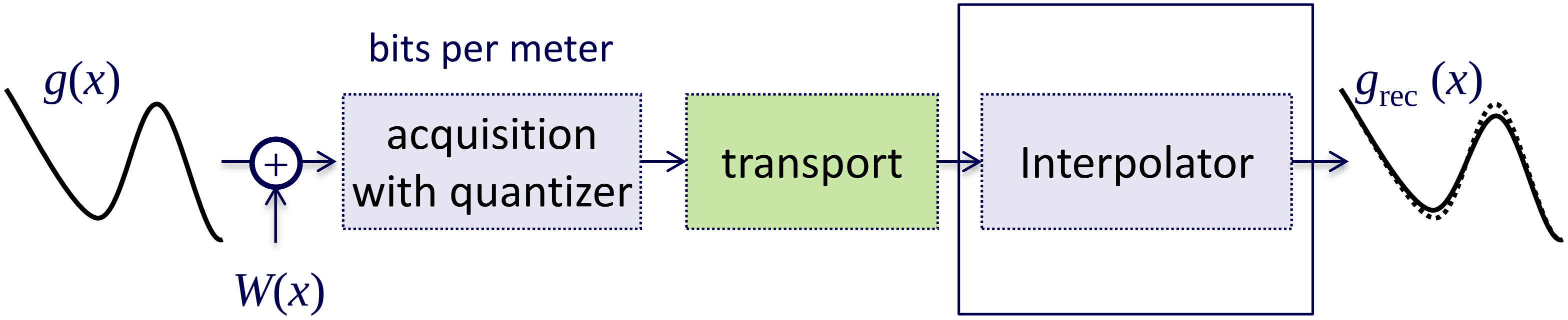
◊ A higher sampling rate (than "essential" Nyquist rate) will combat aliasing and noise – **oversampling**

◊ Distortion would decrease with quantizer precision – **precision**

◊ Sampling location models will include deterministic (and known) locations and statistical (and unknown) locations

◊ Transport will not be addressed

The tutorial roadmap



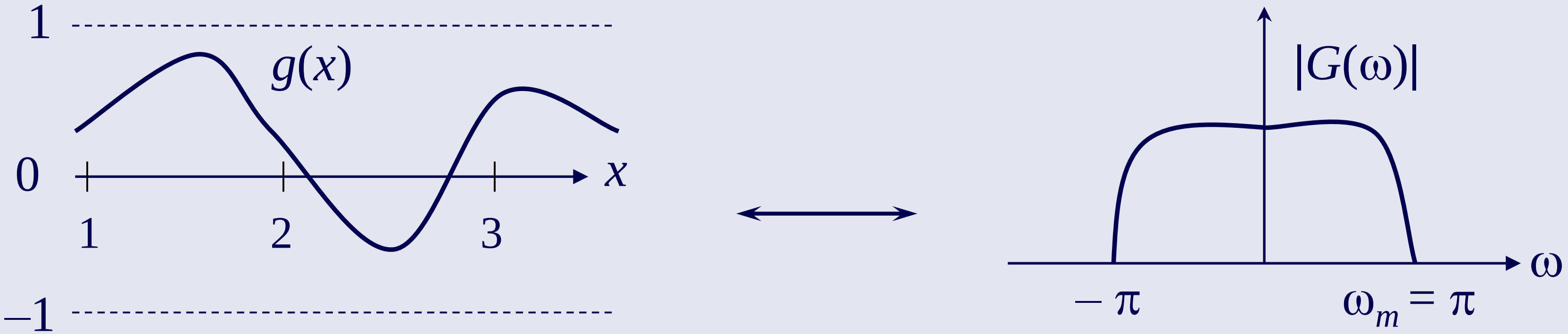
◊ Three part tutorial which deals with

- Sampling with a fixed grid of sensors without measurement noise – tradeoff between quantization and oversampling for smooth fields
- Sampling with fixed grid of sensors with measurement noise – tradeoff between oversampling, quantization, and distortion
- Sampling with unknown but statistically distributed sample locations – tradeoff between oversampling and distortion

Sampling with a fixed grid of sensors without measurement noise – tradeoff between quantization and oversampling

Deterministic bandlimited field model

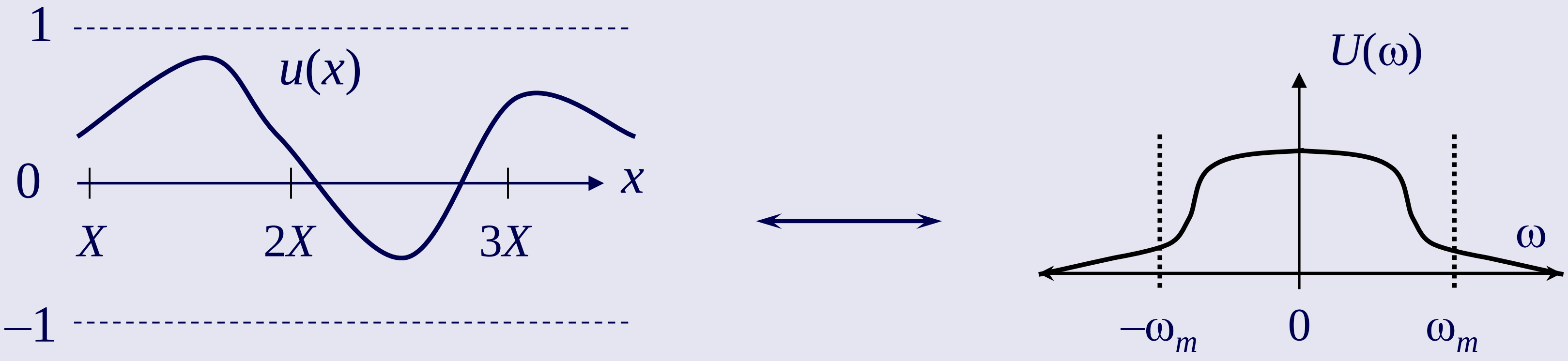
- 1-D real and bounded: $|\, g(x, t)| < 1$ (normalized)
- Fixed temporal snapshot, i.e., consider $g(x) \equiv g(x, t_0)$
- Spatially bandlimited: spatial Nyquist period, $X_{\mathrm{NQ}} = 1$
- Finite energy in L^2 sense (over x)
- **Bernstein's inequality:** $|\, \mathrm{g}'(x) \,| \leq \omega_m \,\|\, g \,\|_\infty = \pi$



Nyquist interval = $X_{\mathrm{NQ}} = \pi / \omega_m = 1$

Deterministic non-bandlimited field model

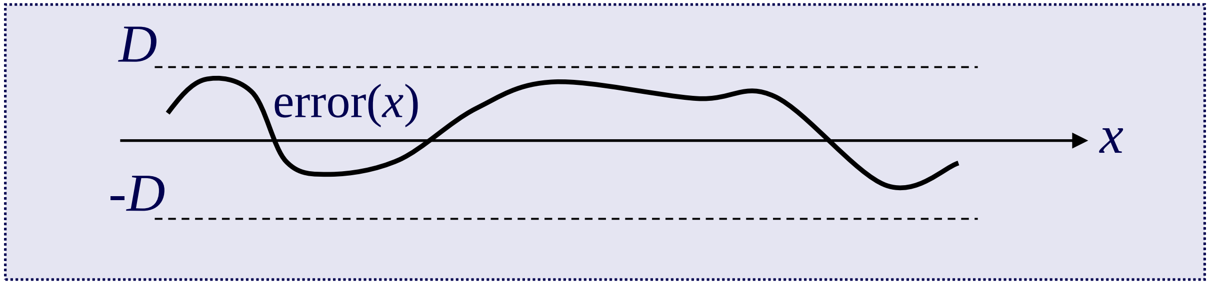
- ◊ 1-D, real, bounded: $|u(x, t)| < 1$ (normalized)
- ◊ Fixed temporal snapshot, i.e., consider $u(x) \equiv u(x, t_0)$
- ◊ Decaying spectrum after a certain (fixed) bandwidth, formally $\int_\omega |\omega U(\omega)| d\omega < \infty$
- ◊ Finite energy in L^2 sense (over x)



Pseudo-Nyquist interval = $X = (\pi / \omega_m)$

Distortion criteria

◊ Maximum pointwise reconstruction error: $D = \sup_x |\, g(x) - g_{\text{rec}}(x)|$



This distortion can be extended to other error criteria, e.g., normalized L^p errors

Part 1: Organization

- Nyquist-style sampling for smooth fields
- Sampling with single-bit sensors
- Bit-conservation principle
- Comments on
    - sampling of multi-dimensional field, and
    - increase in aliasing (with time) in physical fields

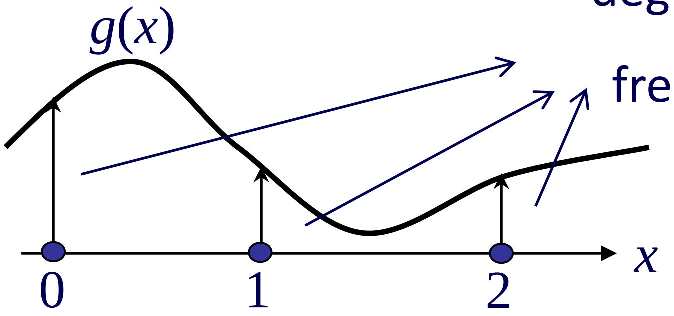
Part 1: Organization

- Nyquist-style sampling for smooth fields
- Sampling with single-bit sensors
- Bit-conservation principle
- Comments on
    - sampling of multi-dimensional field, and
    - increase in aliasing (with time) in physical fields

Shannon interpolation formula and stability

For $g(x)$ bandlimited with $\omega_m = \pi$ it follows that

$$g(x) = \sum_{i=-\infty}^{\infty} g(i)\,\mathrm{sinc}\,(x-i) \quad \Bigg\} \text{ The Shannon-Whittaker-Kotelnikov interpolation formula}$$

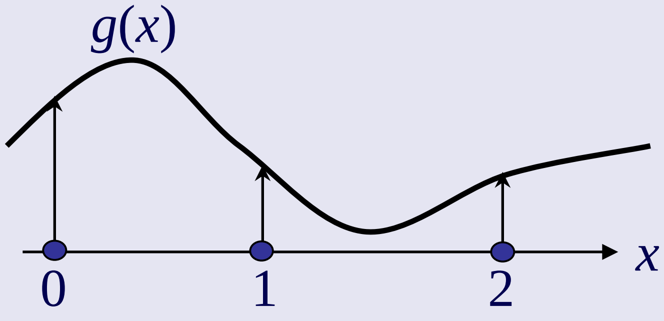


degrees of freedom

The kernel $\mathrm{sinc}(x)$ is square integrable but not absolutely integrable,. This leads to instability in reconstruction

$|\varepsilon_i| \leq \varepsilon$ and $\hat{g}(x) = \sum_{i=-\infty}^{\infty} \{g(i)+\varepsilon_i\} \dfrac{\sin \pi(x-i)}{\pi(x-i)}$ do not imply $|g(x) - \hat{g}(x)| \leq O(\varepsilon)$

Nyquist sampling and reconstruction



$$g(x) = \sum_{i=-\infty}^{\infty} g(i)\,\mathrm{sinc}\,(x-i)$$

◊ **[Shannon's theorem]** The signal $g(x)$ can be reconstructed from samples taken at points $\{iX_{NQ}\}$ where i is an integer

$$g(x) = \sum_{i=-\infty}^{\infty} g(i)\,\mathrm{sinc}\,(x-i)$$

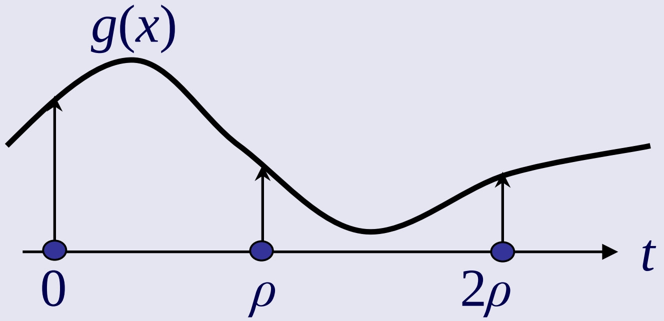
◊ In practice, the samples $g(i)$ will be quantized (say using) k-bit uniform scalar quantizer $Q_k(.)$

Quantization error: $|\,g(i) - Q_k(g(i))| \leq 2^{-k}$

Field reconstruction: $g_{\mathrm{rec}}(x) = \sum_{i=-\infty}^{\infty} Q_k\,(g(i))\,\mathrm{sinc}(x-i)$

Distortion: $D = \sup_x |g(x) - g_{\mathrm{rec}}(x)|$ is unbounded

Bandlimited fields: stable Nyquist sampling

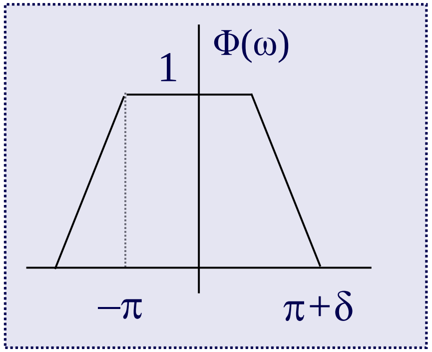


$$g(x) = \sum_{i=-\infty}^{\infty} g(i\rho)\,\varphi(x - i\rho)$$

Stability: $C = \sup_x \sum_{i=-\infty}^{\infty} |\varphi(x - i\rho)| < \infty$

◊ Samples are uniformly spaced slightly closer than the Nyquist points ($\rho < 1$)

$\varphi(x) \quad \leftrightarrow$



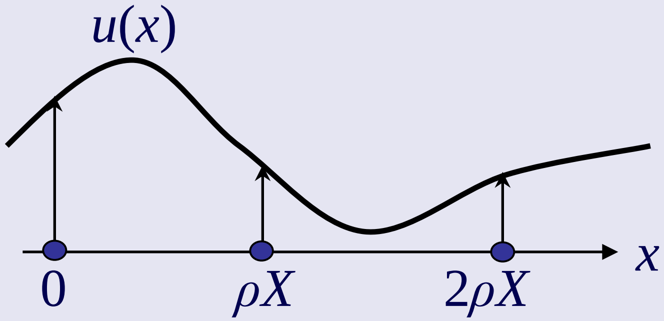
◊ Samples are quantized using a k-bit uniform scalar quantizer $Q_k(.)$

Quantization error: $|g(i\rho) - Q_k(g(i\rho))| \leq 2^{-k}$

Field reconstruction: $g_{\text{rec}}(x) = \sum_{i=-\infty}^{\infty} Q_k(g(i\rho))\,\varphi(x - i\rho)$

Error decay profile: $D \leq C\,2^{-k}$

Nonbandlimited fields: pseudo-Nyquist (PN) sampling

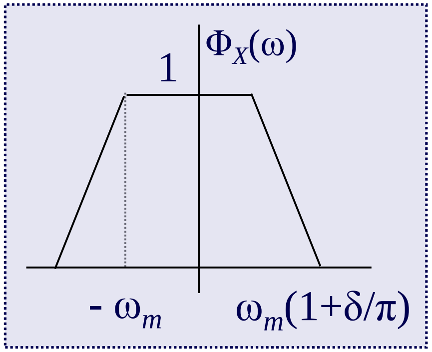


$$u(x) \;!\!= \sum_{i=-\infty}^{\infty} u(i\rho X)\, \varphi_X(x - i\rho X)$$

Stability: $C = \sup_x \sum_{i=-\infty}^{\infty} |\,\varphi_X(x - i\rho X)| < \infty$

◊ Samples are uniformly taken, at every $X = \pi/W$ meters (slightly closer for stability)

$\varphi_X(x) \quad \leftrightarrow$



◊ k-bit uniform scalar quantizer $Q_k(.)$

Quantization error: $|\,u(i\rho X) - Q_k(u(i\rho X))| \leq 2^{-k}$

Signal reconstruction: $u_{\text{rec}}(x) \;=\; \sum_{i=-\infty}^{\infty} Q_k\,(u(i\rho X))\, \varphi_X(x - i\rho X)$

Distortion profile is more involved due to aliasing effects!

PN sampling distortion

Proposition: For non-bandlimited signals $u(x)$, in the PN sampling setup it can be shown that **[Kumar, Ishwar, Ramchandran'2010]**

$$|u(x) - u_{\text{rec}}(x)| \leq \underbrace{A\, 2^{-k}}_{\text{quantization part}} + B \underbrace{\int_{\omega > \omega m} |U(\omega)|\, d\omega}_{\text{aliasing part}}$$

Insights and interpretations:

◊ Under the signal model, the aliasing term $\int_{\omega>\omega m} |U(\omega)|\, d\omega$ decreases to zero as W increases. Recall that $X = \pi / \omega_m$

◊ As k increases, aliasing part dominates. As ω_m increases, quantization dominates

◊ The best strategy is to scale (ω_m, k) together to infinity for sending distortion to zero

Nyquist-style sampling distortions

Distortion: $D = \sup_x |\, g(x) - g_{\text{rec}}(x)|$ or

$$D = \sup_x |\, u(x) - u_{\text{rec}}(x)|$$

Bit-rate: k bits/Nyquist-period or k bits/Pseudo-Nyquist period

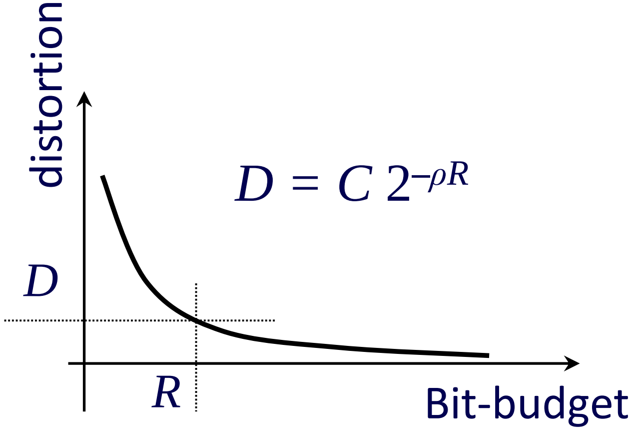
For bandlimited fields:

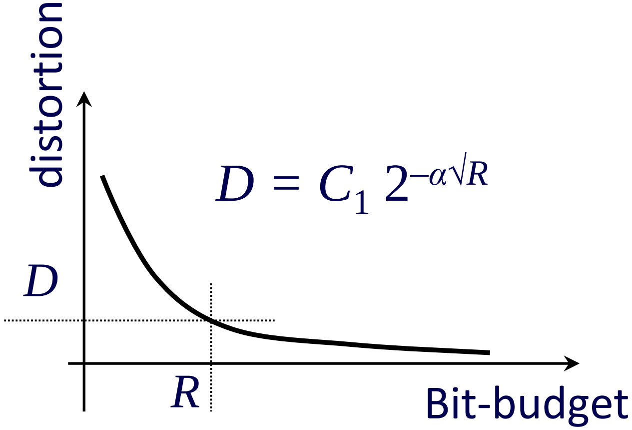
$$D = C\, 2^{-k} = C\, 2^{-\rho R}$$

For non-bandlimited fields:

The distortion depends on spectral decay. For example, if $|U(\omega)| < c_1 \exp(-a|\omega|)$, then $D < C_1\, 2^{-\alpha\sqrt{R}}$

$$|u(x) - u_{\text{rec}}(x)| \leq A\, 2^{-k} + B \int_{\omega > \omega m} |U(\omega)|\, d\omega$$





Comments on the Nyquist-style distortions

Implications:

◊ To decrease the distortion R and hence k must be increased, and expensive and power draining quantizers will be required

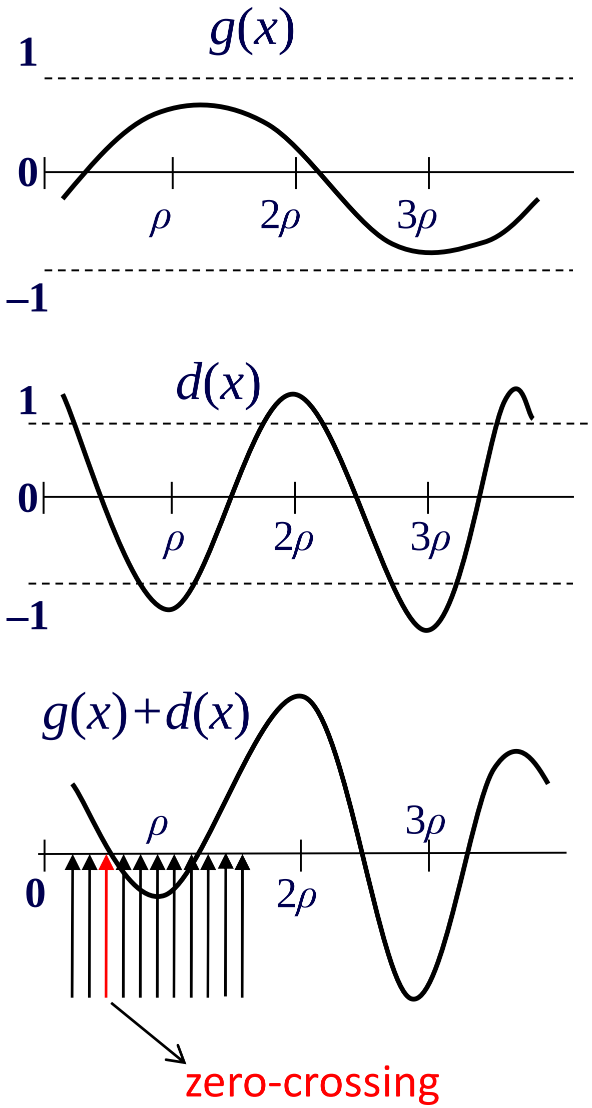
◊ **Optimality:** The exponential distortion-rate for bandlimited fields is optimal (in an order sense) **[Daubechies et al.'2006]**

◊ Can these (order) optimal distortion-rate tradeoffs be obtained with a sampling method tailored towards low(est) precision quantizers?

