High resolution sampling of smooth signals using low-precision quantizers*

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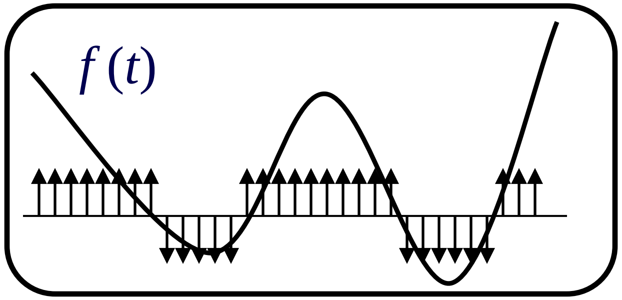
Sampling with low-precision quantizers

◊ **Driving application:** analog to digital conversion of signals

◊ **Physical quantities of interest:** temperature, illumination, humidity, chemical concentration, pollutant, etc. being sensed for digital conversion

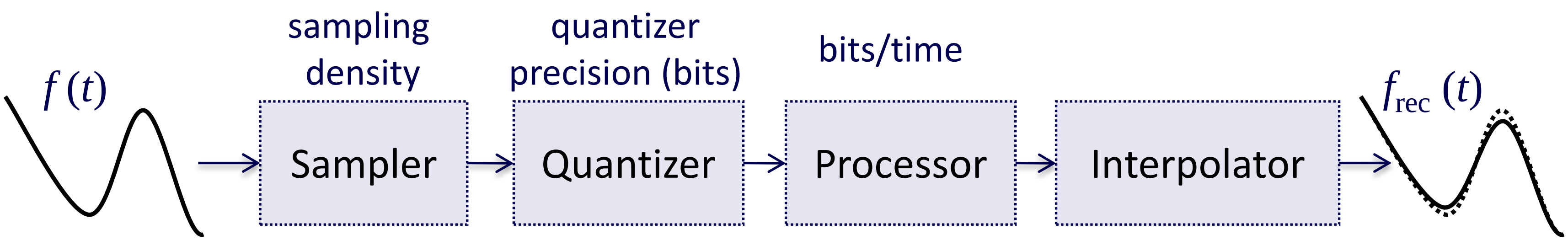
◊ **Feature:** Smoothness due to natural physical laws

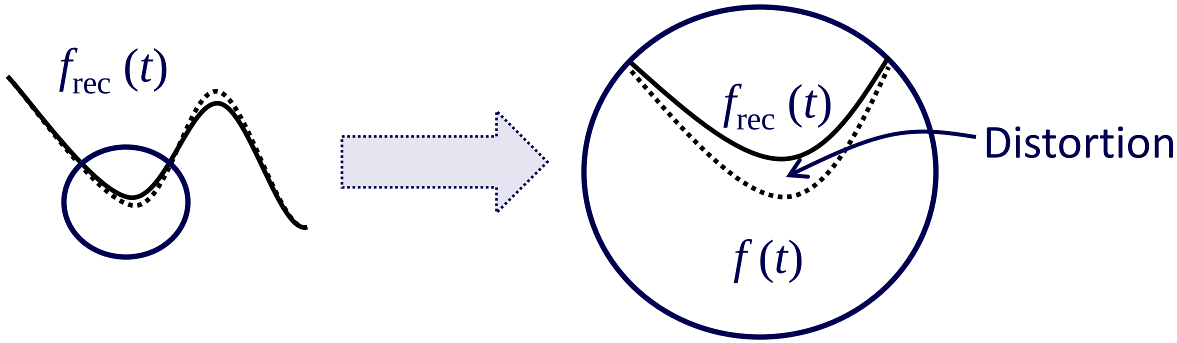


GOAL: high-quality reproduction of a smooth signal by using samples from a fixed precision quantizer

Example: digital recording of voice signal with a microphone

Parameters in sampling of signals





◊ For a fixed target distortion, what is the trade-off between quantizer-**precision**, **bit rate**, and sample-**density**?

◊ For what **classes of signals** these trade-offs will be established?

Main result 1: bit-conservation principle

k-bit A/D, one in time T | b-bit A/D, 2^{k-b+1} in time T | 1-bit A/D, 2^k in time T

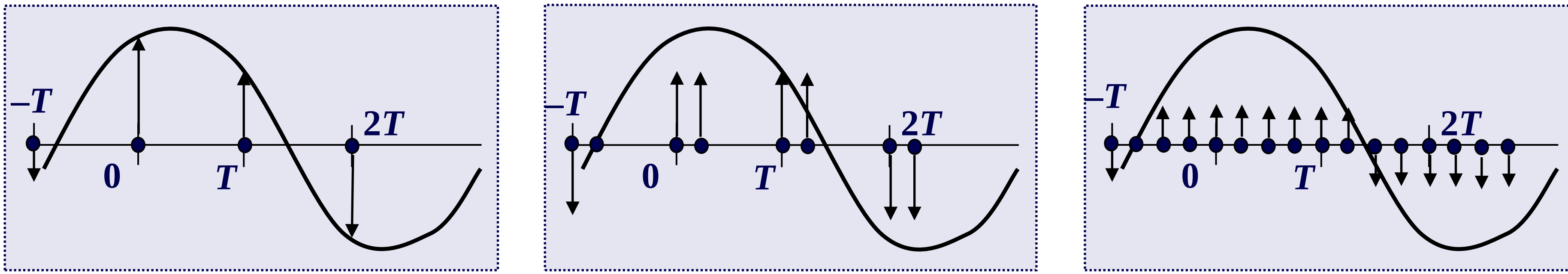


Figure: Bit-conservation trade-off in pictures

[Ishwar, Kumar, & Ramchandran'03]:

For bounded (deterministic or stochastic) bandlimited signals,

> Can trade off sample **density** for quantizer-**precision** without any loss in the *optimal* distortion-rate performance

Similar asymptotic trade-off can be established for smooth non-bandlimited signals $u(t)$ with a decay that satisfies $\int_{\omega} |\omega U(\omega)| d\omega < \infty$

Main result 2: distortion and bit-rate tradeoffs

[Ishwar, Kumar, & Ramchandran'03]: For bounded bandlimited signals (and stationary bandlimited process in the almost sure sense),

- ◊ Target distortion $= D$
- ◊ 1-bit sample density $= O(1/D)$
- ◊ Bit-rate per Nyquist $= O(|\log D|)$

[Kumar, Ishwar, & Ramchandran'04]: For bounded smooth non-bandlimited signals distortion-rate upper bounds can be computed, e.g., for signals with exponentially decaying spectra,

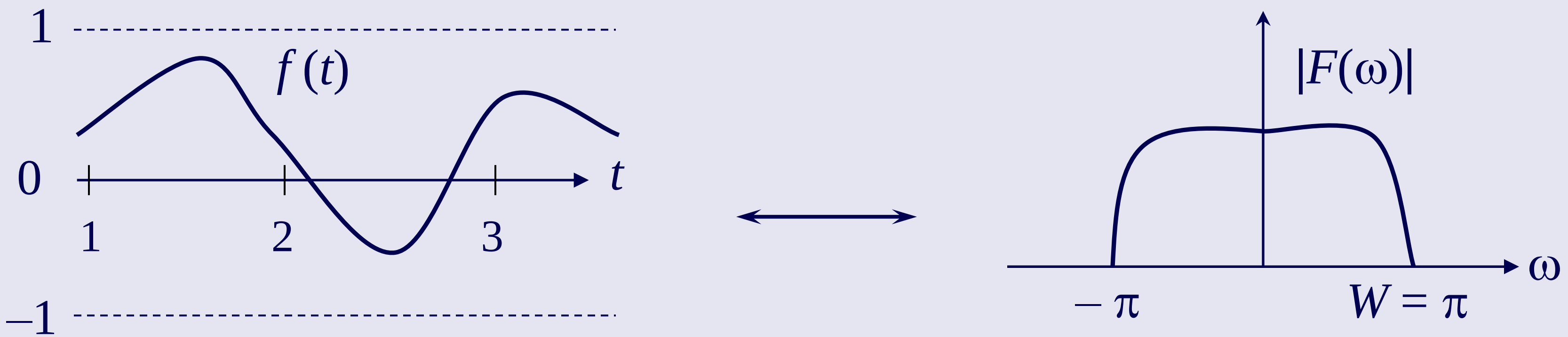
- ◊ Target distortion $= D$
- ◊ 1-bit sample density $= O(|\log D|/D)$ (per unit time)
- ◊ Bit-rate per unit time $= O(|\log D|^2)$

Organization

◊ Introduction and contributions
◊ Sampling of deterministic bandlimited signals
◊ Extensions to stationary bandlimited signals
◊ Extensions to deterministic smooth non-bandlimited signals
◊ Conclusions

Deterministic bandlimited signal model

- 1-D real and bounded: $|f(t)| < 1$ (normalized)
- Temporally bandlimited: temporal Nyquist period, $T_{NQ} = 1$
- Finite energy in L^2 sense
- **Bernstein's inequality:** $|f'(t)| \leq W \| f \|_\infty = \pi$

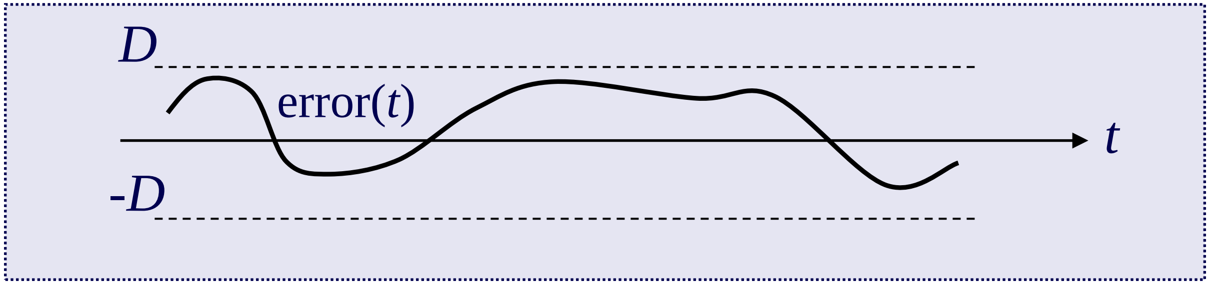


Nyquist interval = $T_{NQ} = \pi / W = 1$

Distortion criteria

Deterministic fields:

◊ Maximum pointwise reconstruction error: $D = \sup_t | f(t) - f_{\text{rec}}(t)|$

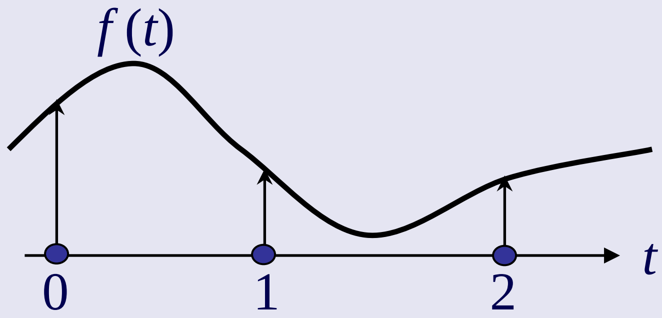


Stochastic fields:

◊ Almost sure bound on reconstruction error: $D = \sup_t |Y(t) - Y_{\text{rec}}(t)|$

This distortion can be extended to other error criteria, e.g., normalized L^p errors

Nyquist sampling and reconstruction



$$f(t) = \sum_{l=-\infty}^{\infty} f(l)\,\mathrm{sinc}\,(t-l)$$

◊ [Nyquist theorem] The signal $f(t)$ can be reconstructed from samples taken at points $\{lT_{NQ}\}$ where l is an integer

$$f(t) = \sum_{l=-\infty}^{\infty} f(l)\,\mathrm{sinc}\,(t-l)$$

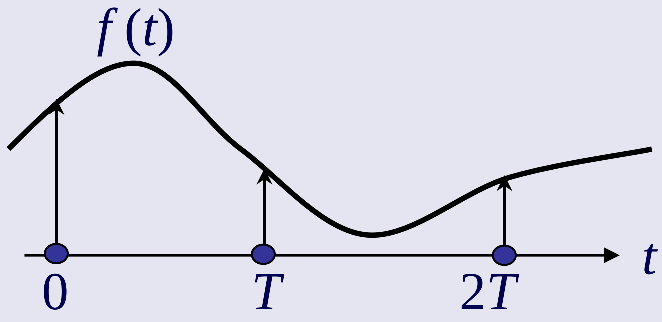
◊ In practice, the samples $f(l)$ will be quantized (say using) k-bit uniform scalar quantizer $Q_k(.)$

Quantization error: $|f(l) - Q_k(f(l))| \leq 2^{-k}$

Field reconstruction: $f_{\mathrm{rec}}(t) = \sum_{l=-\infty}^{\infty} Q_k(f(l))\,\mathrm{sinc}(t-l)$

Distortion: $D = \sup_t |f(t) - f_{\mathrm{rec}}(t)|$ is unbounded

Nyquist sampling and stability

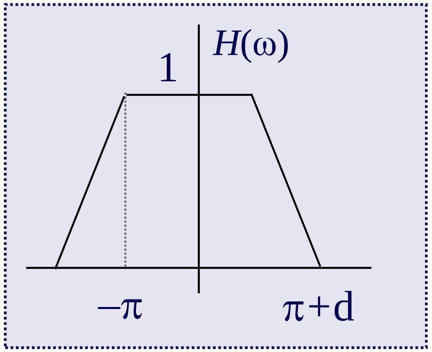


$$f(t) = \sum_{l=-\infty}^{\infty} f(lT)\, h(t - lT)$$

Stability: $C = \sup_t \sum_{l=-\infty}^{\infty} |h(t - lT)| < \infty$

◊ Samples are uniformly spaced slightly closer than the Nyquist points ($T < 1$)

$h(t) \quad \leftrightarrow$



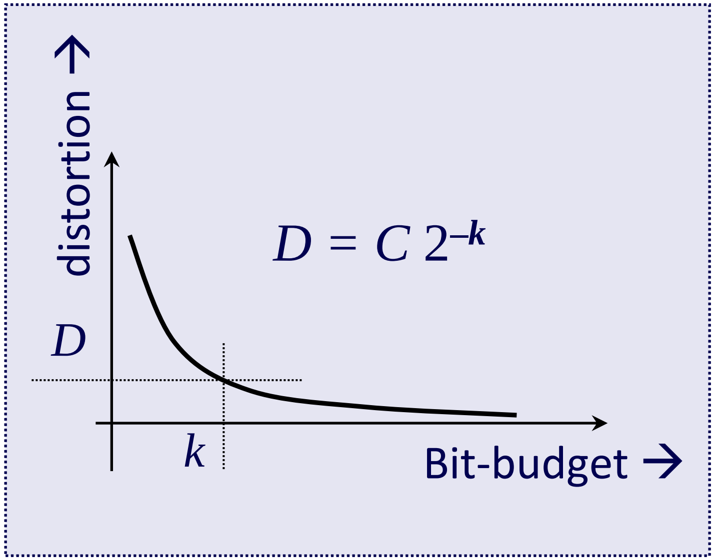
◊ Samples are quantized using a k-bit uniform scalar quantizer $Q_k(.)$

Quantization error: $|f(lT) - Q_k(f(lT))| \le 2^{-k}$

Field reconstruction: $f_{\text{rec}}(t) = \sum_{l=-\infty}^{\infty} Q_k(f(lT))\, h(t - lT)$

Error decay profile: $D \le C\, 2^{-\boldsymbol{k}}$

Comments on reconstruction distortion



Distortion: $D = \sup_t |f(t) - f_{\mathrm{rec}}(t)|$

Bit-rate: k bits/Nyquist-period (baseline precision)