Part 1: Organization

- Nyquist-style sampling for smooth fields
- Sampling with single-bit sensors
- Bit-conservation principle
- Comments on
    - sampling of multi-dimensional field, and
    - increase in aliasing (with time) in physical fields

1-bit dithered sampling [Cvetkovic & Daubechies'00]



◊ A known dither field forces a zero-crossing in every (Pseudo) Nyquist interval

◊ 2^k one-bit quantizers spaced $\tau = \rho/2^k$ apart

=> k-bit spatial resolution

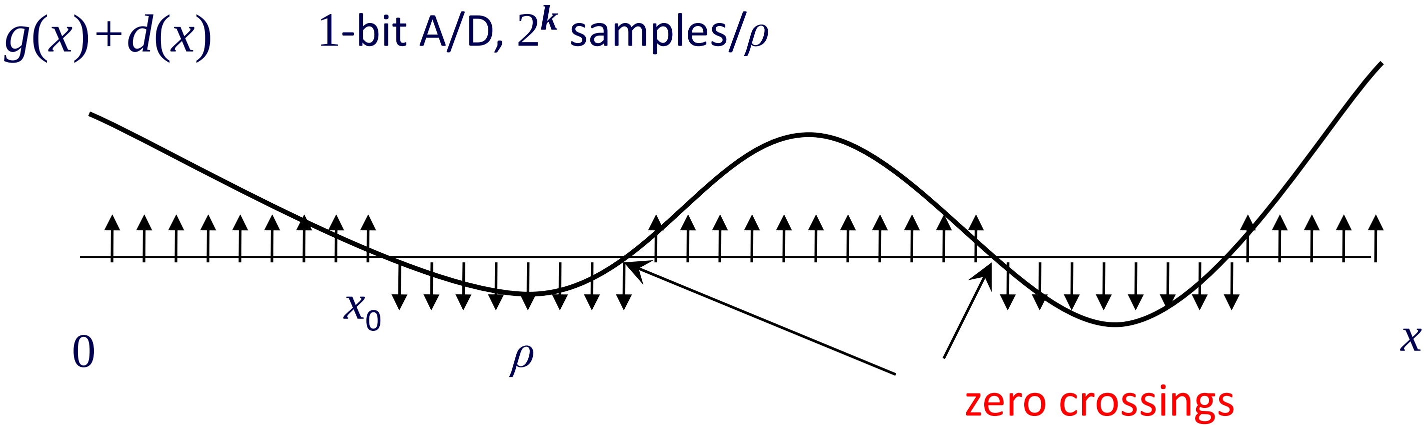
◊ k bits per Nyquist interval are required to index the location of the **first** zero-crossing

◊ By design, the slope of dither field is bounded

◊ Recall that $g'(x)$ is finite because field is bandlimited

◊ Similar technique works for non-bandlimited fields with ρ scaled to ρX

1-bit dithered sampling: sample accuracy



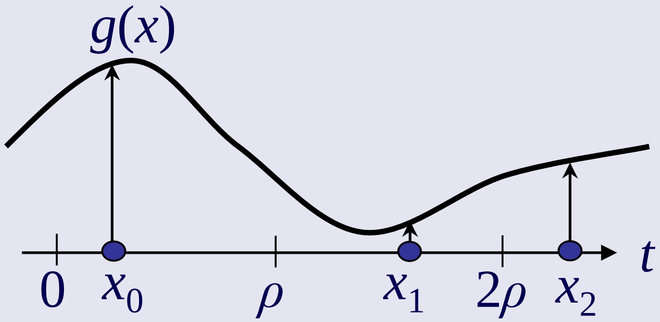
Sample spacing: $\tau = \rho/2^k$

Bounded slope of the smooth field and the dither imply that

if $g_{\text{dith, est}}(x_0) = -d(x_0)$, then for bandlimited fields

$$|g(x_0) - g_{\text{dith, est}}(x_0)| \leq \| g'(x) + d'(x)\|_\infty \, \tau/2$$
$$= ((\pi + \Delta)\rho/2)\, 2^{-k}$$

1-bit dithered sampling distortion: bandlimited fields



$$g(x) = \sum_{i=-\infty}^{\infty} g(x_i)\,\psi_i(x-x_i)$$

Stability: $C' = \sup_x \sum_{i=-\infty}^{\infty} |\psi_i(x-x_i)| < \infty$

If $|\, g(x_i) + d(x_i)| < ((\pi + \Delta)\rho/2)\, 2^{-k}$ and

$$g_{\text{dith, rec}}(x) = \sum_{i=-\infty}^{\infty} -d(x_i)\,\psi_i(x-x_i),$$ then

the distortion decay is: $D \leq C'\, 2^{-k}$ **[Cvetkovic & Daubechies'00]**

Corresponding result for nonbandlimited fields

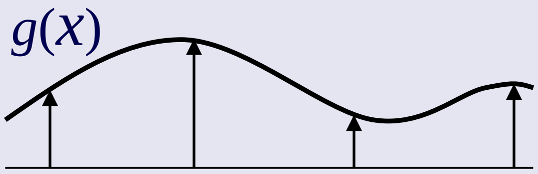
Proposition: For non-bandlimited signals $u(x)$, in the 1-bit dithered sampling setup it can be shown that **[Kumar, Ishwar, Ramchandran'2010]**

$$|u(x) - u_{\text{dith,rec}}(x)| \leq \underbrace{A\, 2^{-k}}_{\text{quantization part}} + B \underbrace{\int_{\omega > \omega m} |U(\omega)|\, d\omega}_{\text{aliasing part}} + \underbrace{B'\, 2^{-k}/\omega_m}_{\text{negligible term}}$$

Example: (exponentially decaying spectrum) The distortion depends on spectral decay. For example, if $|U(\omega)| < c_1 \exp(-a|\omega|)$, then $D < C_2\, 2^{-\alpha\sqrt{R}}$

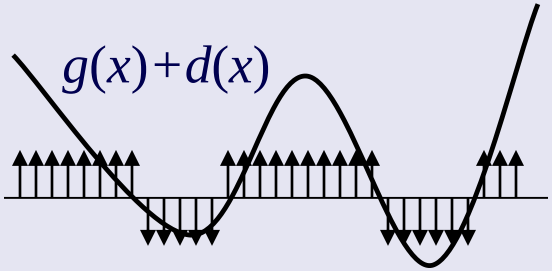
Two extreme scenarios

k-bit A/D, 1 sample per Nyquist



k-bit Nyquist sampling

1-bit A/D, 2^k samples per Nyquist



1-bit dithered oversampling

Similar results are observed for smooth non-bandlimited fields

Question: Can we trade-off between the number of samples/Nyquist and the quantizer-precision for a given (R, D) pair?

Part 1: Organization

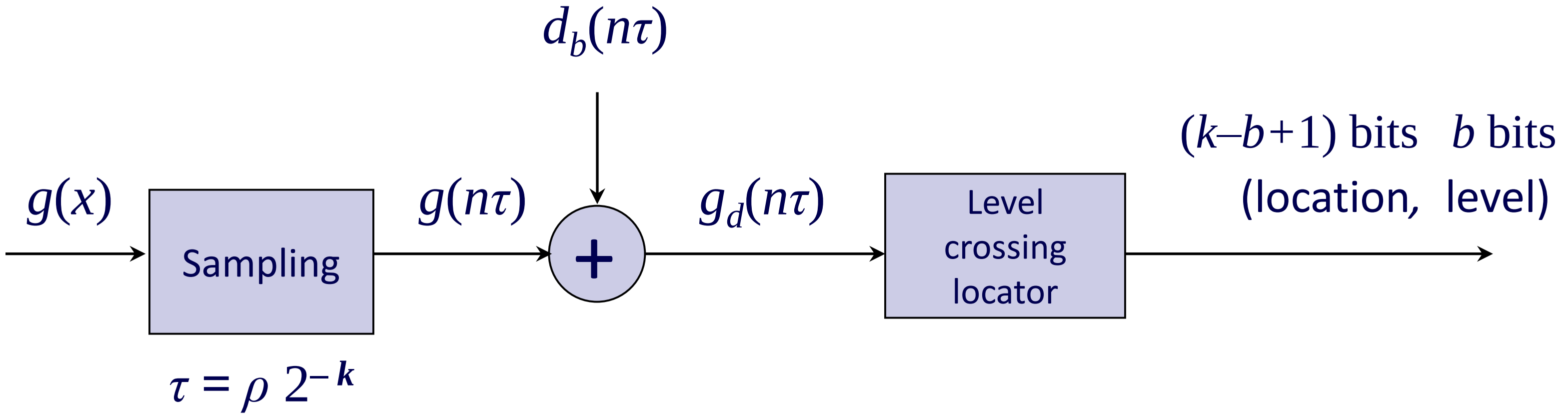
- Nyquist-style sampling for smooth fields
- Sampling with single-bit sensors
- Bit-conservation principle
- Comments on
  - sampling of multi-dimensional field, and
  - increase in aliasing (with time) in physical fields

b-bit dithered oversampling [Kumar-Ishwar-Ramchandran'03]

- 1 level
- Smooth dither: $d(x)$
- $g_d(x) = g(x) + d(x)$
- zero crossing in $(0, \rho)$
- time-instant of crossing

- 2^b–1 levels
- Smooth dither: $d_{\boldsymbol{b}}(x)$
- $g_d(x) = g(x) + d_{\boldsymbol{b}}(x)$
- level crossing in $(0, \rho/2^{(\boldsymbol{b}-1)})$
- (location, level) of crossing

Same distortion-rate characteristics as in the 1-bit dithered sampling case



$d_b(n\tau)$

$g(x)$

Sampling

$\tau = \rho\, 2^{-\boldsymbol{k}}$

$g(n\tau)$

+

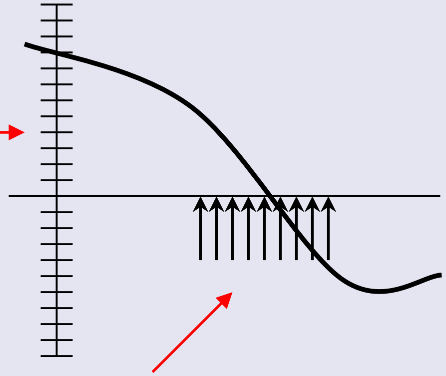
$g_d(n\tau)$

Level crossing locator

$(k–b+1)$ bits b bits

(location, level)

Interpretation



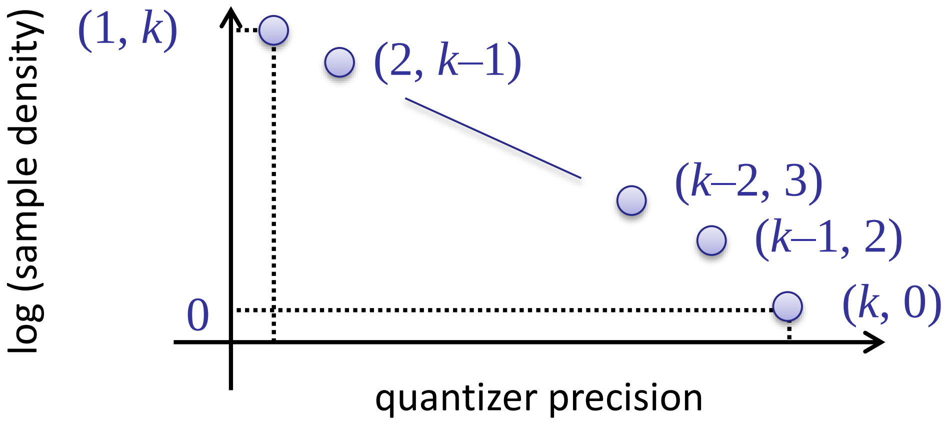
Nyquist sampling exhausts the bit budget in recording amplitude event

b-bit dithered sampling is a tradeoff between these two extremes

1-bit dithered sampling exhausts the bit budget in recording spatial event

Bit-conservation principle [Kumar-Ishwar-Ramchandran'03]

``Conservation of bits'' principle: Let k be the number of bits available per Nyquist interval. For each $1 \le b < k$ there exists a sampling scheme with 2^{k-b+1}, b-bit samples per Nyquist-interval that achieves a distortion of the order of $O(2^{-k})$



For non-bandlimited fields, since our results are not known to be optimal, it results in a oversampling/quantization trade-off law

Note:

$D \sim O(1/\mathrm{poly}(R_{Nyquist}))$ for 1-bit Σ-Δ conversion versus

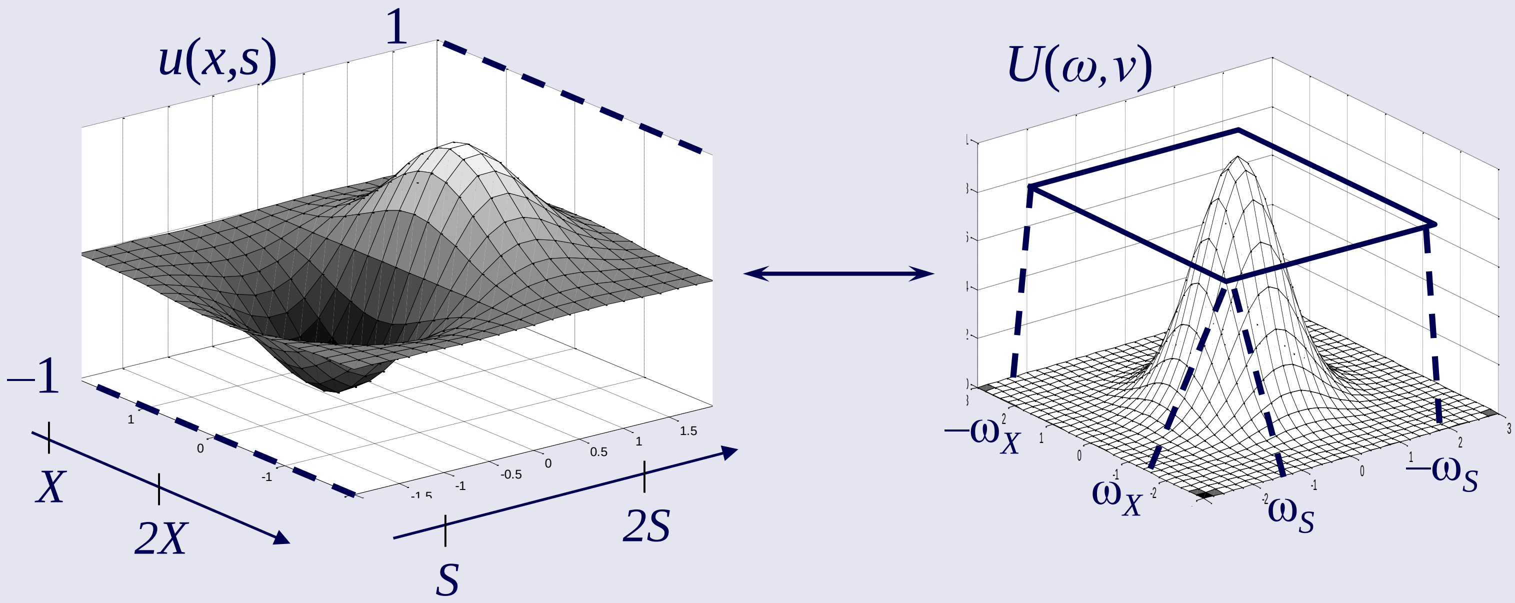
$D \sim O(1/\exp(R_{Nyquist}))$ for Nyquist sampling and b-bit dithered sampling

Part 1: Organization

- Nyquist-style sampling for smooth fields
- Sampling with single-bit sensors
- Bit-conservation principle
- Comments on
    - sampling of multi-dimensional field, and
    - increase in aliasing (with time) in physical fields

Two-dimensional field model

- 2-D, real, bounded: $|u(x, s)| < 1$ (normalized)
- Fixed temporal snapshot, i.e., consider $u(x, s) \equiv u(x, s, t_0)$
- Decaying spectrum after a certain (fixed) bandwidth, formally $\int_\omega |\omega U(\omega,v)| d\omega < \infty$
- Finite energy in L^2 sense (over x, s)



PN region = $X, S = (\pi / \omega_X), (\pi / \omega_S)$

What about two-dimensional fields?

Main Result: For non-bandlimited fields $u(x, s) \equiv u(x, s, t_0)$, with k information-bits in every PN interval (rectangle), it can be shown that there is a reconstruction $u_{\mathrm{rec}}(x, s)$ **[Athawale-Kumar'2012]**

$$|u(x, s) - u_{\mathrm{rec}}(x, s)| \leq \underbrace{A\, 2^{-k}}_{\text{quantization part}} + B \underbrace{\int_{\omega > \omega m \text{ or } \nu > \nu m}}_{\text{aliasing part}} |U(\omega, v)|\, \mathrm{d}\omega\, \mathrm{d}v$$

This result requires that field is bounded and one sensor with k-bits/sample or 2^k sensors with 1-bit/sample are available

Acquisition of evolving physical fields

- ◊ Many spatial phenomena result in a smooth field due to underlying physical laws (temperature, humidity, pollution)
- ◊ A Nyquist-style sampling approach is natural for such field's acquisition

At any time t_0, recall that sampling with aliasing results in

$$|u(x, t_0) - u_{\text{rec}}(x, t_0)| \leq A\, 2^{-k} + B \int_{\omega > \omega m} |U(\omega, t_0)|\, d\omega$$

Will this aliasing error term increase with time when a physical field evolves according to a partial differential equation based spatio-temporal law?

Constant coefficient PDE based evolution

Our work addresses constant coefficient partial differential equation based evolution of physical field $u(x, t)$, with enough initial conditions:

$$a_2\, u_{tt}(x, t) + a_1\, u_t(x, t) + a_0 u(x, t) = b_1\, u_x(x, t) + b_2\, u_{xx}(x, t);$$

$$u(x, 0) = u_0(x); \quad u_t(x, 0) = u_1(x);$$

The spatial Fourier spectrum of $u(x, t)$ is obtained using transform domain

$$a_2\, U_{tt}(\omega, t) + a_1\, U_t(\omega, t) + (a_0 - \mathrm{j} b_1 \omega + b_2 \omega^2)U(\omega, t) = 0$$

$$U(\omega, 0) = U_0(\omega); \quad U_t(\omega, 0) = U_1(\omega);$$

At this point, it becomes a pole-analysis (RHP) problem as a function of frequency and can be solved using Agashe's algorithm **[Sharma-Kumar'2015]**

For the second order PDE above, assuming $a_2 = 1$ without loss of generality

$$a_1 > 0,\ a_0 > 0,\ b_2 > (b_1^2/a_1^2)$$

Summary of results

[Ishwar, Kumar, and Ramchandran'11]: For bounded bandlimited signals (and stationary bandlimited process in the almost sure sense),

◊ Target distortion = D

◊ 1-bit sample density = $O(1/D)$

◊ Bit-rate per Nyquist = $O(|\log D|)$

[Kumar, Ishwar, and Ramchandran'10]: For bounded smooth non-bandlimited signals distortion-rate upper bounds can be computed, e.g., for signals with exponentially decaying spectra,

◊ Target distortion = D

◊ 1-bit sample density = $O(|\log D|/D)$ (per unit time)

◊ Bit-rate per unit time = $O(|\log D|^2)$

Summary of results

[Athawale and Kumar’2012] For bounded two dimensional smooth fields, upper bounds on distortion were computed for Nyquist-style sampling and single-bit sensor assisted sampling

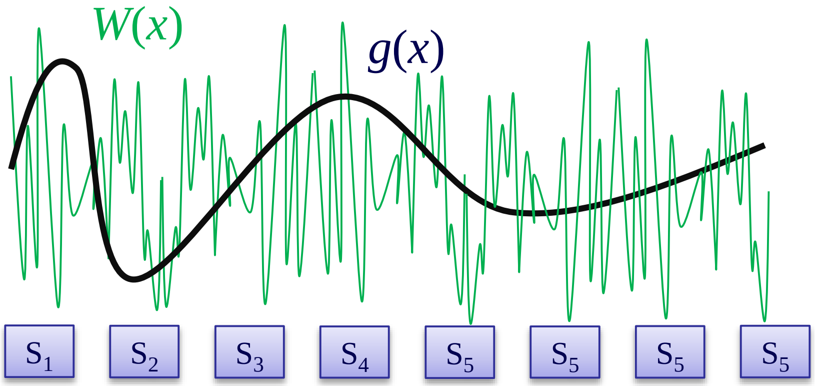
[Sharma and Kumar’2015] Evolution of aliasing-error term in sampling of non-bandlimited fields was examined. A procedure, using techniques from control theory, was established to determine increase in the aliasing error term with time for linear constant coefficient PDEs

For the second order PDE above, assuming $a_2 = 1$ without loss of generality

$$a_1 > 0,\ a_0 > 0,\ b_2 > (b_1^2/a_1^2)$$

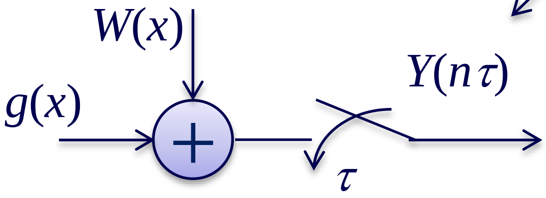
Sampling with fixed grid of sensors with measurement noise – tradeoff between oversampling and distortion (precision-indifference)

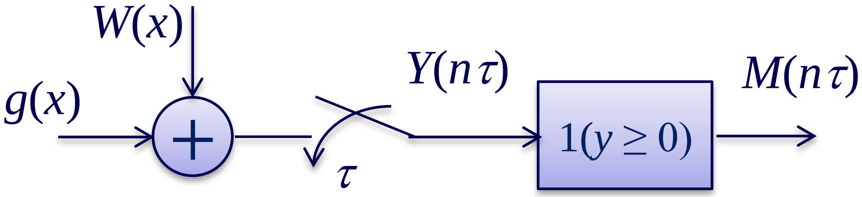
Sampling bandlimited field in Gaussian noise



Consider an array of sensors sampling a bandlimited field in additive and independent Gaussian (measurement) noise. The noise variance is finite

In particular, acquisition with full precision samples and acquisition with single-bit samples will be compared for maximum (over x) mean-squared error

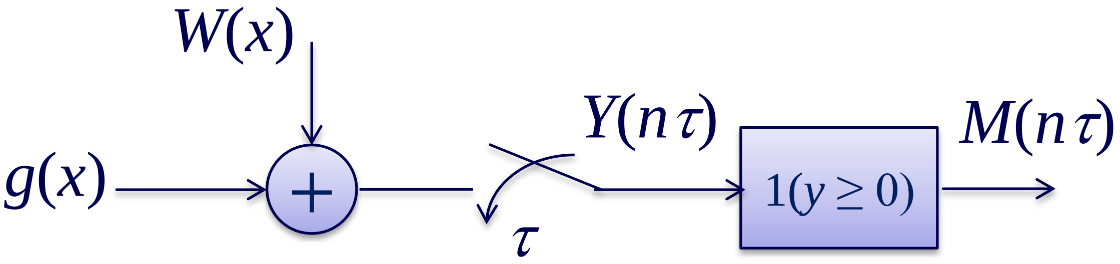




Theoretical abstraction of the problems

These sampling problems can be abstracted into the following

$g(x) + W(x)$ is a field to be sampled through precision-limited or single-bit quantizers (comparators)



We will assume that $g(x)$ is bandlimited in a finite bandwidth and $W(x)$ is an additive independent Gaussian noise process

Part 2: Organization

- ◊ Introduction to the problem
- ◊ Field/signal model
- ◊ Insights into main result using degrees of freedom
- ◊ Analysis of proposed single-bit quantization scheme
- ◊ Extensions and future work

Part 2: Organization

◊ Introduction to the problem

◊ Field/signal model

◊ Insights into main result using degrees of freedom

◊ Analysis of proposed single-bit quantization scheme

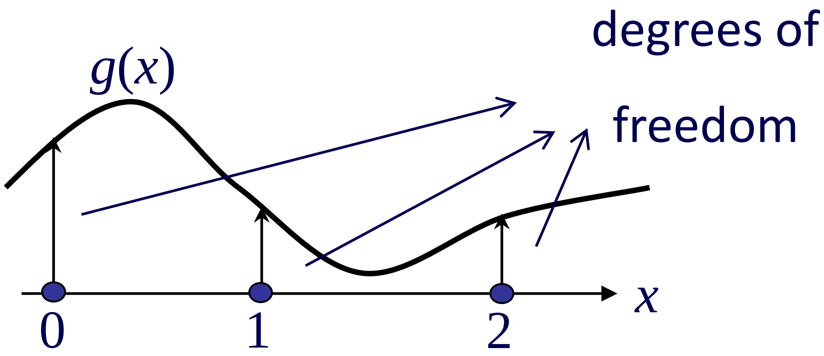
◊ Extensions and future work

Shannon interpolation formula and stability

For $g(x)$ bandlimited with $\omega_m = \pi$ it follows that

$$g(x) = \sum_{i=-\infty}^{\infty} g(i)\,\mathrm{sinc}\,(x-i) \quad \Bigg\}$$

The Shannon-Whittaker-Kotelnikov interpolation formula



The kernel $\mathrm{sinc}(x)$ is square integrable but not absolutely integrable,. This leads to instability in reconstruction

$|\varepsilon_i| \le \varepsilon$ and $\hat{g}(x) = \sum_{i=-\infty}^{\infty} \{g(i)+\varepsilon_i\} \dfrac{\sin \pi(x-i)}{\pi(x-i)}$ do not imply $|g(x) - \hat{g}(x)| \le O(\varepsilon)$

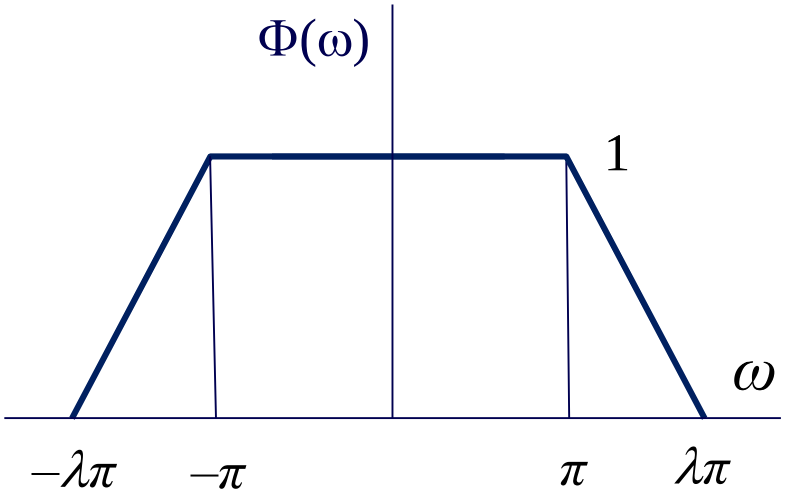
Zakai-sense bandlimited signal model

Define $\phi(x)$ as follows for $\lambda > 1$

and $a = (\lambda - 1)/2$

$$\phi(x) = \frac{1}{\pi a x^2} \sin((\pi + a)x) \sin(ax)$$

$$\phi(0) = 1 + \frac{a}{\pi}$$

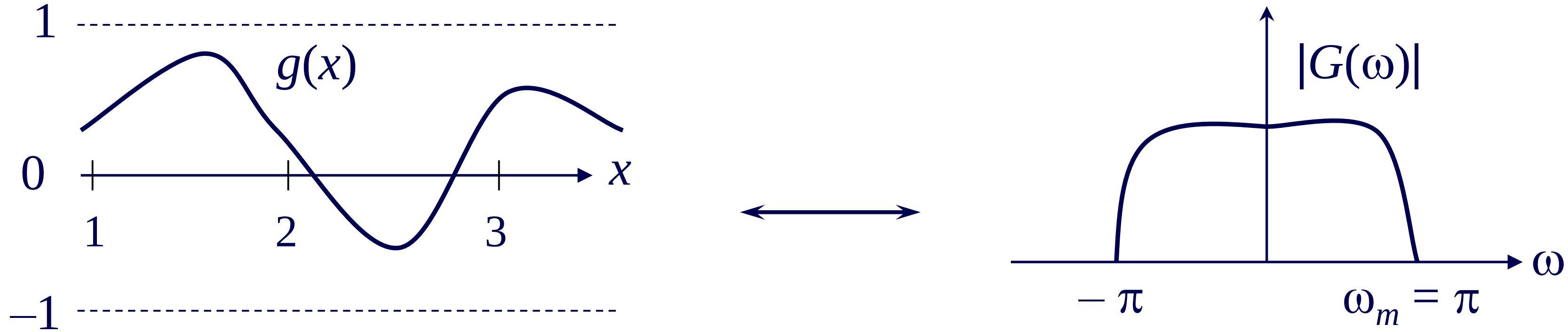


A subset of Zakai class of bandlimited signals is our signal model

$BL = \{g(x)$: $|g(x)| \leq 1$ and $g(x) \star \phi(x) = g(x)$, for all t real$\}$

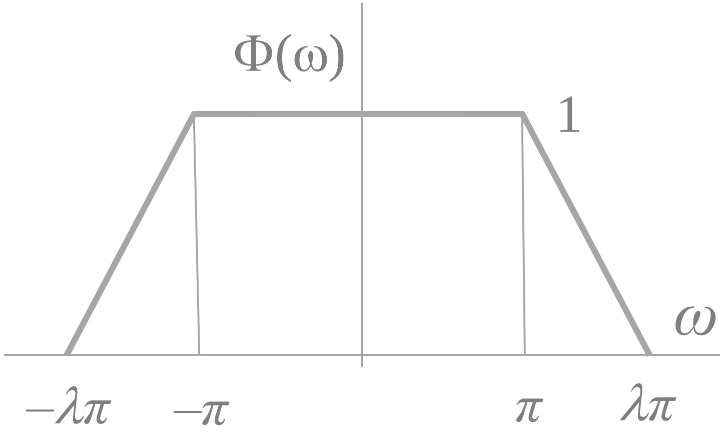
The kernel $\phi(x)$ is square and absolutely integrable, which aids in worst-case or pointwise error analysis

$L^2(R)$ bandlimited $\subset$ Zakai bandlimited



Nyquist interval = $X_{NQ} = \pi / \omega_m = 1$

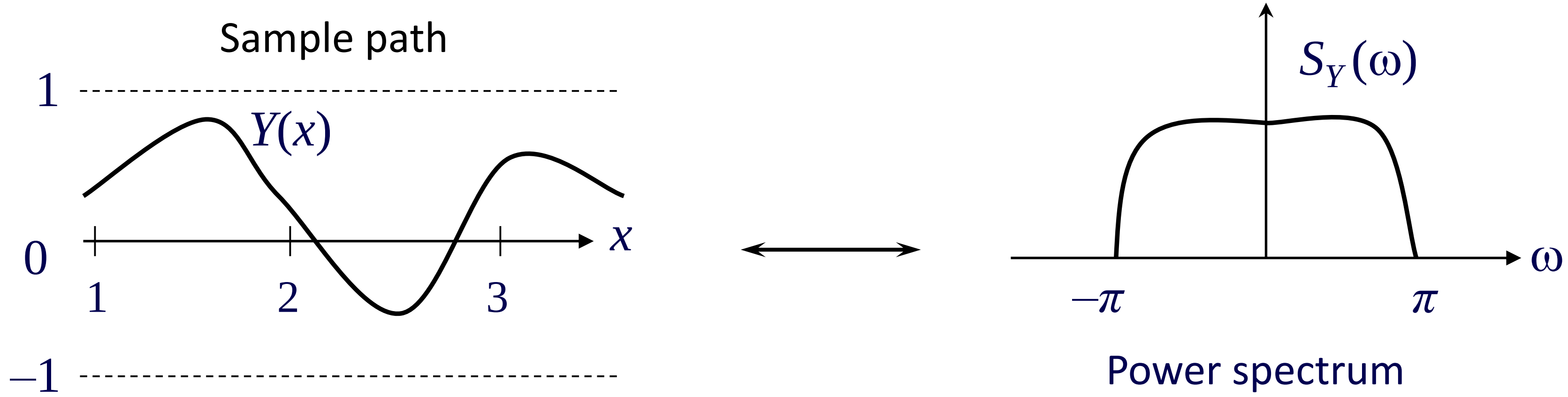
- 1-D real, continuous, and $|g(x)| < 1$
- Nyquist period, $X_{NQ} = 1$
- Finite energy in L^2 sense



Then $g(x)$ belongs to Zakai class of bandlimited signals since

$$g(x) \star \phi(x) = FT^{-1}[G(\omega)\Phi(\omega)] = FT^{-1}[G(\omega)] = g(x), \text{ for all } x \text{ real}$$

Stationary bandlimited $\subset$ Zakai bandlimited



Nyquist interval = $X_{NQ} = \pi / \omega_m$

Power spectrum

◊ 1-D, real valued, wide-sense stationary signals with amplitude sample paths bounded by 1.

◊ Autocorrelation function is finite-energy bandlimited with $T_{NQ} = 1$

◊ See [Zakai'65] and [Masry'76] ,

$$Y(x) = Y(x) \star \phi(x),$$

almost surely for WSS bandlimited $Y(x)$

Properties of Zakai sense bandlimited signals

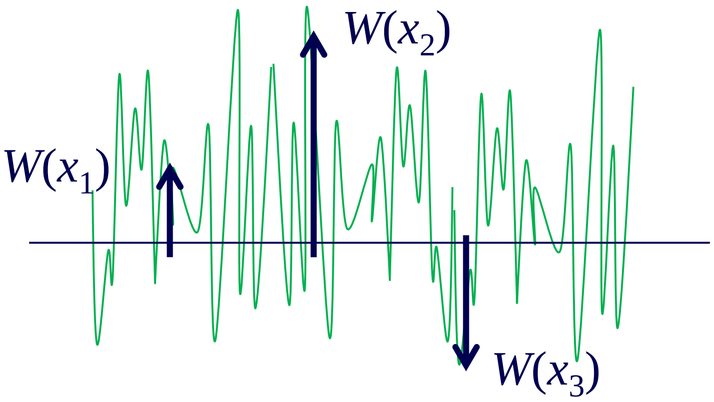
- ◊ Any result that applies to bounded-amplitude Zakai sense bandlimited signals will also apply to bounded-amplitude finite energy bandlimited signals as well as bounded stationary bandlimited signals
- ◊ Since $\phi(x)$ is smooth and absolutely integrable, $g(x) = g(x) \star \phi(x)$ can be used to establish smoothness of $g(x)$
- ◊ Zakai sense bandlimited signals admit a sampling theorem with a stability factor $\rho < 1$. We have

$$g(x) = \lambda \sum_{i \in \mathbb{Z}} g\left(\frac{i}{\lambda}\right) \phi\left(x - \frac{i}{\lambda}\right)$$

Noise model and mean-squared distortion

The noise $W(x)$ is assumed to be an independent Gaussian process. That is,

$W(x_1), W(x_2), \ldots, W(x_n)$ are independent and identically distributed $N(0, \sigma^2)$



For example, $W(x_1), W(x_2), W(x_3)$ are independent for any x_1, x_2, x_3

Distortion considered for statistical signals is maximum mean-squared error

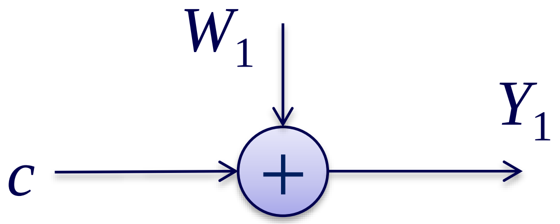
$$D_{\text{rec}} := \sup_{x \in \mathbb{R}} D_{\text{rec}}(x) = \sup_{x \in \mathbb{R}} \mathbb{E}\left[\widehat{G}_{\text{rec}}(x) - g(x)\right]^2$$

Part 2: Organization

- ◊ Introduction to the problem
- ◊ Field/signal model
- ◊ Insights into main result using degrees of freedom
- ◊ Analysis of proposed single-bit quantization scheme
- ◊ Extensions and future work

Estimation of a constant from one reading

Consider the problem of estimating a bounded constant (one degree of freedom) in one reading with additive Gaussian noise

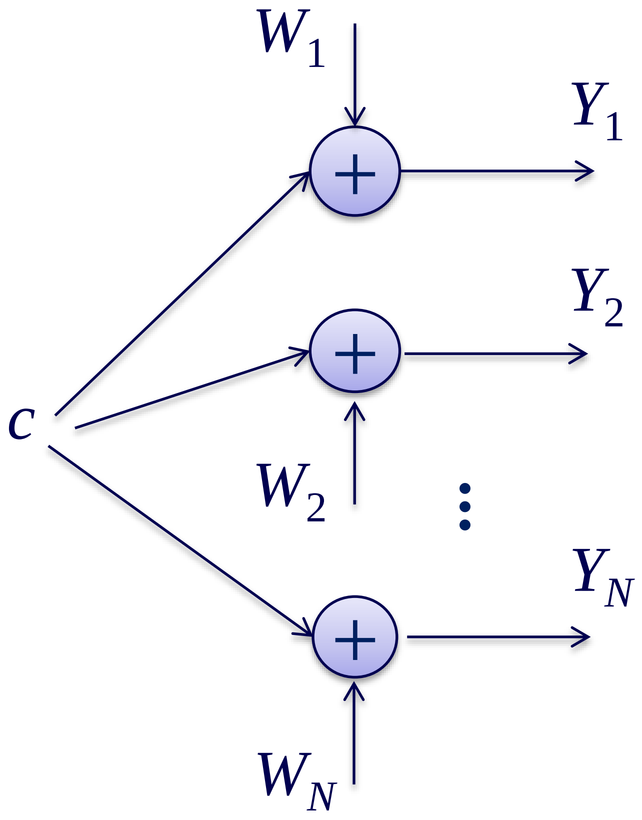


That is estimate c from $Y_1 = c + W_1$

There will be a mean-squared error $\geq \sigma^2$ regardless of the procedure adopted (see Cramer-Rao lower bound)