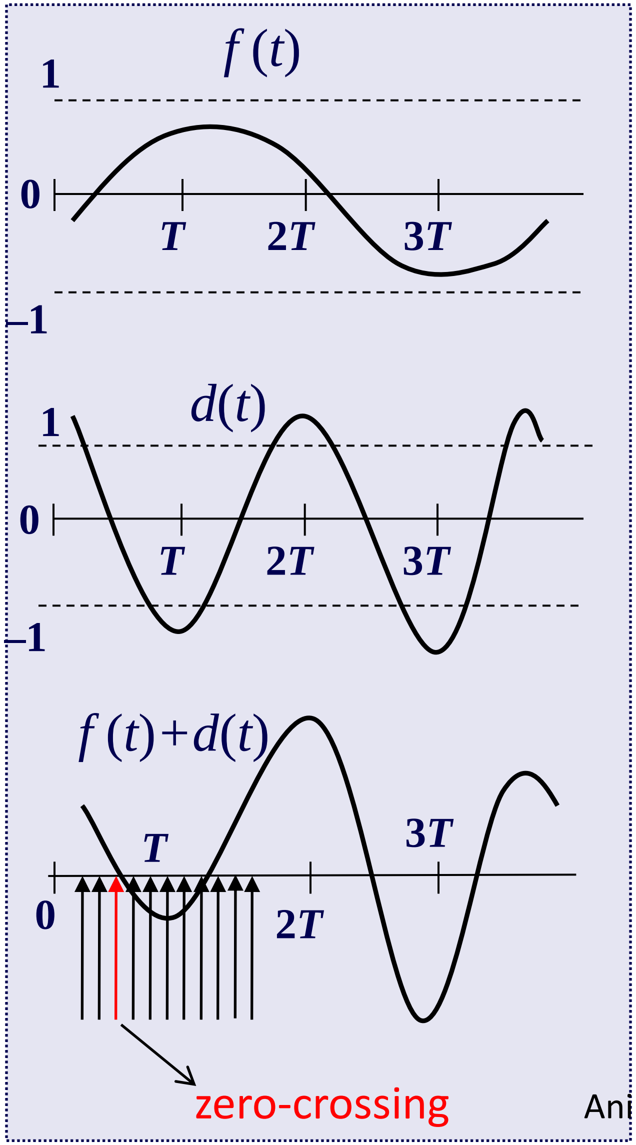
Optimality: The exponential distortion-rate is optimal (in an order sense)

[Daubechies et al.'2006]

Implications: (i) To decrease distortion k must be increased, and (ii) expensive and power draining quantizer will be required

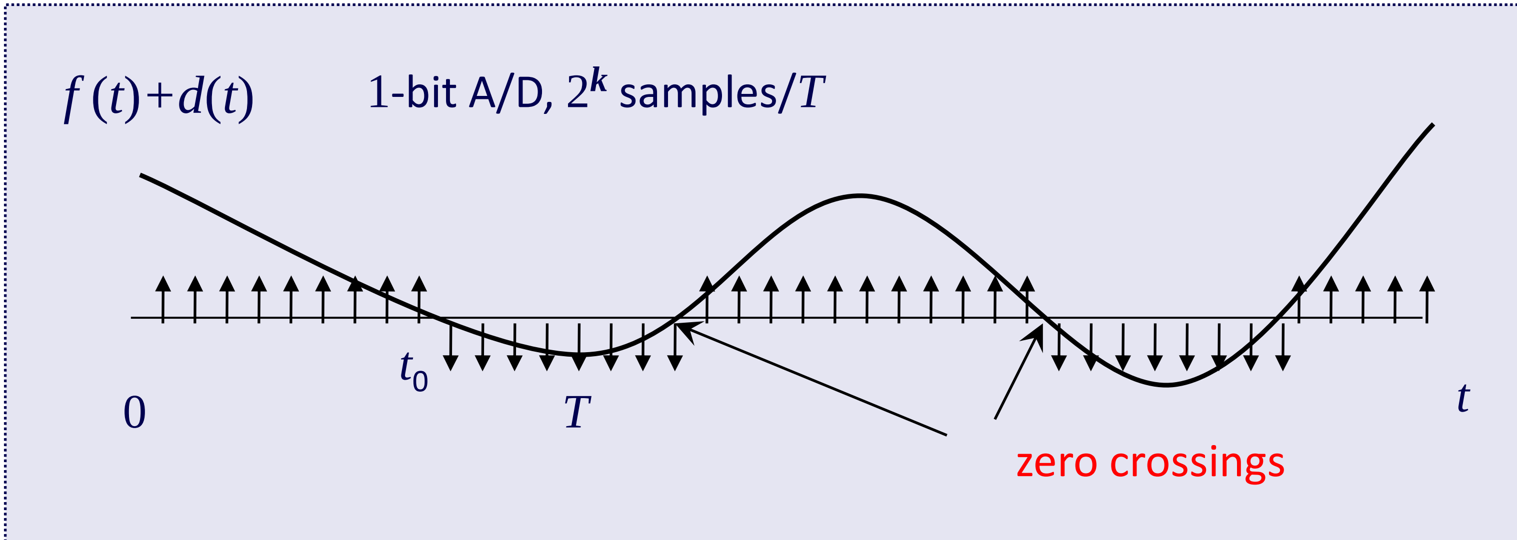
◊ Can these (order) optimal distortion-rate tradeoffs be obtained with a sampling method tailored towards low(est) precision quantizers?

1-bit dithered sampling [Cvetkovic & Daubechies'00]



- ◊ A known dither signal forces a zero-crossing in every Nyquist interval
- ◊ 2^k one-bit quantizers spaced $\tau = T/2^k$ apart => k-bit spatial resolution
- ◊ k bits per Nyquist interval are required to index the location of the **first** zero-crossing
- ◊ By design, the slope of dither function is bounded
- ◊ Recall that $f'(t) \leq \pi$, by Bernstein's inequality

1-bit dithered sampling: sample accuracy



Sample spacing: $\tau = T / 2^k$

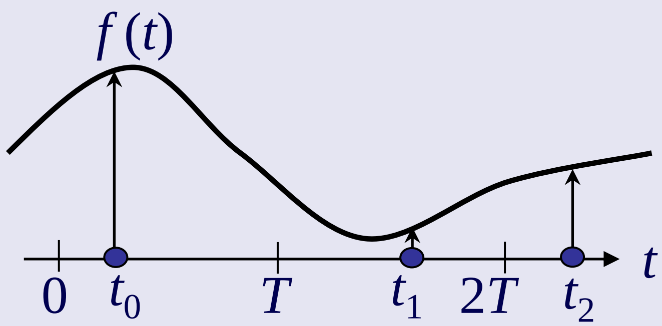
Bounded slope of the bandlimited signal and the dither imply that

if $f_{\text{dith, est}}(t_0) = -d(t_0)$,

$$| f(t_0) - f_{\text{dith, est}}(t_0)| \;=\leq\; \| f'(t) + d'(t)\|_\infty \; \tau/2$$

$$= \; ((\pi + \Delta)\, T/2)\; 2^{-k}$$

1-bit dithered sampling: distortion analysis



$$f(t) = \sum_{l=-\infty}^{\infty} f(t_l)\, g_l(t - t_l)$$

Stability: $C' = \sup_t \sum_{l=-\infty}^{\infty} |g_l(t - t_l)| < \infty$

If $|\, f(t_l) + d(t_l)| < ((\pi + \Delta)\, T/2)\, 2^{-k}$ and

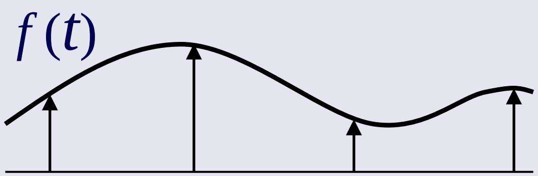
$$f_{\text{dith, rec}}(t) = \sum_{l=-\infty}^{\infty} -d(t_l)\, g_l(t - t_l)\text{, then}$$

the distortion decay is: $D \leq C'\, 2^{-k}$

[Cvetkovic & Daubechies'00]

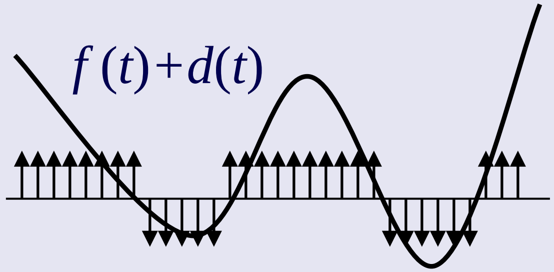
Two extreme scenarios

k-bit A/D, 1 sample per Nyquist



k-bit Nyquist sampling

1-bit A/D, 2^k samples per Nyquist



1-bit dithered oversampling

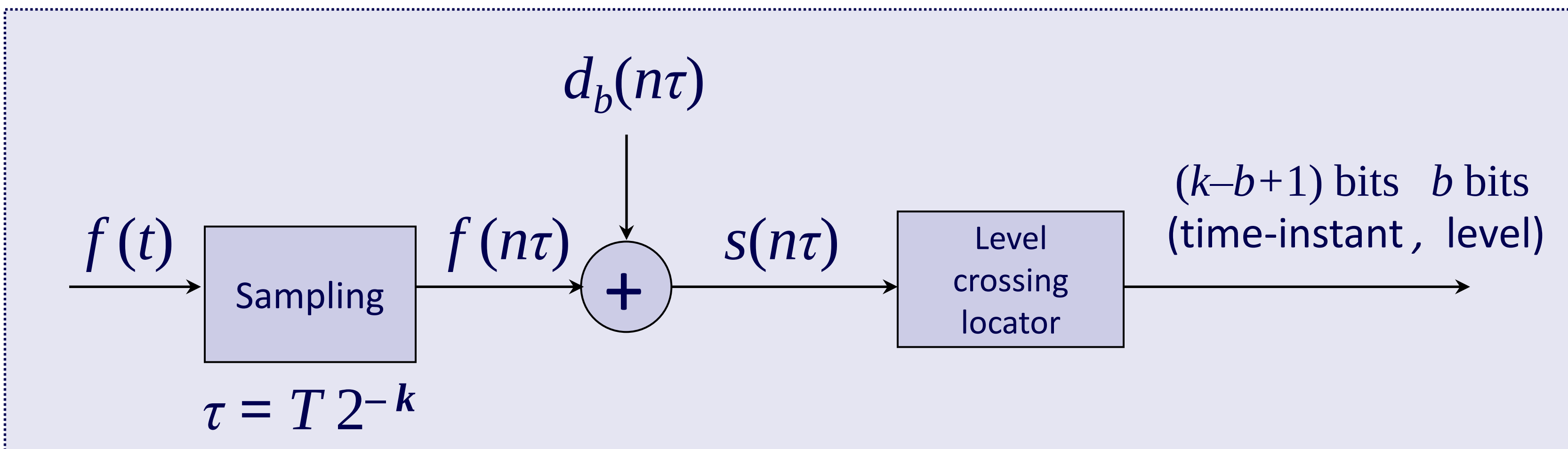
Question: Can we trade-off between the number of samples/Nyquist and the quantizer-precision for a given (k, D) pair?

b-bit dithered oversampling [Kumar-Ishwar-Ramchandran'03]

- 1 level
- Smooth dither: $d(t)$
- $s(t) = f(t) + d(t)$
- zero crossing in $(0, T)$
- time-instant of crossing