Oversampling is needed to reduce the mean-squared error

Estimation of a constant with oversampling



◊ Now consider the problem of estimating a **bounded** constant (one degree of freedom) in additive independent (i.i.d.) Gaussian noise

$$Y_1 = c + W_1, \ldots, Y_N = c + W_N$$

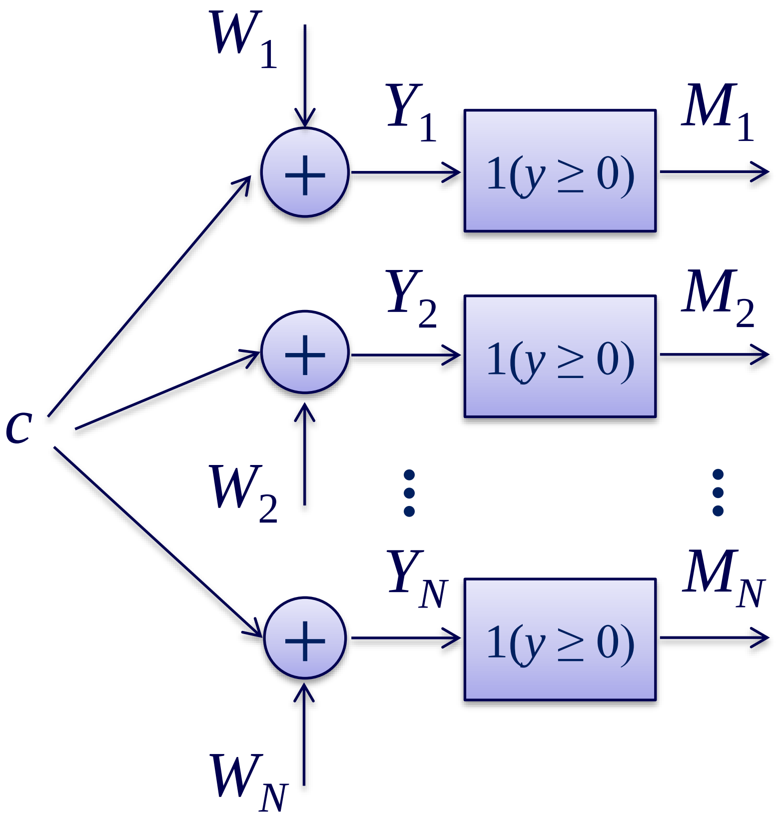
◊ It is known that the best mean-squared error estimate is the avg of $Y_1, Y_2, \ldots, Y_N$

◊ And mean-squared error between $(Y_1 + \ldots Y_N)/N$ and c is $O(1/N)$

Unquantized samples, oversampling by N, and mean-squared error is $O(1/N)$

Estimation and quantization (nonlinearity)

Now consider the same problem in the presence of single-bit quantization



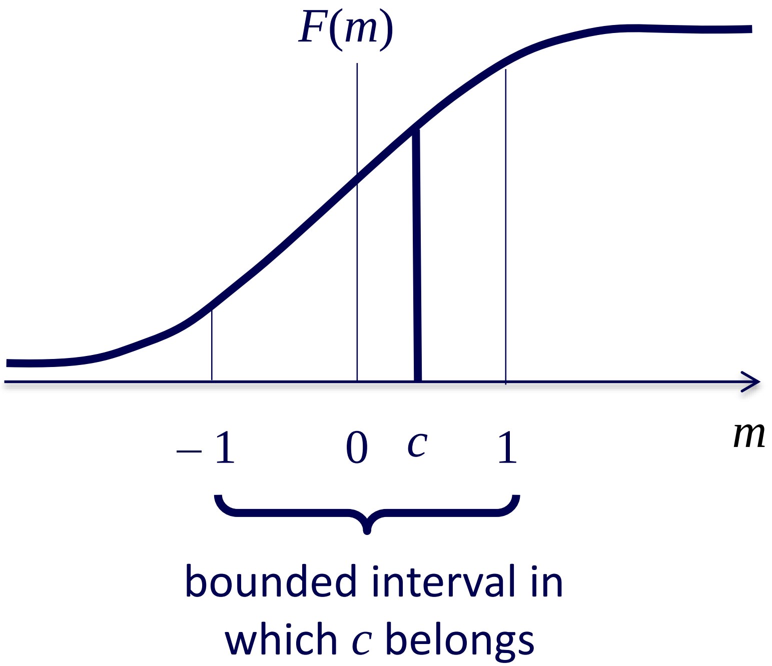
$M_1 = 1(c + W_1 \geq 0)$, $M_2 = 1(c + W_2 \geq 0)$, $\ldots$, $M_N = 1(c + W_N \geq 0)$

◊ These variables are i.i.d. Bernoulli($F(c)$), where $F(x)$ is the cumulative distribution function of W

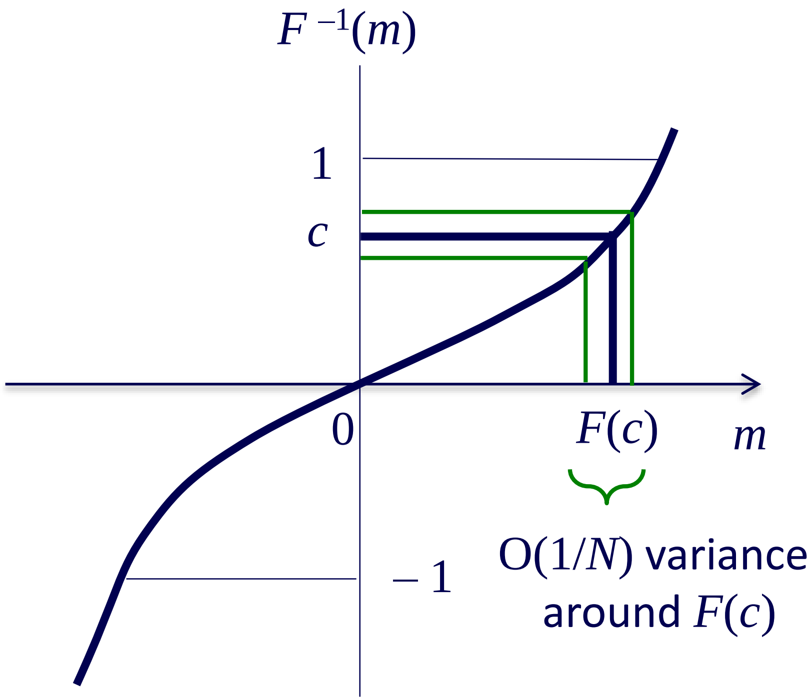
◊ It is known that the average of M_1, M_2, $\ldots$, M_N converges with variance $O(1/N)$ to $F(c)$

Quantized samples, oversampling by N, and mean-squared error in $F(c)$ is $O(1/N)$

The delta-method



An estimate for c is desired, but an estimate for $F(c)$ is present with small variance of $O(1/N)$



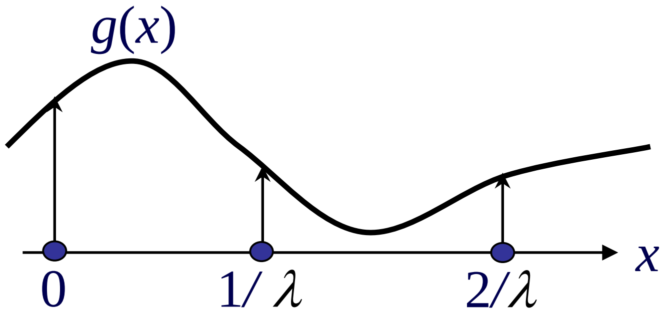
If $F^{-1}(m)$ has a finite slope, then

$F^{-1}[(M_1 + \ldots + M_N)/N]$ converges to

$F^{-1}[F(c)] = c$ with a variance of $O(1/N)$

Thus “**precision indifference**” holds while estimating one degree of freedom in Gaussian noise

Noisy samples and bandlimited signals



$$g(x) = \lambda \sum_{i \in \mathbb{Z}} g\left(\frac{i}{\lambda}\right) \phi\left(x - \frac{i}{\lambda}\right)$$

◊ Samples are uniformly spaced slightly closer than the Nyquist points ($\lambda > 1$)

◊ Thus, there is one degree of freedom every Nyquist interval in $g(x)$

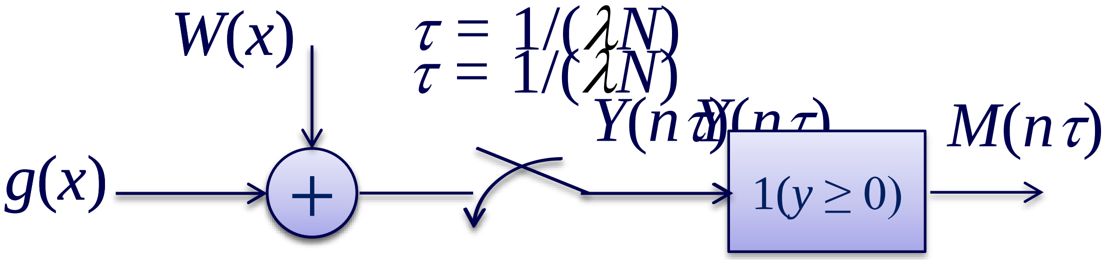
◊ If we oversample $g(x) + W(x)$ by a factor of N, there are N noisy reading for each degree of freedom on an average

◊ Thus we expect optimal distortion to be $O(1/N)$ with perfect samples!

Key result that will be shown next

A distortion of $O(1/N)$ is achievable with single-bit quantized samples!

This improves the previously known bound of $O(1/N^{2/3})$ **[Masry 1981]**



The Zakai class of bandlimited signals will be the signal model

$$BL = \{g(x) \colon |g(x)| \leq 1 \text{ and } g(x) \star \phi(x) = g(x), \text{ for all } x \text{ real}\}$$

The results will apply to finite energy bandlimited signals as well as stationary bandlimited signals

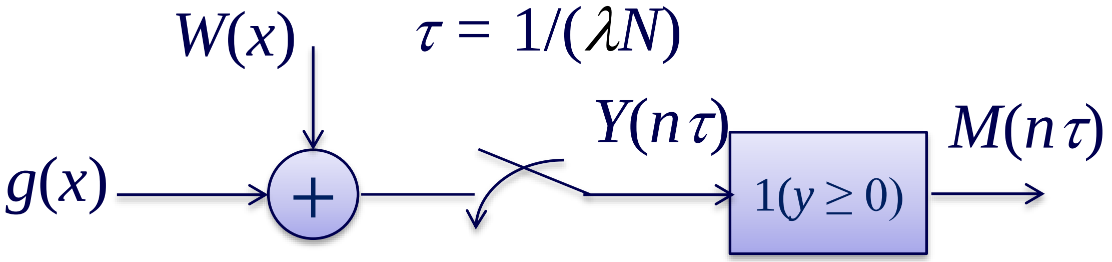
Related work

- **[Masry'1981]** Single-bit quantization of smooth signals, with additive noise as a dither. His analysis results in an mean-squared bound of $O(1/N^{2/3})$
- **[Wang-Ishwar'2009]** and **[Masry-Ishwar'2009]** Acquisition of a finite-support field in bounded noise. Mean-squared error bounds were established

Part 2: Organization

- ◊ Introduction to the problem
- ◊ Field/signal model
- ◊ Insights into main result using degrees of freedom
- ◊ Analysis of proposed single-bit quantization scheme
- ◊ Extensions and future work

Interpolation of quantized samples



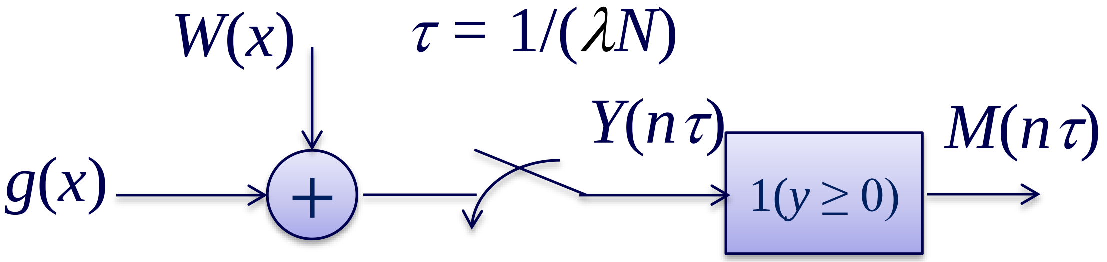
Define an interpolation from the quantized single-bit samples $M(n\tau)$

$$H_N(x) = \tau \sum_{n=-\infty}^{\infty} (M(n\tau) - ½)\phi(x - n\tau)$$

Note that $E[M(x)] = F(g(x))$

$$\tau \sum_{n=-\infty}^{\infty} (M(n\tau) - ½)\phi(x - n\tau) \xrightarrow{\text{LLN}} \tau \sum_{n=-\infty}^{\infty} (F(g(n\tau)) - ½)\phi(x - n\tau) \xrightarrow{\text{Smooth-ness}} (F(g(x)) - ½) \star \phi(x)$$

Interpolation of quantized samples



Remember that

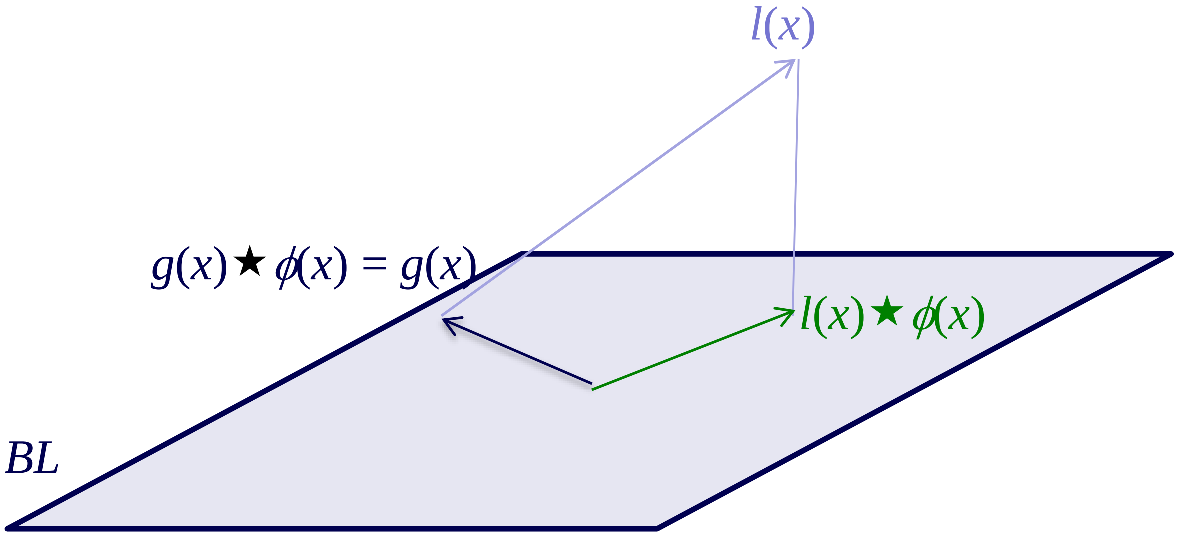
$$H_N(x) = \tau \sum_{n=-\infty}^{\infty} (M(n\tau) - ½)\phi(x - n\tau)$$

Proposition: Let $l(x) = (F(g(x)) - ½)$ and $H_N(x)$ be as defined above. Then,

$$\sup_{x \in \mathbb{R}} \mathbb{E}\left[|H_N(x) - l(x) \star \phi(x)|^2\right] \leq \frac{C_2}{N}$$

where C_2 does not depend on $g(x)$

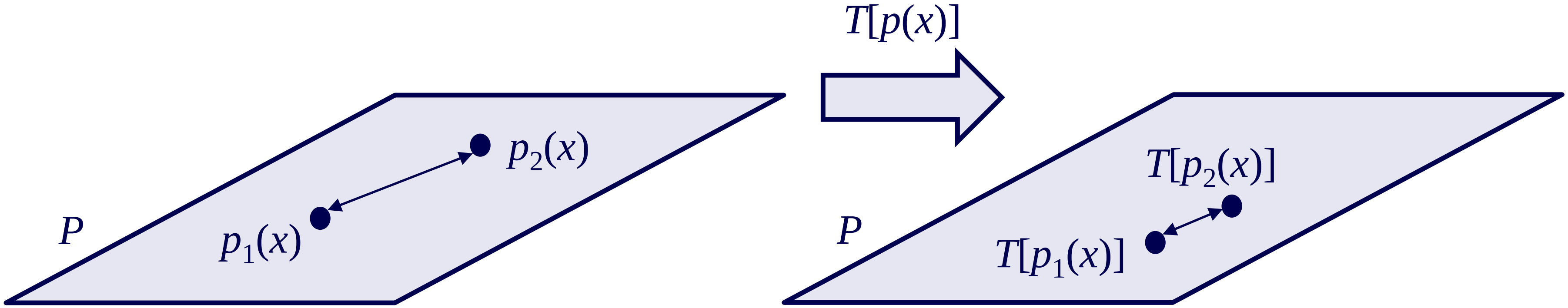
Invertibility of the limit



For the set BL the signal $l(x)\star\phi(x)$ is invertible and $g(x)$ can be obtained uniquely from it (in a pointwise or L^∞ sense). The proof follows using Banach's contraction theorem

$L^2(R)$ version of this problem has been considered by Landau and Miranker'1961. It cannot be used directly since statistical noise is not in L^2

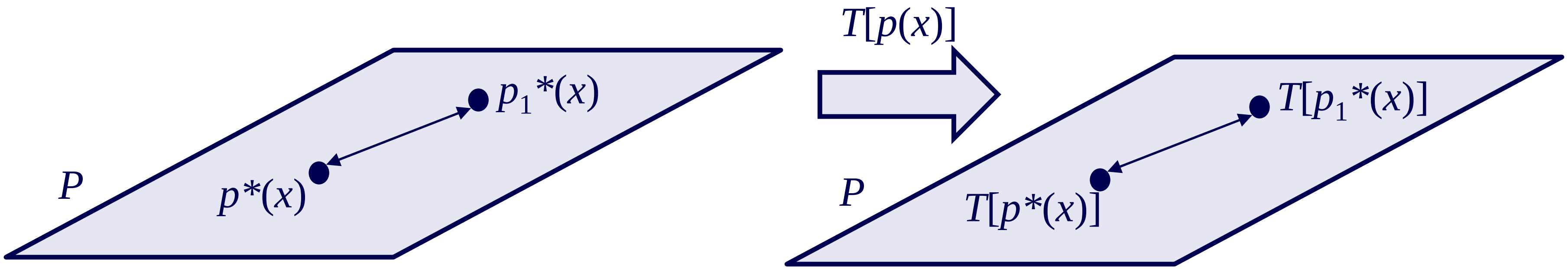
Banach’s contraction theorem



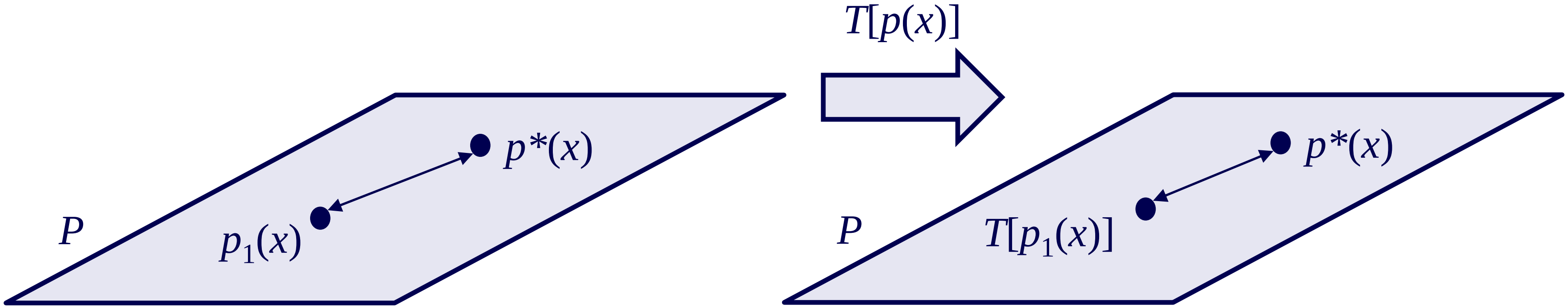
Ingredients: Banach’s contraction theorem needs a closed set P, a map T, a distance metric dist(p_1, p_2) and contraction for any $p_1(x)$, $p_2(x)$ in P

$$\mathrm{dist}(T[p_1], T[p_2]) \leq \alpha\, \mathrm{dist}(p_1, p_2) \text{ with } 0 < \alpha < 1$$

Result: Then $T[p(x)] = p(x)$ has a unique solution $p^*(x)$ in P.



Banach's contraction theorem (contd.)



Fact: $p_i(x) = T[p_{i\text{-}1}(x)]$ sets up a recursive procedure to approach $p^*(x)$

And $p^*(x) = p_0(x) + [p_1(x) - p_0(x)] + [p_2(x) - p_1(x)] + [p_3(x) - p_2(x)] + \dots$

$$= p_0(x) + [p_1(x) - p_0(x)] + T[p_1(x) - p_0(x)] + T^2[p_1(x) - p_0(x)] + \dots$$

Bandlimited space and projection

Recall, for $g(x)$ bandlimited with $\omega_m = \pi$ it is known that

$$g(x) = \sum_{i=-\infty}^{\infty} g(l) \frac{\sin \pi(x-i)}{\pi(x-i)}$$

The kernel $\mathrm{sinc}(x) := \sin(\pi x)/\pi x$ is square integrable and its shifts span the set of square integrable bandlimited signals

For any square integrable signal $y(x)$

$$y(x) \star \frac{\sin(\pi x)}{\pi x} = FT^{-1}[Y(\omega) \cdot \mathbb{1}(|\omega| \leq \pi)]$$

This convolution is a **projection**, in the squared error norm, on the space of signals bandlimited to $\omega_m = \pi$

Landau and Miranker's contraction formula

Consider $L^2\text{-}BL = \{g(x)\text{: } G(\omega)$ with support on $[-\pi, \pi]\}$

For this class of signal, the following fixed-point and recursion formula leads to $g(x)$ from $h(x) = [F(g(x)) - ½] \star \text{sinc}(x)$

$$g(x) = \beta\, h(x) + (g(x) - \beta\, l(x)) \star \text{sinc}(x)$$

Fixed-point equation

$$g_{i+1}(x) = \beta\, h(x) + (g_i(x) - \beta\, l_i(x)) \star \text{sinc}(x)$$

Recursion formula
(Picard iteration)

Landau & Miranker [1961]

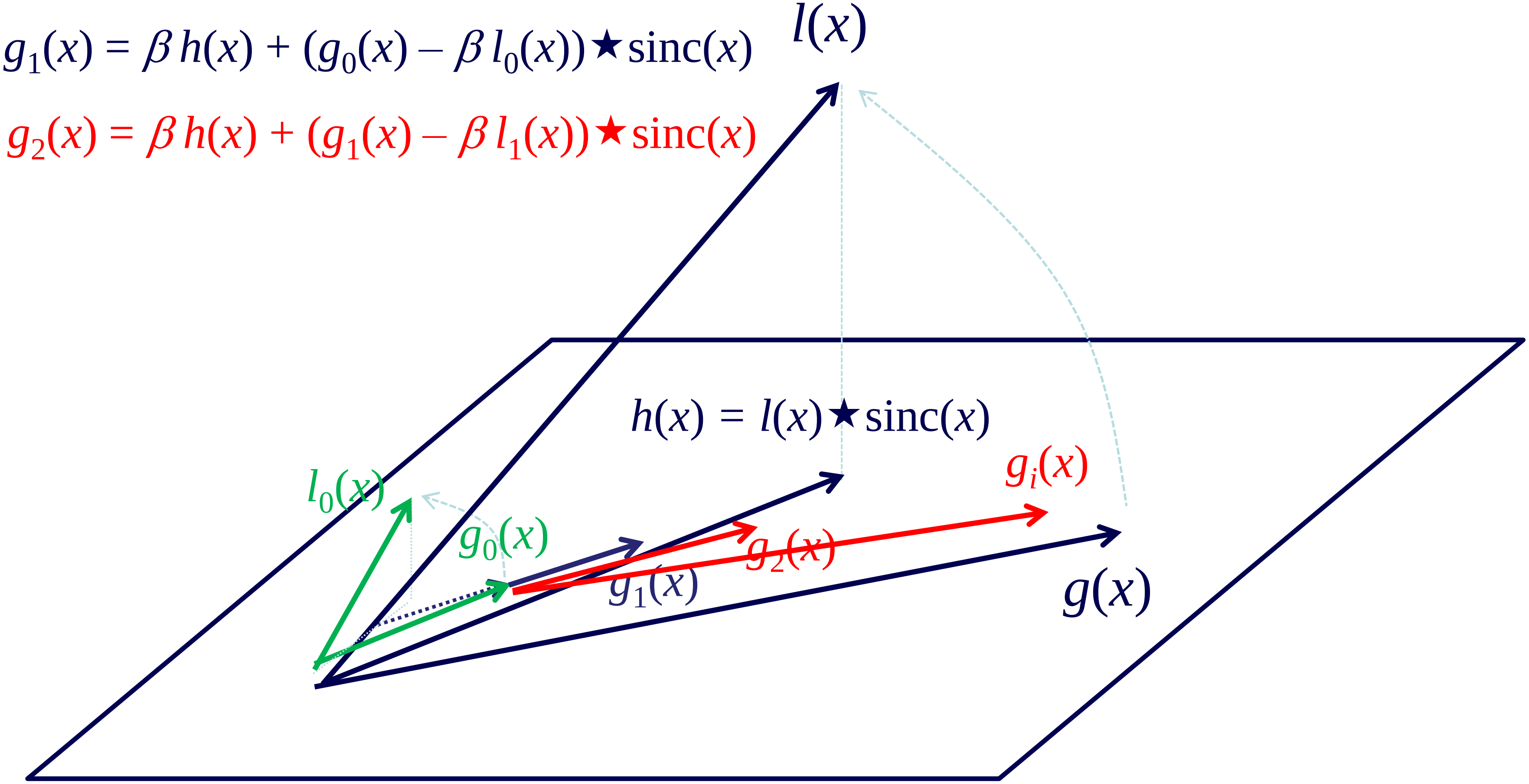
The technique works since $L^2\text{-}BL$ is a closed subset and there is a value of β for which the recursion formula is a contraction (Banach's contraction mapping theorem)

Landau and Miranker's recursion in pictures

$g_0(x) = \beta\, h(x)$

$g_1(x) = \beta\, h(x) + (g_0(x) - \beta\, l_0(x)) \bigstar \mathrm{sinc}(x)$

$g_2(x) = \beta\, h(x) + (g_1(x) - \beta\, l_1(x)) \bigstar \mathrm{sinc}(x)$



Ingredients of contraction for our problem

Let BL_{bdd} be the set defined as

$$BL_{\text{bdd}} = \{p(x) \colon |p(x)| \leq C_\phi \text{ and } p(x) \star [\rho\phi(x/\rho)] = p(x), \text{ for all } t \text{ real}\}$$

where $\rho\phi(x/\rho)$ has slightly larger bandwidth than $\phi(x)$

The distance metric is the maximum pointwise difference (L^∞ error)

Define $\text{Clip}[c] = \text{sgn}(c)$ if $|c| > 1$ and $\text{Clip}[c] = c$, otherwise

$$T[g(x)] = \text{Clip}\Big[\mu\, l(x) \star \phi(x) + [g(x) - \mu\, l(x)] \star \phi(x)\Big] \star \phi(x)$$

Fixed-point equation

Set $g_0(x) = 0$ and

$$g_{i+1}(x) = \text{Clip}\Big[\mu\, l(x) \star \phi(x) + [g_i(x) - \mu\, l_i(x)] \star \phi(x)\Big] \star \phi(x)$$

Recursion formula

It turns out that there is a value of μ such that T is a contraction on BL_{bdd}

So $g(x)$ can be obtained from $l(x)$ by using T and recursion on BL_{bdd}

Final step of the proof

But $H_N(x)$ an approximation of $l(x) = F(g(x)) - ½)$ is available! However, contraction is stable to perturbations

The modified recursive map is

$$T[G_{i+1}(x)] = \mathrm{Clip}\{\mu\, H_N(x) \star \phi(x) + [G_i(x) - \mu\,(F(G_i(x)) - 1/2) \star \phi(x)\ ]\} \star \phi(x)$$

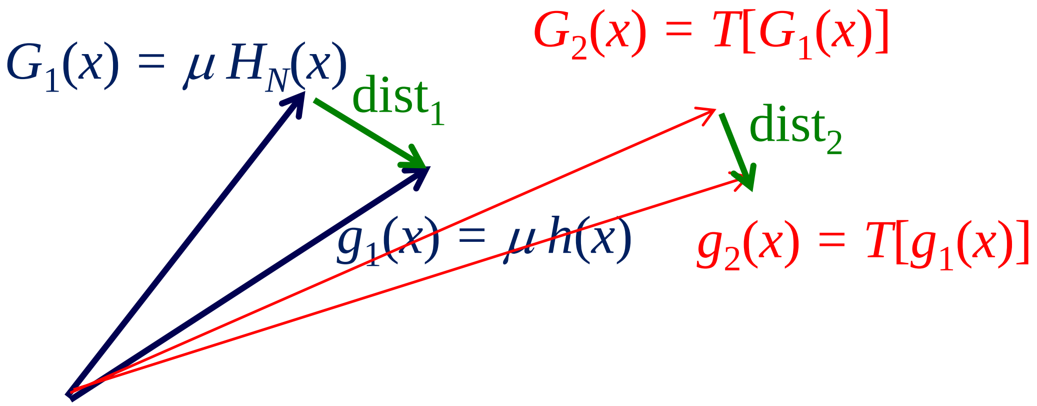
Theorem: Let $G_0(x) = 0$. Let $\hat{G}_{\text{1-bit}}(x)$ be the limit of $G_i(x)$. Then, the above recursion results in

$$D_{\text{1-bit}} := \sup_{x \in \mathbb{R}} \mathbb{E}\left[\widehat{G}_{\text{1-bit}}(x) - g(x)\right]^2 = O(1/N)$$

which establishes the precision-indifference principle for bandlimited signals in additive independent Gaussian noise

Picture for the final step of proof

$H_N(x)$ an approximation of $l(x) = F(g(x)) - \frac{1}{2})$ is available. Contraction is stable to perturbations



By contraction property, … $\text{dist}_3 \le \alpha\,\text{dist}_2 \le \alpha^2\,\text{dist}_1$

Thus, by triangle inequality, the maximum error can be shown to be sum of all dist_i's, or $\text{dist}_1/(1\text{-}\alpha)$

Part 2: Organization

- ◊ Introduction to the problem
- ◊ Field/signal model
- ◊ Insights into main result using degrees of freedom
- ◊ Analysis of proposed single-bit quantization scheme
- ◊ Extensions and future work

Precision indifference for polynomial fields

Field model: Consider the class of bounded polynomials with support on [-1, 1] and with a finite maximum degree. The field is observed with single-bit sensors having additive independent Gaussian noise in measurement

Main Result: There exists an estimate $V_{\text{1-bit}}(x)$ for the polynomial field $v(x)$ such that

$$D_{\text{1-bit}} := \sup_{x \in [-1,1]} \mathbb{E}\left[\widehat{V}_{\text{1-bit}}(x) - v(x)\right]^2 = O(1/N)$$

which establishes the precision-indifference principle for polynomial fields in additive independent Gaussian noise

In review: sampling of polynomial signals on a finite support **[Bhatt-Kumar' (under review)]**

Summary

For bounded-amplitude bandlimited fields sampled in the presence of (additive) independent Gaussian process with oversampling N and mean-squared distortion

◊ Optimal distortion is expected to be

$$= \mathrm{O}(1/N)$$

◊ Distortion achievable with single-bit quantized readings

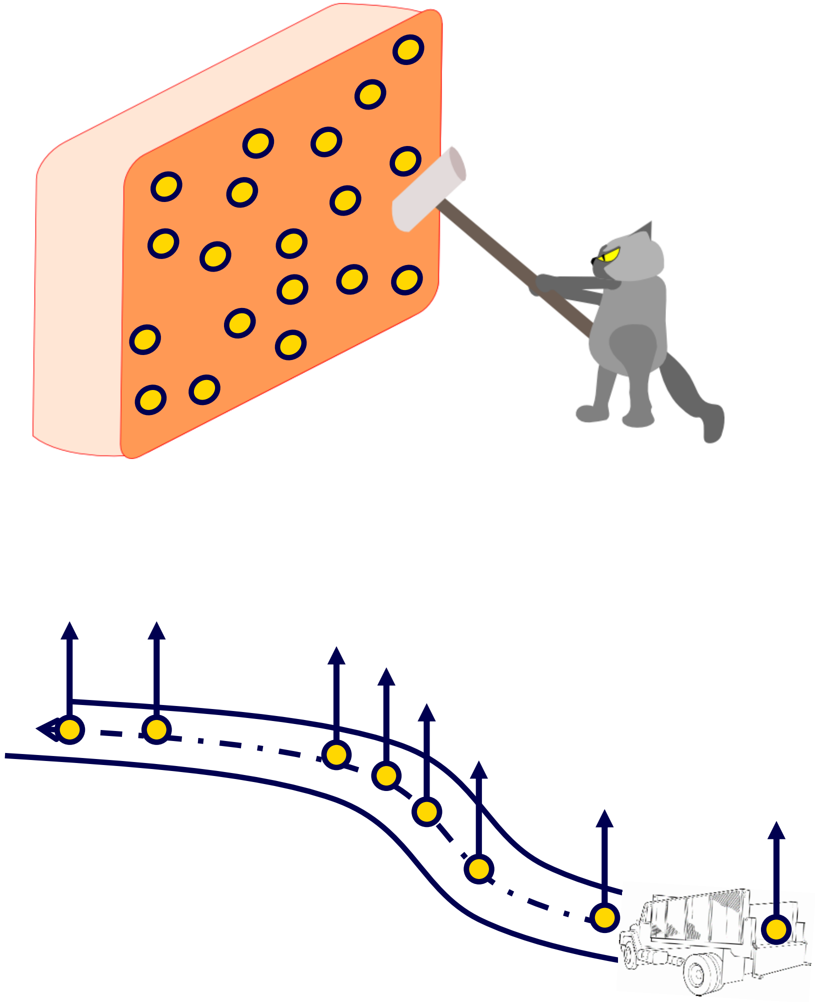
$$= \mathrm{O}(1/N)$$

Future work

- Our estimate is not minimum risk. Fast algorithms for finding Maximum likelihood estimates, which will also be accurate up to $O(1/N)$, will be useful
- Extension of these results to more classes of fields (FRI, finite-support, orthogonal spaces, non-bandlimited fields)

Sampling with unknown but statistically distributed sample locations – tradeoff between oversampling and distortion

Spatial sampling with “unknown” location



- Sampling with fixed array of randomly deployed sensors:
  - **Key issues:** nonuniform unknown locations, quantization, noise, temporal variation
  - **Signal structure:** bandlimited, smooth

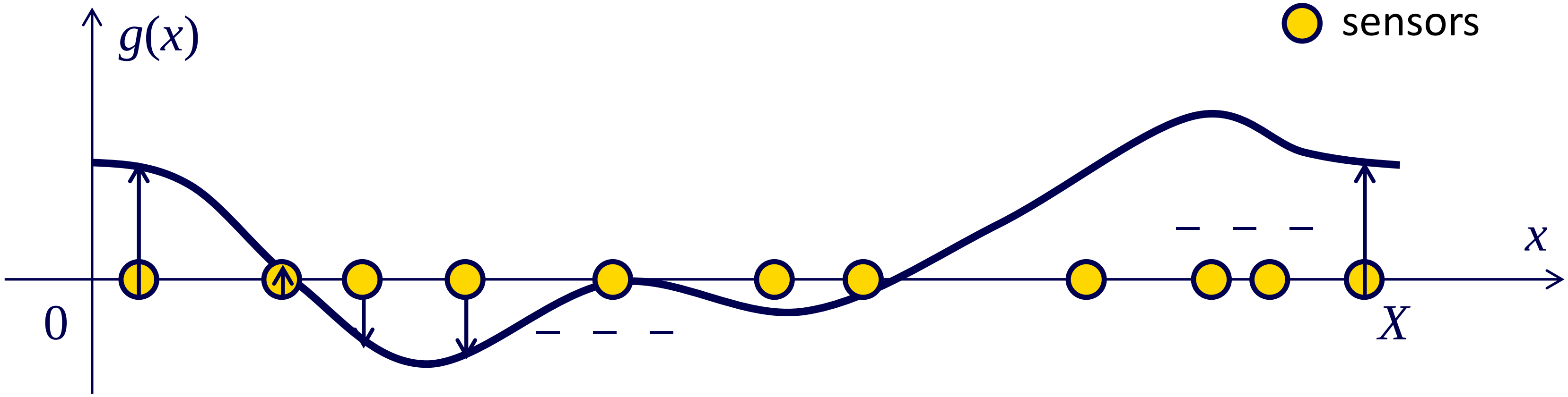
Part 3: Organization

- Sampling model, field model, and distortion
- Field estimation without any knowledge of sampling location
- Field estimation when order of samples is known
- Field estimation with order of samples known in the presence of measurement noise
- Future work

Part 3: Organization

- ◊ Sampling model, field model, and distortion
- ◊ Field estimation without any knowledge of sampling location
- ◊ Field estimation when order of samples is known
- ◊ Field estimation with order of samples known in the presence of measurement noise
- ◊ Future work