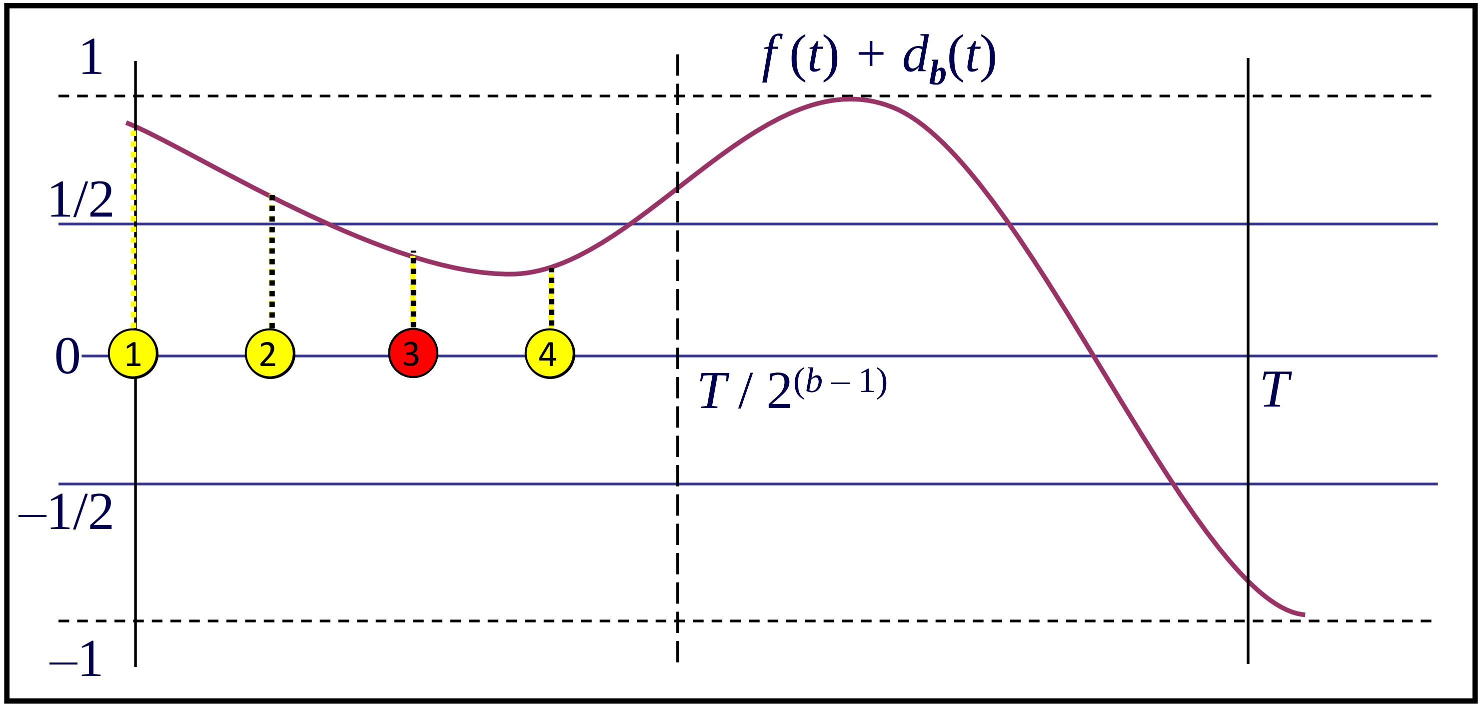
- 2^b-1 levels
- Smooth dither: $d_b(t)$
- $s(t) = f(t) + d_b(t)$
- level crossing in $(0, T / 2^{(b-1)})$
- (time-instant, level) of crossing

Same distortion-rate characteristics as in the 1-bit dithered sampling case



An example of b-bit dithered sampling

$k = 3$, $b = 2$, $2^b - 1 = 3$ levels: $\{0, +½, -½\}$, $2^{k-b+1} = 4$ samples



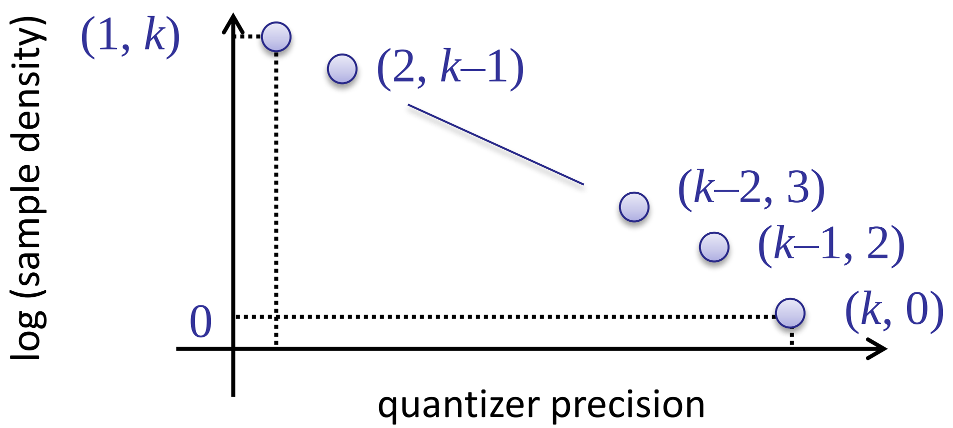
past value	> ½ →	> ½ →	> ½
	1	2	3
current value		> ½	< ½

Thus the sample for $[0, T]$

is recorded as $(3, ½)$

Bit-conservation principle [Kumar-Ishwar-Ramchandran'03]