Sampling model



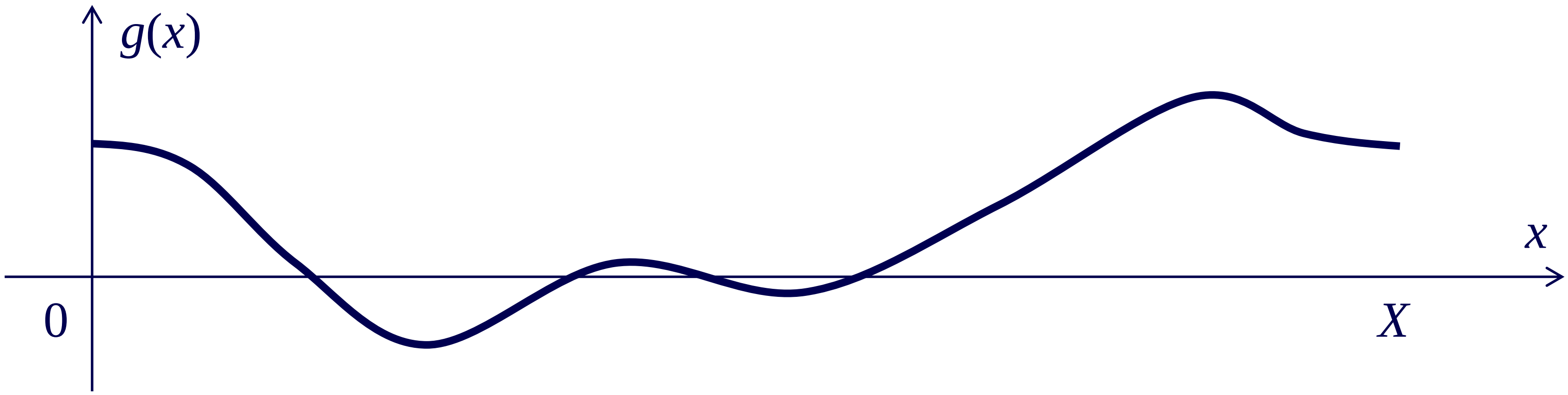
Motivated by the smart-dust paradigm, where a lot of sensors are “scattered” in a region, we consider **random** deployment of sensor for sampling the field

There are two possible scenarios:

◊ When the sensor locations are random but known

◊ When the sensor locations are **unknown** but their statistical distribution is known

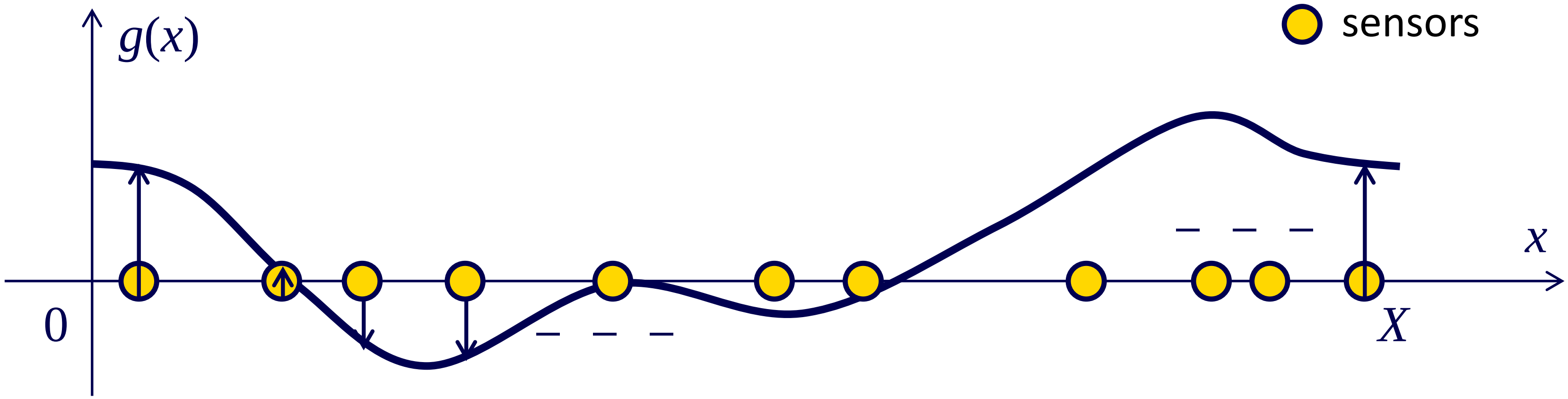
Spatial acquisition problem of interest



Consider the acquisition problem, where a smooth field in a finite interval has to be sampled or estimated

Example: acquisition of spatial fields with sensors

Sampling model

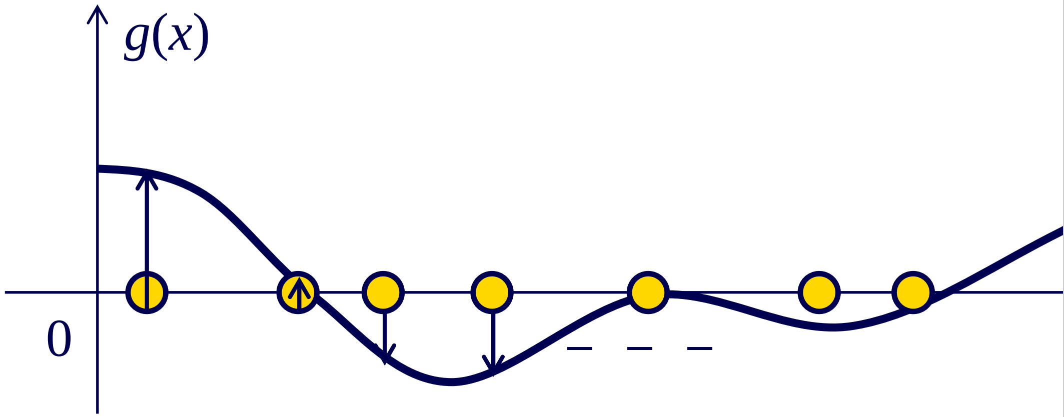


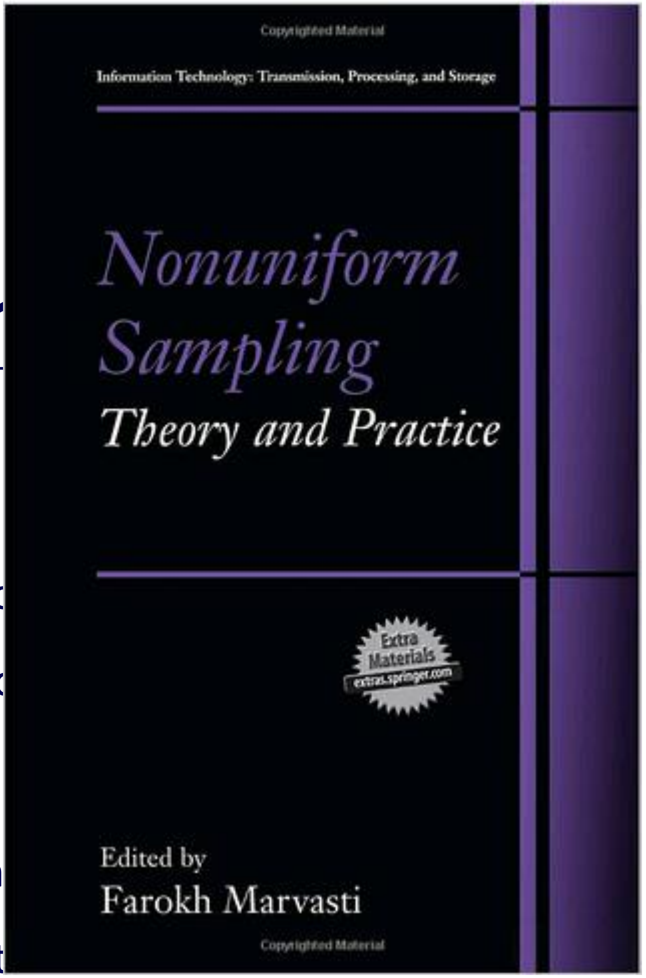
Motivated by the smart-dust paradigm, where a lot of sensors are "scattered" in a region, we consider **random** deployment of sensor for sampling the field

There are two possible scenarios:

- ◊ When the sensor locations are random but known
- ◊ When the sensor locations are **unknown** but their statistical distribution is known

Sampling model





Information Technology: Transmission, Processing, and Storage

Nonuniform Sampling
Theory and Practice

Edited by
Farokh Marvasti

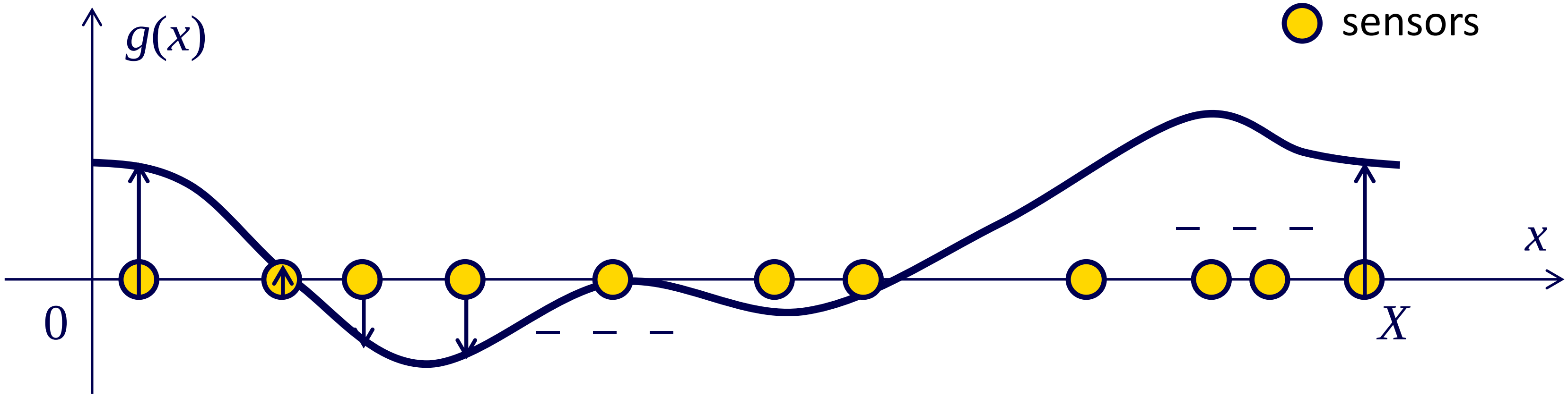
Motivated by the smart-dust paradigm, where a lot

region, we consider **random** deployment of sensor f

There are two possible scenarios:

- ◊ When the sensor locations are random but kn
- ◊ When the sensor locations are **unknown** but t
  is known

Sampling model

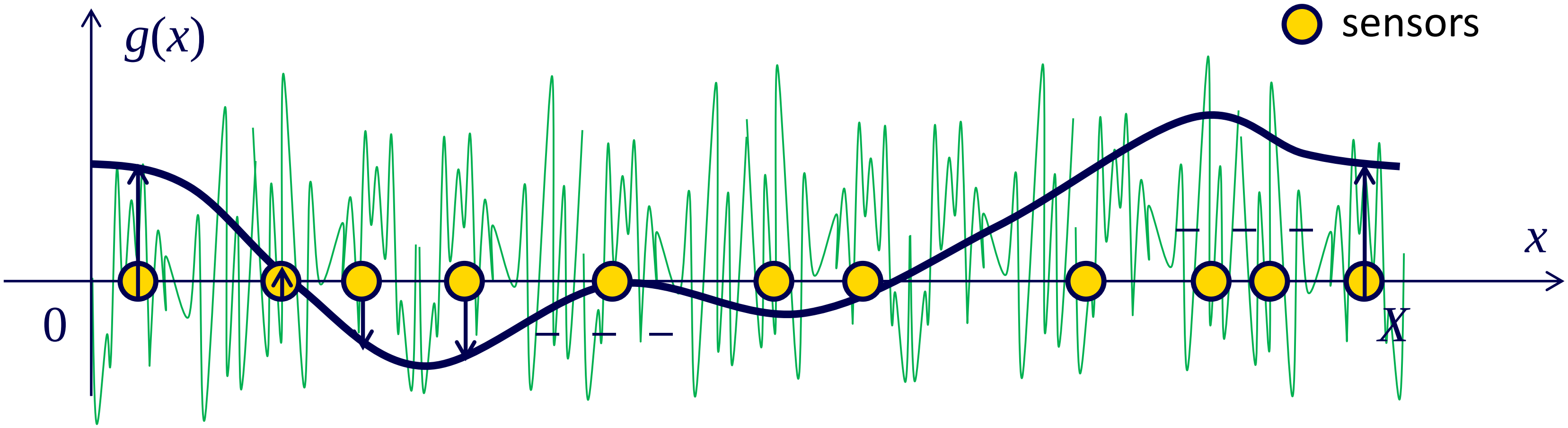


Motivated by the smart-dust paradigm, where a lot of sensors are “scattered” in a region, we consider **random** deployment of sensor for sampling the field

There are two possible scenarios:

- ◊ When the sensor locations are random but known
- ◊ When the sensor locations are **unknown** but their statistical distribution is known

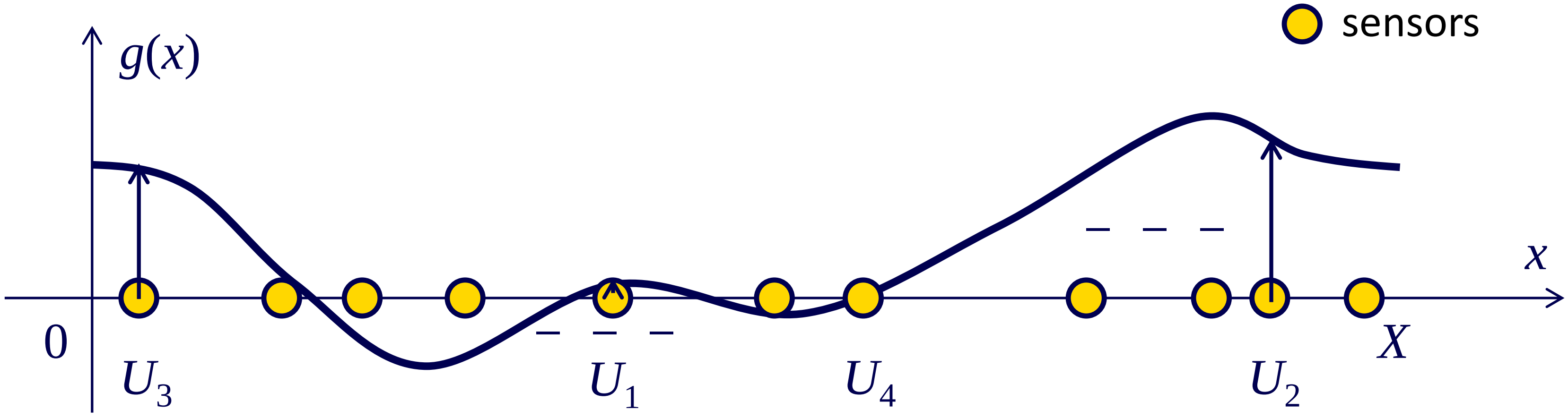
Field model with/without measurement noise



For the same model on random deployment of sensors for sampling the field with unknown sensor locations

- ◊ When the sensor measurements are not affected by noise
- ◊ When the sensor measurements are affected by additive noise with finite variance

Field and sensor-locations models

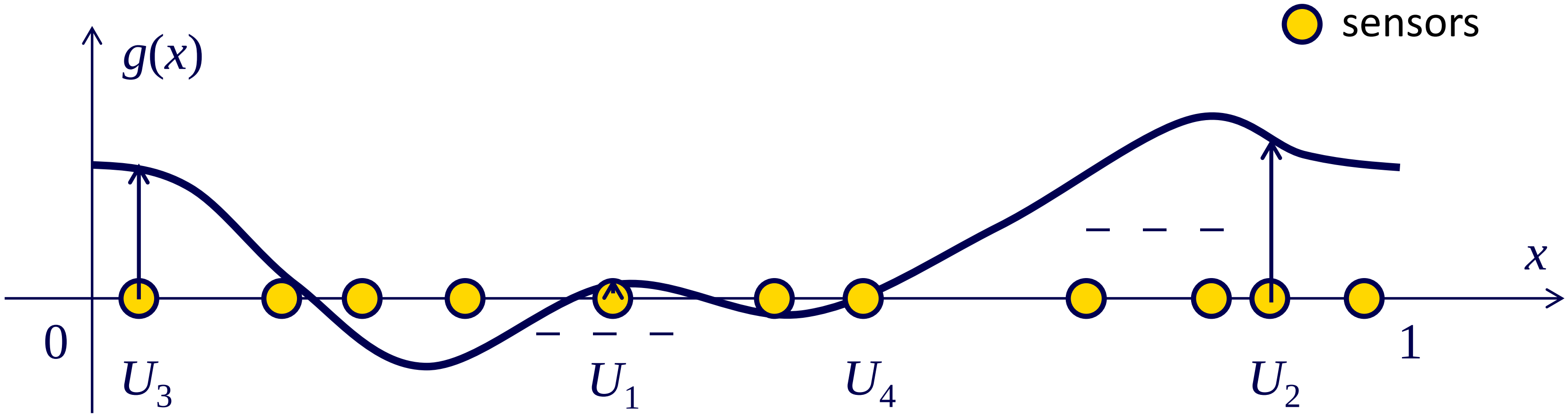


Sensor locations are **unknown** but their statistical distribution is known. For this work, $U_1^n = (U_1, U_2, \ldots, U_n)$ are i.i.d. Unif$[0,X]$

We assume that a periodic extension of the field $g(x)$ is bandlimited, that is, $g(x)$ is given by a finite number of Fourier series coefficients, (WLOG) $|g(x)| \leq 1$, and $X = 1$

$$g(x) = \sum_{n=-b}^{b} a_n \exp(j2\pi n x)$$

Observations made and distortion criterion



$\mathbf{G}^T = (g(U_1), g(U_2), \ldots, g(U_n))$ is collected without the knowledge of $(U_1, U_2, \ldots, U_n)$

We wish to estimate $g(x)$ and measure the performance of estimate against the average mean-squared error, i.e., if $\hat{G}(x)$ is the estimate then

$$D := \|\widehat{G} - g\|_2^2 := \int_0^1 |\widehat{G}(x) - g(x)|^2 \mathrm{d}x$$

Main results

◊ A bandlimited field *cannot be* uniquely determined with (perfect) samples obtained at statistically distributed locations, even if the number of samples is infinite

◊ If the **order** (left to right) of sample locations is known, a consistent estimate $\hat{G}(x)$ for the field of interest can be obtained

- Distortion and weak convergence results are established for this estimate $\hat{G}(x)$
- Distortion results are also established for an estimate of $g(x)$ in the case when field is affected by additive independent noise with finite variance

Related work

◊ Recovery of (narrowband) discrete-time bandlimited signals from samples taken at unknown locations **[Marziliano and Vetterli'2000]**

◊ Recovery of a bandlimited signal from a finite number of ordered nonuniform samples at unknown sampling locations **[Browning'2007]**.

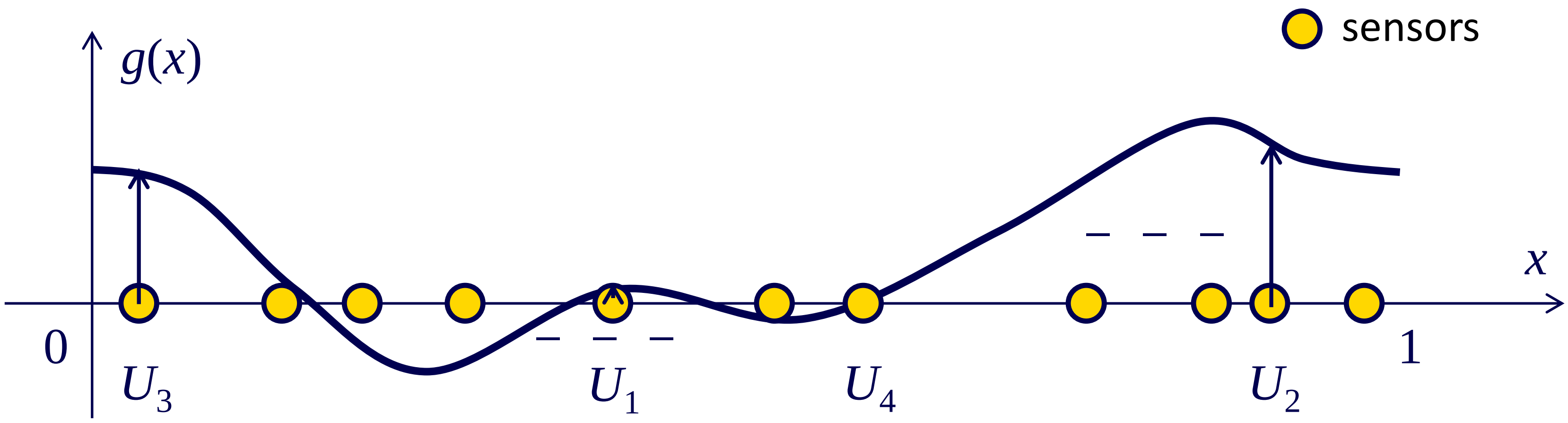
◊ Estimation of periodic bandlimited signals in the presence of random sampling location under two models **[Nordio, Chiasserini, and Viterbo'2008]**

- Reconstruction of bandlimited signal affected by noise at random but known locations
- Estimation of bandlimited signal from noisy samples on a location set obtained by random perturbation of equi-spaced deterministic grid

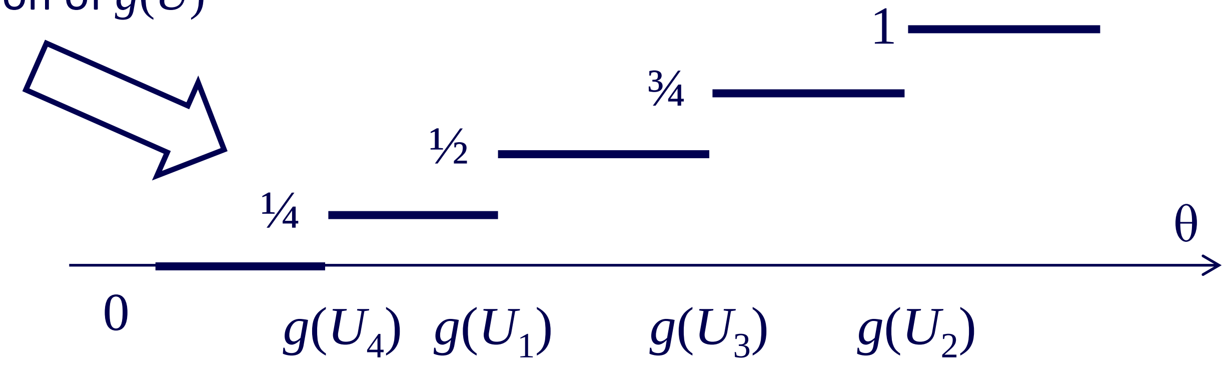
Part 3: Organization

- ◊ Sampling model, field model, and distortion
- ◊ Field estimation without any knowledge of sampling location
- ◊ Field estimation when order of samples is known
- ◊ Field estimation with order of samples known in the presence of measurement noise
- ◊ Future work

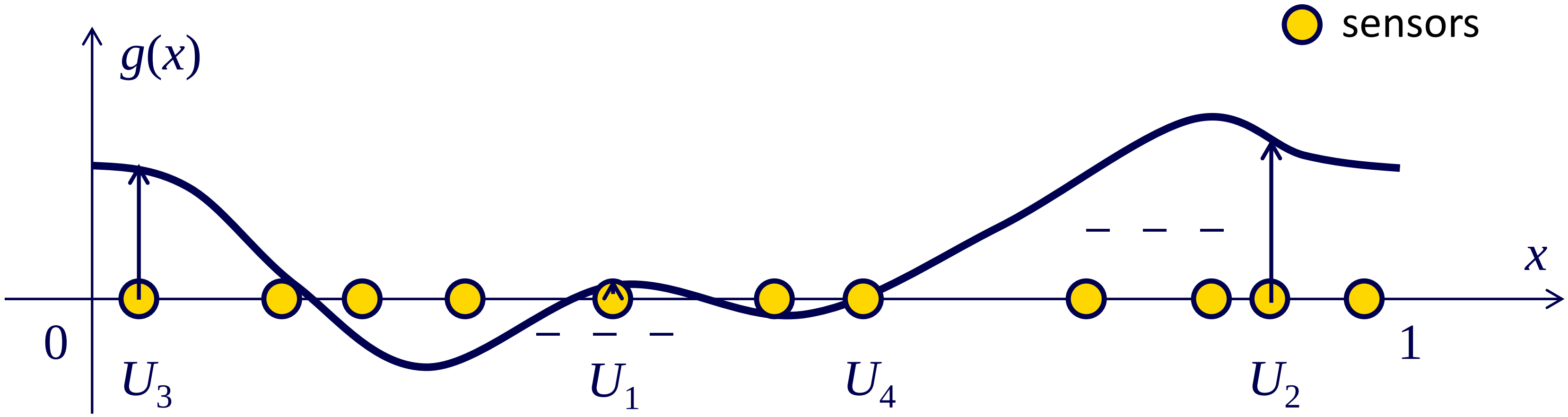
It is impossible to infer $g(x)$ from $g(U_1^\infty)$



Effectively, we are just collecting the empirical distribution or histogram of $g(U_1)$, $g(U_2)$, …, $g(U_n)$ and, in the limit of large n, the task is to estimate $g(x)$ from the distribution of $g(U)$



It is impossible to infer $g(x)$ from $g(U_1^\infty)$



Consider the statistic

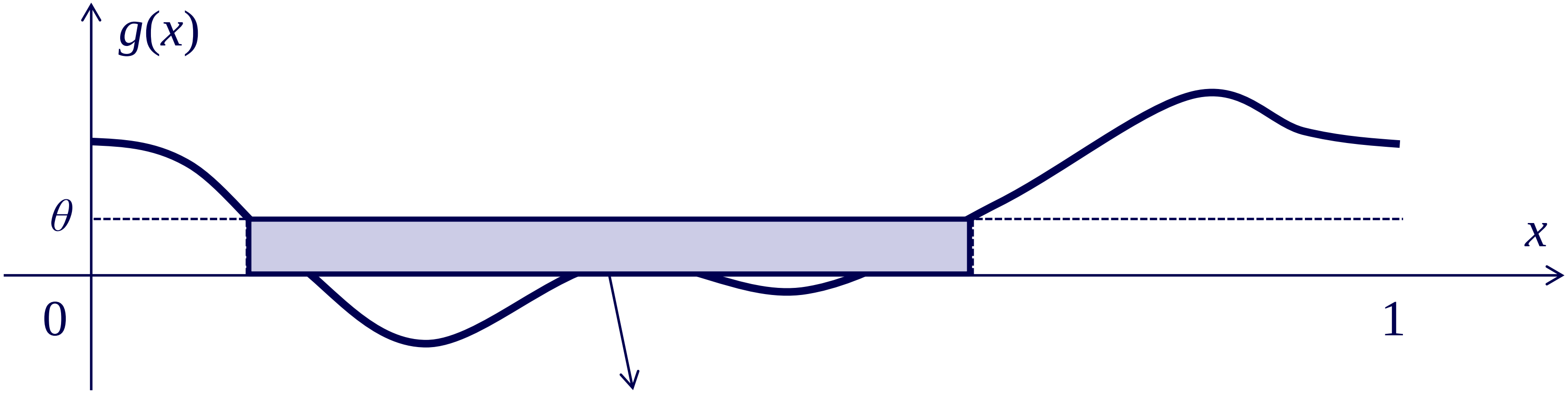
$$F_{g,n}(\theta) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}[g(U_i) \leq \theta]$$

◊ Then $F_{g,n}(\theta)$, x in set of reals and $g(U_1)$, $g(U_2)$, …, $g(U_n)$ are statistically equivalent

◊ By the Glivenko Cantelli theorem, $F_{g,n}(\theta)$ converges almost surely to Prob($g(U) \leq \theta$) for each θ in set of real numbers **[van der Vaart'1998]**

It is impossible to infer $g(x)$ from U_1^∞

So what does Prob($g(U) \leq \theta$), for x in set of real numbers, looks like?



Level-Set: $\{u\text{: g}(u) \leq \theta\}$

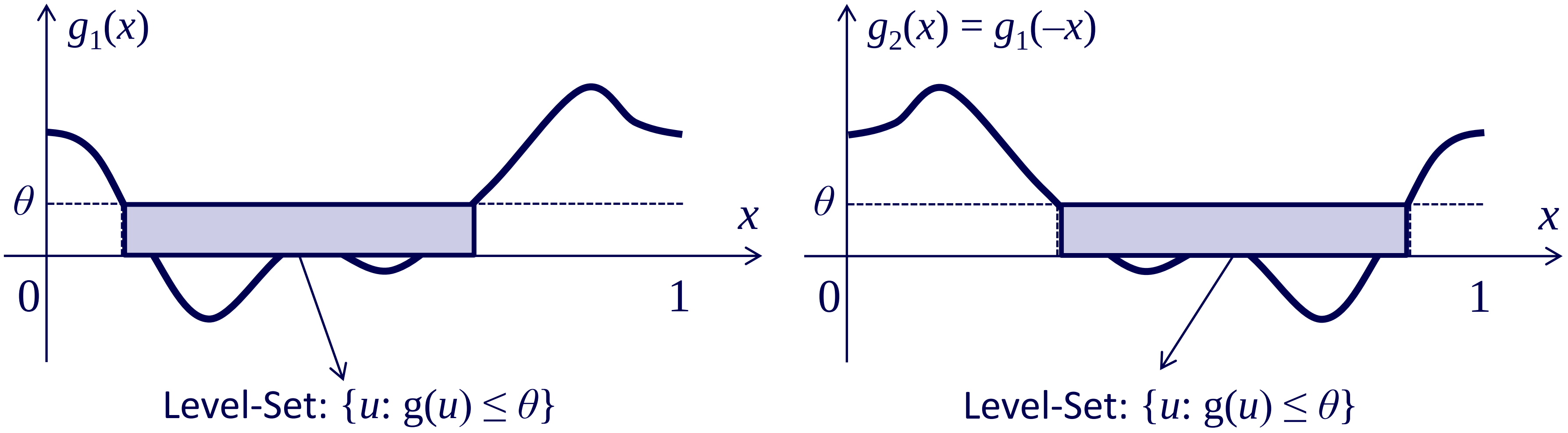
◊ Prob($g(U) \leq \theta$) for each θ is the probability of U belonging in the level-set. Thus, it is simply the length (measure) of level-set

◊ We will now illustrate that two different fields $g_1(x) \neq g_2(x)$ can still lead to

Prob($g_1(U) \leq \theta$) = Prob($g_2(U) \leq \theta$)

Graphical proof of first result

$g_1(x) \neq g_2(x)$ does not imply Prob($g_1(U) \leq \theta$) = Prob($g_2(U) \leq \theta$)



◊ The length (measure) of the level-sets is the same in the two cases for every θ

◊ As a recap, we showed that the Glivenko Cantelli theorem's limit, obtained from a statistical equivalent of observed samples, is the same for two different signals. Thus, the observed samples alone do not lead to a unique reconstruction of the field

Part 3: Organization

- Sampling model, field model, and distortion
- Field estimation without any knowledge of sampling location
- Field estimation when order of samples is known
- Field estimation with order of samples known in the presence of measurement noise
- Future work

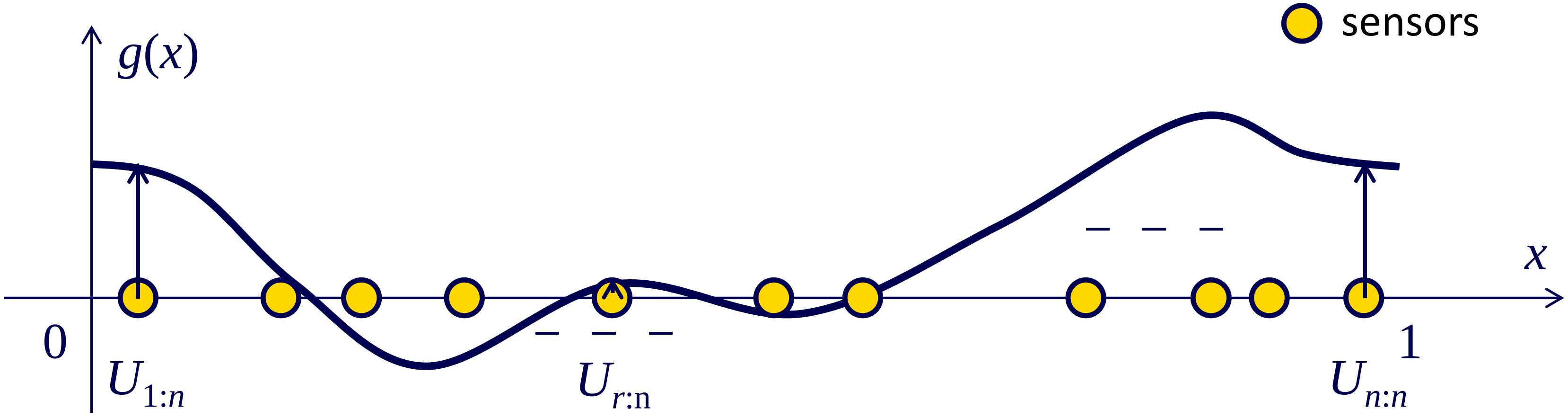
Working with ordered samples

◊ If the **order** (left to right) of sample locations is known, a consistent estimate $\hat{G}(x)$ for the field of interest can be obtained

◊ Recall that

$$g(x) = \sum_{n=-b}^{b} a_n \exp(j2\pi nx)$$

◊ Thus, due to bandlimitedness, there are $(2b+1)$ parameters to be learned or estimated



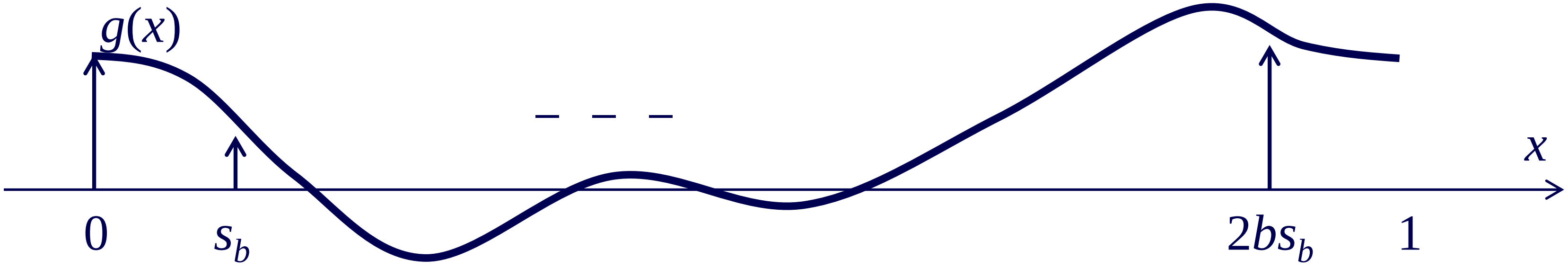
Using field samples to get the Fourier series

From $(2b+1)$ equi-spaced samples of the field, the $(2b+1)$ Fourier series coefficients (and hence the field) can be obtained as follows

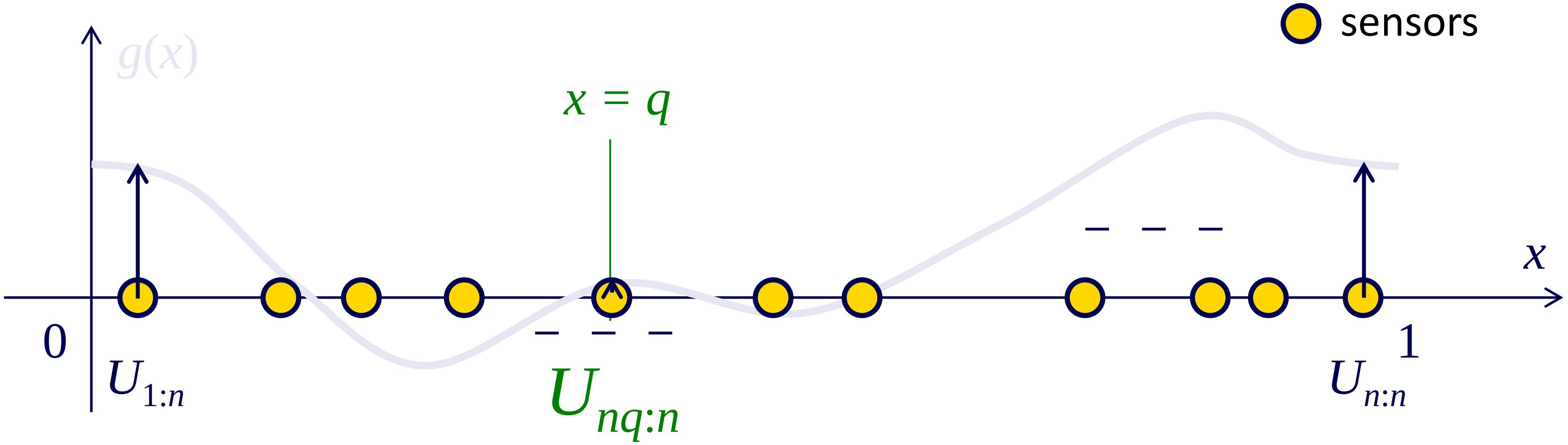
$$\begin{bmatrix} g(0) \\ g(s_b) \\ \vdots \\ g(2bs_b) \end{bmatrix} = \begin{bmatrix} 1 & \ldots & 1 \\ \phi_{-b} & \ldots & \phi_b \\ \vdots & & \vdots \\ (\phi_{-b})^{2b} & \ldots & (\phi_b)^{2b} \end{bmatrix} \begin{bmatrix} a_{-b} \\ a_{-b+1} \\ \vdots \\ a_b \end{bmatrix}$$

where $s_b = 1/(2b+1)$ and $\phi_b = \exp(j2\pi k s_b) = \exp(j2\pi k/(2b+1))$. In matrix notation and upon inversion

$$\vec{a} = (\Phi_b)^{-1}\vec{g} = \frac{1}{(2b+1)}\Phi_b^{\dagger}\vec{g}$$



Approximation of the field samples



It is known that $U_{nq:n}$ converges to p in many ways (in L^2, in almost-sure sense, and in weak-law) **[David and Nagaraja'2003]**

In the absence of field values $g(0)$, $g(s_b)$, …, $g(2bs_b)$, we use $\vec{G} = (g(U_{1:n}), g(U_{nsb:n}), \ldots, g(U_{2bsbn:n}))$, to define the Fourier series estimate and field estimate as follows

$$\vec{A} := [\widehat{A}_{-b}, \widehat{A}_{-b+1}, \ldots, \widehat{A}_b]^T := \frac{1}{(2b+1)}\Phi_b^{\dagger}\vec{G} \quad \text{and} \quad \widehat{G}(x) = \sum_{n=-b}^{b} \widehat{A}_n \exp(j2\pi n x)$$

Consistency of our estimate

Define $\overrightarrow{G} = (g(U_{1:n}),\ g(U_{nsb:n}),\ \ldots,\ g(U_{2bsbn:n}))$, and the Fourier series and field estimates as

$$\vec{A} = \frac{1}{(2b+1)}\Phi_b^{\dagger}\vec{G} \quad \text{and} \quad \widehat{G}(x) = \sum_{n=-b}^{b} \widehat{A}_n \exp(j2\pi nx)$$

Key ideas:

◊ For $r = [nq] + 1$, $U_{r:n} \rightarrow q$ almost surely

◊ That is $U_{[nsb]:n} \rightarrow s_b$ almost surely, $U_{[2nsb]:n} \rightarrow 2s_b$ almost surely, etc.

◊ By continuity of $g(x)$, $g(U_{[nsb]:n}) \rightarrow g(s_b)$ almost surely, $g(U_{[2nsb]:n}) \rightarrow g(2s_b)$ almost surely, etc.

◊ Finally, the estimates $\overrightarrow{A}$ and $\hat{G}(x)$ are bounded-coefficient finite linear combination of $g(U_{1:n}), g(U_{[nsb]:n}), \ldots, g(U_{[2b\ nsb]:n})$

Mean-squared error performance

If $r \approx [nq]$ then the second moment of $(U_{r:n} - q)$ satisfies

$$\left.\right\} \begin{aligned} n\mathbb{E}[(U_{r:n} - q)^2] &= q(1-q)\mathbb{E}(Z^2) + O(\sqrt{1/N}) \\ &\leq \frac{1}{4} + O(\sqrt{1/N}) \end{aligned}$$

[David and Nagaraja'2003]

Keep in mind that

$$\vec{A} = \frac{1}{(2b+1)} \Phi_b^{\dagger} \vec{G} \quad \text{and} \quad \widehat{G}(x) = \sum_{n=-b}^{b} \widehat{A}_n \exp(j2\pi n x)$$

Then the following mean-squared result holds for the estimate $\hat{G}(x)$

Theorem 2: $$n\mathbb{E}\left[\left\|\widehat{G} - g\right\|_2^2\right] \leq \pi^2 b^2 (2b+1)\left[1 + O(1/\sqrt{N})\right]$$

Key ideas:

◊ The matrix Φ_b has entries with magnitude $|(\phi_j)^k| = 1$. The signal's derivative $g'(x)$ is bounded. As a result, linear approximations can be used to get the above bound

◊ Observe that the mean-squared error decreases as $O(1/N)$

Weak-convergence of the estimate $\hat{G}(x)$

Fact: If $0< q_1 < q_2 < \ldots < q_{(2b+1)} < 1$ and $(r_i/n - q_i) = o(1/\sqrt{n})$ for each i. Then,

$$\sqrt{n}\left[U_{r_1:n} - q_1, \ldots, U_{r_{2b+1}} - q_{2b+1}\right]^T \xrightarrow{d} \mathcal{N}\left(\vec{0}, K_U\right)$$

where $[K_U]_{i,i'} = q_i(1 - q_{i'})$ for $i \leq i'$. **[David and Nagaraja'2003]**

Once again

$$\vec{A} = \frac{1}{(2b+1)}\Phi_b^\dagger \vec{G} \quad \text{and} \quad \widehat{G}(x) = \sum_{n=-b}^{b} \widehat{A}_n \exp(j2\pi n x)$$

Then the following mean-squared result holds for the estimate $\hat{G}(x)$

Theorem 2: $$\sqrt{n}\left(\widehat{G}(x) - g(x)\right) \xrightarrow{d} \mathcal{N}\left(\vec{0}, K_G(x)\right)$$

where, the variance $K_G(x)$ depends on K_U, the derivative of $g(x)$, and Φ_b

Key ideas:

◊ $(U_{1:n}, U_{sbn:n}, \ldots, U_{2bsbn:n})$ converges to a Gaussian vector

Weak-convergence of the estimate $\hat{G}(x)$

Theorem 2: $$\sqrt{n}\left(\widehat{G}(x) - g(x)\right) \xrightarrow{d} \mathcal{N}\left(\vec{0}, K_G(x)\right)$$

where, the variance $K_G(x)$ depends on K_U, the derivative of $g(x)$, and Φ_b

Key ideas (contd.):

◊ Since $g(x)$ is smooth, therefore $\overrightarrow{G} = g(U_{1:n}), g(U_{sbn:n}), \ldots, g(U_{2bsbn:n})$ converges to a Gaussian vector by the Delta method **[van der Vaart'1998]**

◊ Since the map from $\overrightarrow{G} = (g(U_{1:n}), g(U_{sbn:n}), \ldots, g(U_{2bsbn:n}))$ to $\hat{G}(x)$ is linear, therefore, Gaussian distribution is preserved

Part 3: Organization

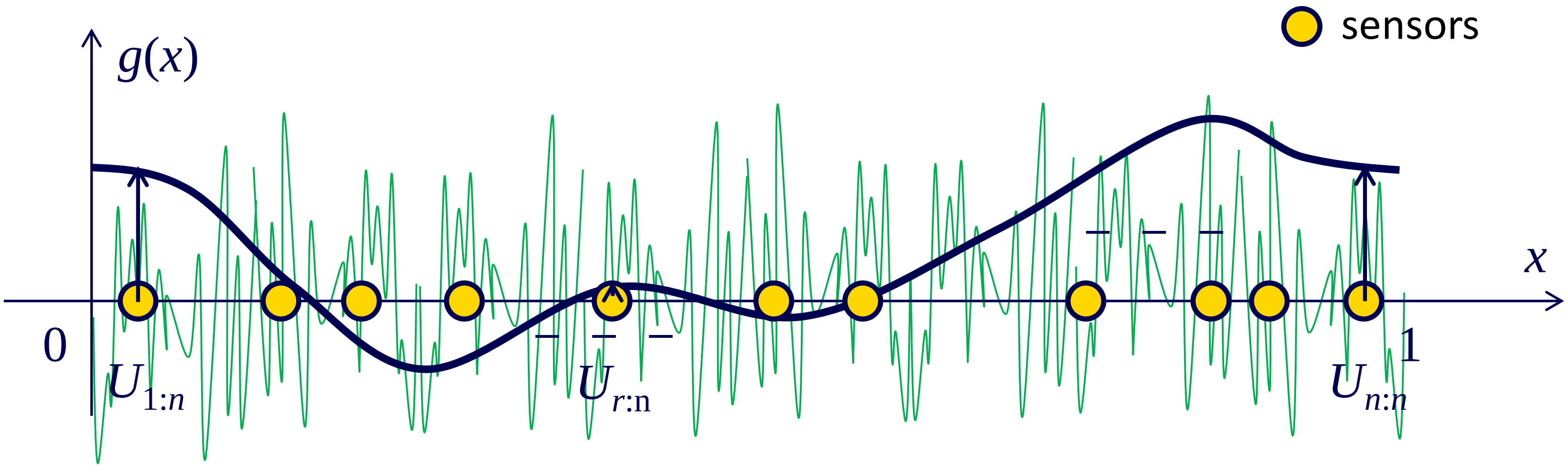
- ◊ Sampling model, field model, and distortion
- ◊ Field estimation without any knowledge of sampling location
- ◊ Field estimation when order of samples is known
- ◊ Field estimation with order of samples known in the presence of measurement noise
- ◊ Future work

Ordered field samples in additive indep noise

◊ If the **order** (left to right) of sample locations is known and field is affected by independent measurement noise, a consistent estimate $\hat{G}(x)$ for the field of interest can be obtained as follows

$$Y(x) = g(x) + W(x) = \sum_{n=-b}^{b} a_n \exp(j2\pi nx) + W(x)$$

◊ Due to bandlimitedness, there are $(2b+1)$ parameters to be learned or estimated



Fourier series coefficient estimates

- It is assumed that b is known
- The ordered samples $Y(U_{1:n})$, …, $Y(U_{n:n})$ are available, but the values of $U_{1:n}$, …, $Un:n$ are not known. The following estimate can be used

$$
\begin{aligned}
\widehat{A}_i &= \frac{1}{n}\sum_{\beta=1}^{n} Y(U_{\beta:n})\exp(-j2\pi i\beta/n) \\
&= \frac{1}{n}\sum_{\beta=1}^{n} g(U_{\beta:n})\exp(-j2\pi i\beta/n) + \frac{1}{n}\sum_{\beta=1}^{n} W(U_{\beta:n})\exp(-j2\pi i\beta/n) \\
&\approx \int_0^1 g(x)\exp(-j2\pi ix)\mathrm{d}x + \frac{1}{n}\sum_{\beta=1}^{n} W(U_{\beta:n})\exp(-j2\pi i\beta/n) \\
&= a_i + \boxed{\frac{1}{n}\sum_{\beta=1}^{n} W(U_{\beta:n})\exp(-j2\pi i\beta/n)}
\end{aligned}
$$

Main result

Theorem: Let Fourier series coefficient estimates for $g(x)$ be obtained as

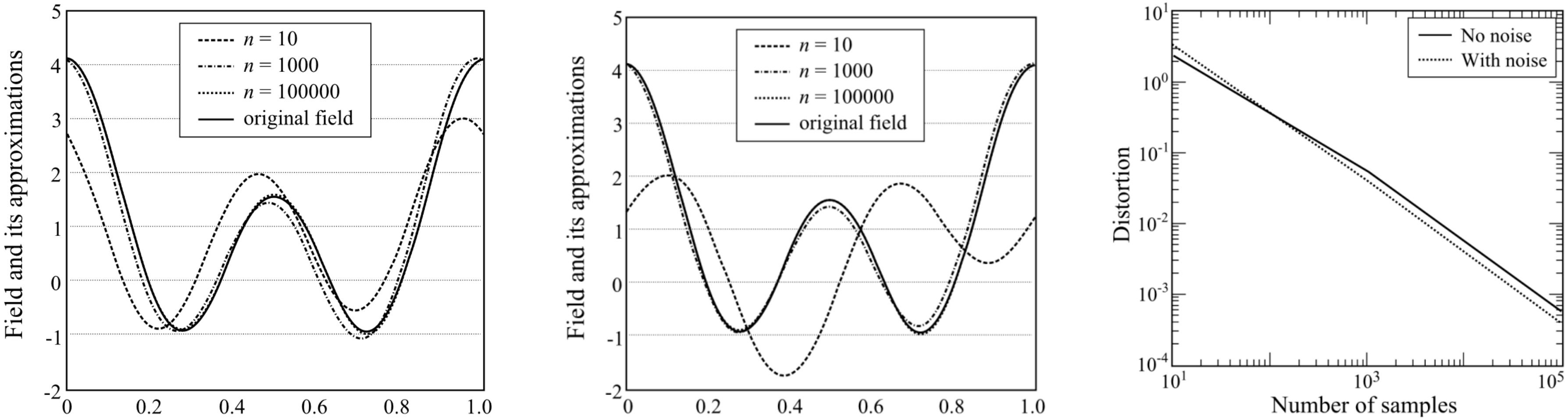
$$\widehat{A}_i = \frac{1}{n}\sum_{\beta=1}^{n} Y(U_{\beta:n}) \exp(-j2\pi i\beta/n)$$

Then the average mean-squared error (distortion) between $g(x)$ and its estimate $G(x)$ with Fourier series coefficients above is bounded by

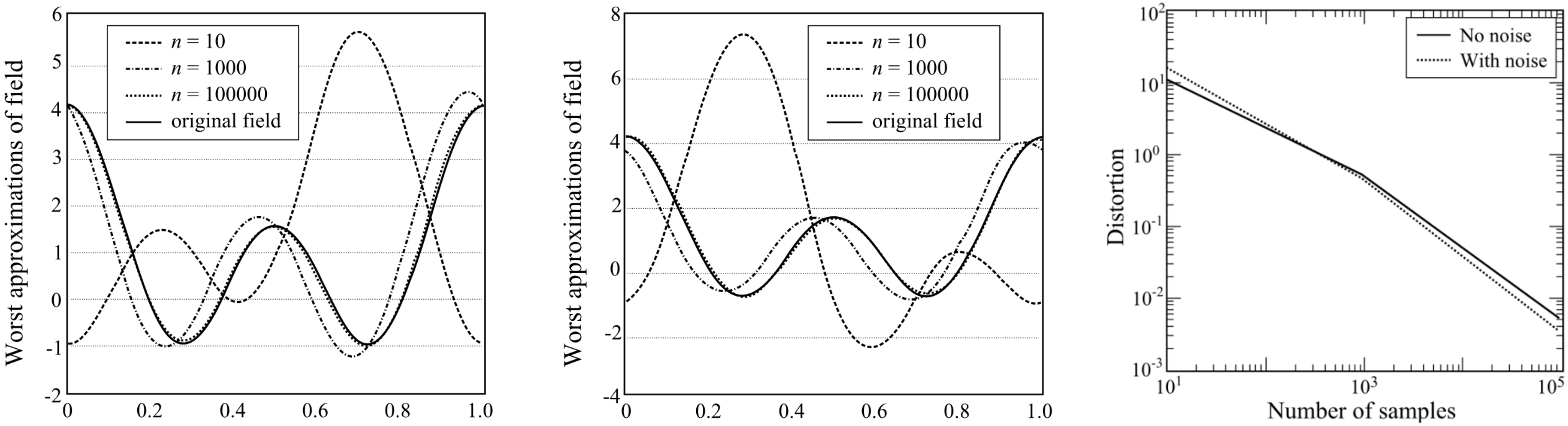
$$\begin{aligned}\mathbb{E}\left|G(x) - g(x)\right|^2 &= \sum_{i=-b}^{b} \mathbb{E}\left|\widehat{A}_i - a_i\right|^2 \\ &\leq (2b+1)\left[\frac{\pi^2 b^2}{n} + O\left(\frac{1}{n\sqrt{n}}\right) + \frac{16\pi^2 b^2}{n^2} + \frac{\sigma^2}{n}\right] \\ &= O\left(\frac{1}{n}\right).\end{aligned}$$

where is the σ^2 variance of the additive noise

Simulation results



Fourier series coefficients are given by [0.9134, 0.6324, 1.0000, 0.6324, 0.9134]



Part 3: Organization

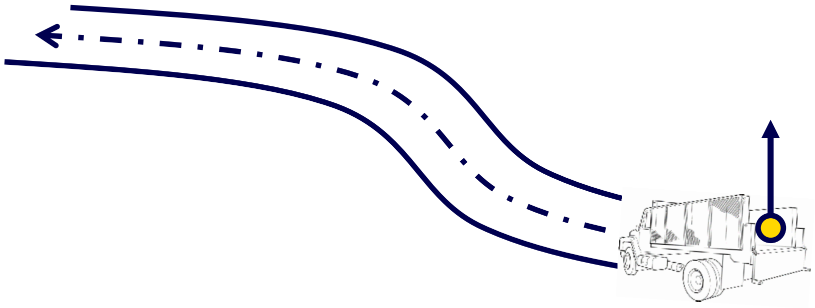
- Sampling model, field model, and distortion
- Field estimation without any knowledge of sampling location
- Field estimation when order of samples is known
- Field estimation with order of samples known in the presence of measurement noise
- Future work

Future work

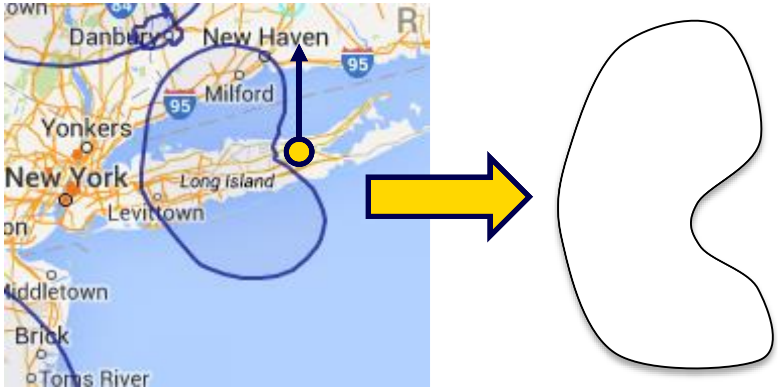
- Estimates are not minimum risk. Or, techniques for finding Maximum likelihood estimates will be useful
- It is unclear if $O(1/N)$ distortions obtained are optimal
- Extension of these results to more classes of fields (FRI, finite-support, orthogonal spaces, non-bandlimited fields)
- What is the effect of quantization?

Concluding remarks

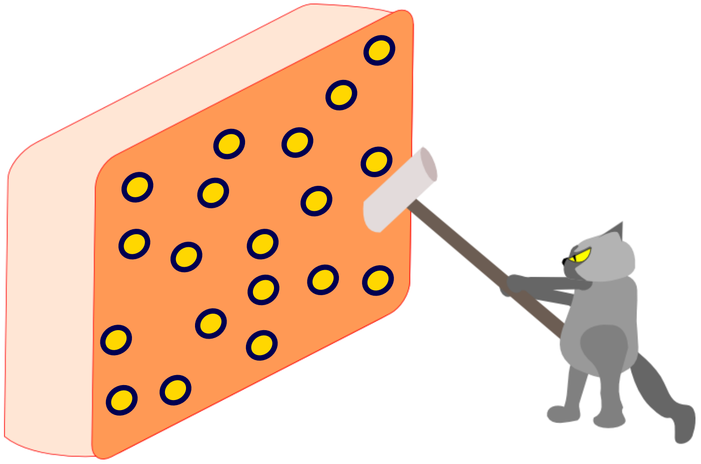
Emission monitoring with sensors



Sampling along a path with vehicle



Coverage region for TV transmitters



Randomly sprayed smart-dust/paint

Concluding remarks

1. **Adversary:** No prefilter, quantization (poor precision), measurement noise
2. **Friend:** oversampling, low-cost of sensors
3. **Wisdom:** nonlinearity in quantization, field structure, parametric understanding of fields, functional analysis
4. **Net result:** difficult problems, entertaining results (interplay of oversampling, distortion, quantization precision, rate, field's smoothness structure)

Acknowledgments

Prof. Kannan Ramchandran (EECS, Univ of California, Berkeley)

Prof. Prakash Ishwar (ECE, Boston University)

Prof. Vinod Prabhakaran (STCS, TIFR Bombay)

Akshta Athawale (BTech 2012, IIT Bombay)

Alankrita Bhatt (Btech student, IIT Kanpur)

Karthik Sharma (Dual Degree student, IIT Bombay)

Selected publications

1. Z. Cvetkovic, I. Daubechies, and B. F. Logan Jr., “Single-Bit Oversampled A/D Conversion With Exponential Accuracy in the Bit Rate”, IEEE Transactions on Information Theory
2. A. Kumar, P. Ishwar, and K. Ramchandran, “High-resolution distributed sampling of bandlimited fields with low-precision sensors,” IEEE Transactions on Information Theory
3. A. Kumar, P. Ishwar, and K. Ramchandran, “Dithered A/D conversion of smooth non-bandlimited signals,” IEEE Transactions on Signal Processing.
4. A. Athawale and A. Kumar, “On Dithered A/D Conversion of Multidimensional Smooth Non-bandlimited Signals”, Proceedings of SPCOM 2012, IISc Bangalore, India
5. E. Masry, “The reconstruction of analog signals from the sign of their noisy samples”, IEEE Transactions on Information Theory
6. A. Kumar and V. Prabhakaran, “Estimation of Bandlimited Signals From the Signs of Noisy Samples”, Proceedings of ICASSP 2013, Vancouver, Canada
7. A. Kumar, “On Bandlimited Signal Reconstruction From the Distribution of Unknown Sampling Locations”, To Appear in IEEE Transactions on Signal Processing

Selected publications

8. K. Sharma and A. Kumar, “Sampling Smooth Spatio-temporal Physical Fields: When will the Aliasing Error Increase with Time?” Proceedings of ICASSP 2015, Brisbane, Australia
9. A. Bhatt and A. Kumar, “Sampling Polynomial Fields with Measurement Noise and Single-Bit Quantizers”, Submitted to EUSIPCO 2015 for review
10. J. Unnikrishnan and M. Vetterli, “Sampling and reconstruction of spatial fields using mobile sensors”, IEEE Transactions on Signal Processing

Other works of interest

- **[Wang-Ishwar’2009]** and **[Masry-Ishwar’2009]** Acquisition of a finite-support field in bounded noise. Mean-squared error bounds were established
- **[Jayakrishnan-Vetterli’2013]** Acquisition of fields using mobile sensors
- **[Ranieri-Vetterli’2013]** on sampling of fields with physical evolution
- **[Baltasar-Konsbruck-Vetterli’2004]** for physics based distributed sensing
- **[Rabbat-Nowak’2002]** for binary field’s boundary estimation