``**Conservation of bits'' principle:** Let k be the number of bits available per Nyquist interval. For each $1 \leq b < k$ there exists a sampling scheme with 2^{k-b+1}, b-bit samples per Nyquist-interval that achieves a distortion of the order of $O(2^{-k})$



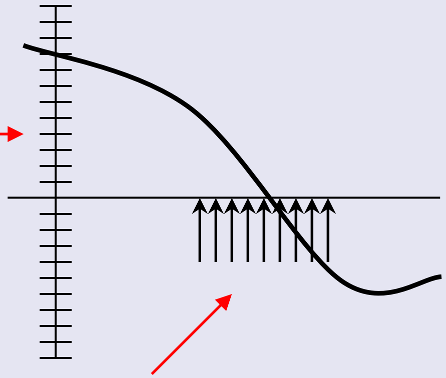
Note:

$D \sim O(1/\mathrm{poly}(R_{Nyquist}))$ for 1-bit Σ-Δ conversion versus

$D \sim O(1/\exp(R_{Nyquist}))$ for Nyquist sampling and b-bit dithered sampling

Interpretation

Nyquist sampling exhausts the bit budget in recording amplitude event



b-bit dithered sampling is a tradeoff between these two extremes

1-bit dithered sampling exhausts the bit budget in recording time event

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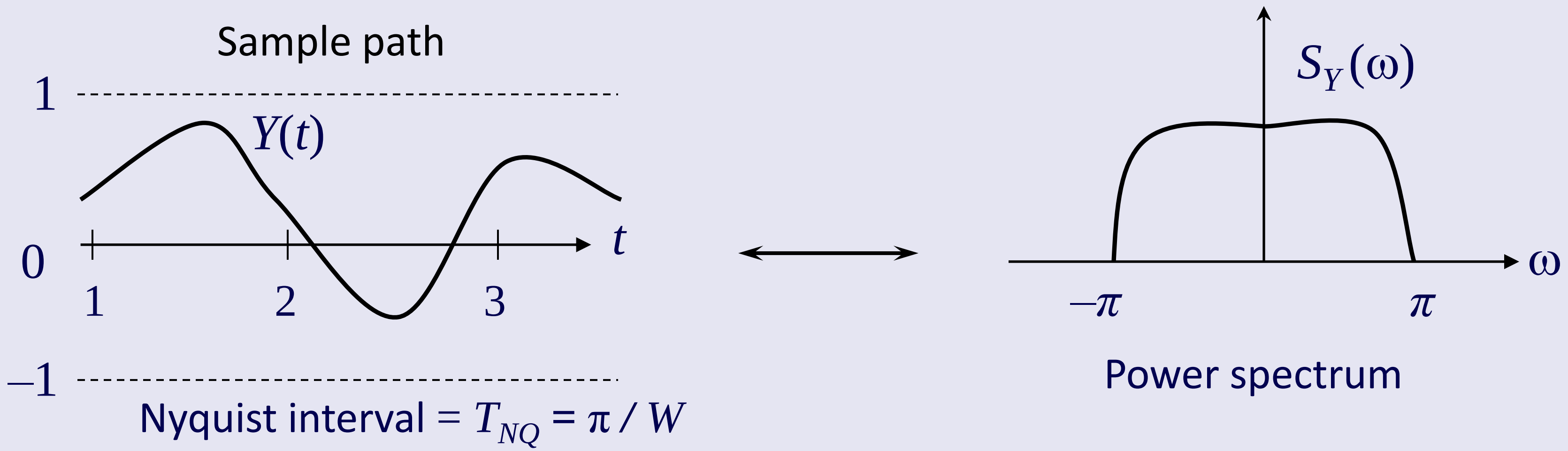
Stochastic bandlimited signal model

- ◊ 1-D, real valued, wide-sense stationary
- ◊ Bounded amplitude sample paths (normalized to 1)
- ◊ Temporally bandlimited autocorrelation function: Nyquist period is T_{NQ} with $T_{NQ} = 1$
- ◊ Finite energy autocorrelation function



Proof techniques for WSS bandlimited signals

- ◊ Deterministic results were for bounded bandlimited signals with finite energy
- ◊ Almost all sample paths of WSS bandlimited process will not, in general, have this property
- ◊ Even if we dither a bounded WSS bandlimited process, we have to deal with random zero-crossing (level-crossing) locations that have complicated and non-linear dependency on the sampled process
- ◊ Thus, results of deterministic case cannot be directly extended to the stochastic case
- ◊ **Key Idea:** Take an **almost-sure proof** approach and prove results for every sample path of the stationary bandlimited signal

Method: stochastic proof technique (1)

Deterministic case:

1. Locate zero crossings (collect samples)
2. Interpolate with kernels $g_l(t - t_l)$ (requires stable interpolation results for the stochastic case)
3. Utilize per sample accuracy, obtained by signal's smoothness

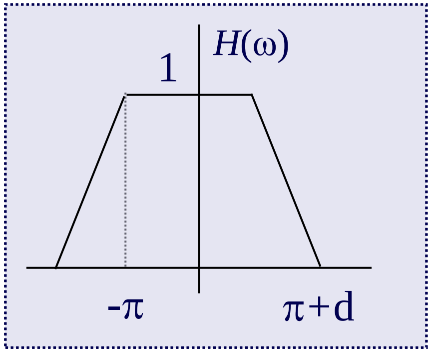
The first step can be repeated for the stochastic case. However, distortion analysis is required to complete the proof

Method: stochastic proof technique (2)

Key steps:

- Extend non-uniform stable sampling result of Cvetkovic & Daubechies to WSS process by using Zakai sense bandlimited signal theory [Zakai'65]
- Using [Zakai'65] and [Masry'76] ,

$$Y(t) = Y(t) * h(t), \qquad h(t) \leftrightarrow$$



almost surely for WSS bandlimited $Y(t)$

- Since $h(t)$ is smooth, above property can be used to establish smoothness of $Y(t)$
- Bit-conservation principle can be extended in a similar fashion

Details can be found in our journal submission (review pending) at www.arxiv.org

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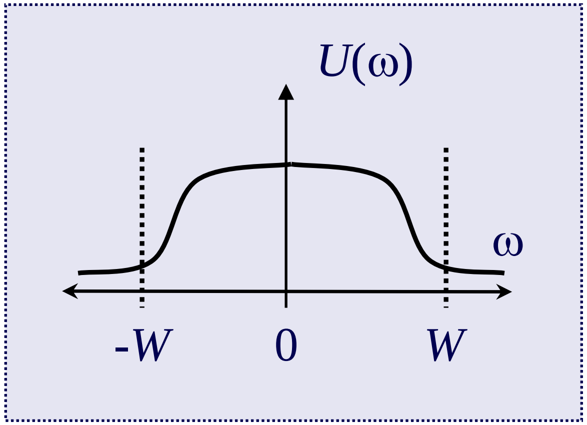
What about smooth non-bandlimited signals?



◊ Consider smooth non-bandlimited signals with $\int_{\omega} |\omega U(\omega)| d\omega < \infty$

◊ Consider the case when $u(t)$ is not prefiltered by a lowpass antialiasing filter

- For example, In a spatially distributed sampling set-up, anti-alias pre-filtering is *impossible* and aliasing error is *inevitable*

Combating against aliasing distortion:

◊ With large W pretend that the field is bandlimited (for sampling/reconstruction)

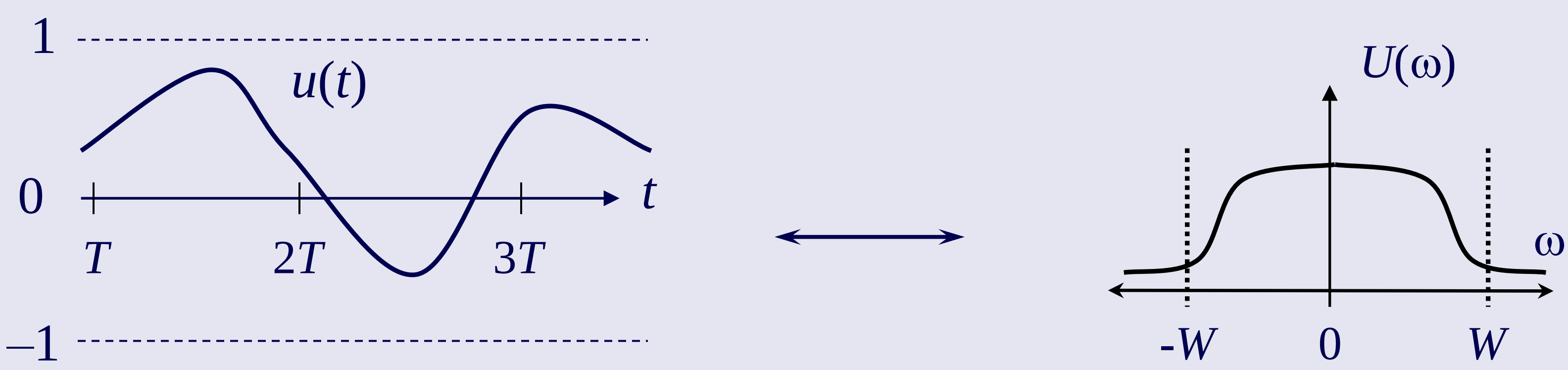
◊ Make W sufficiently large so that aliasing error is comparable to quantization error

Deterministic non-bandlimited signal model

◊ 1-D, real, bounded: $|u(t)| < 1$ (normalized)

◊ Decaying spectrum after a certain (fixed) bandwidth, formally $\int_{\omega} |\omega U(\omega)| d\omega < \infty$

◊ Finite energy in L^2 sense



Pseudo-Nyquist interval = $T = \pi / W$

Main results for non-bandlimited signals

◊ Upper bounds on the distortion D, while accounting for the aliasing and quantization errors terms, are established

◊ Using these upper bounds and by selection of the reconstruction bandwidth W, the distortion-rate tradeoff can be computed

◊ **Achievable trade-off law:** For high bit-rates, any distortion-rate pair $(D, R(D))$ is achievable by a discrete trade-off between quantizer-precision and sample density (*similar to* bit-conservation principle)

[Kumar-Ishwar-Ramchandran'04]

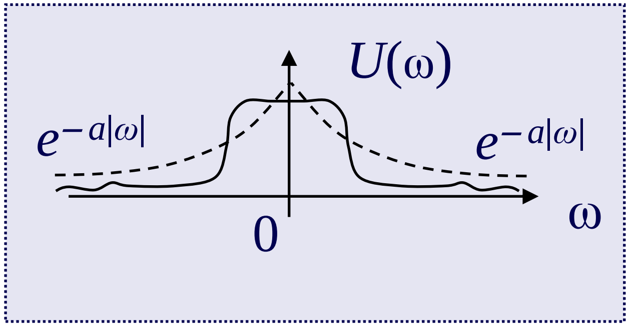
Non-bandlimited case: illustrative example

◊ It can be shown that

$$|u(t) - u_{\text{rec}}(t)| \le A\, 2^{-k} + B \int_{\omega > W} |U(\omega)|\, d\omega$$

quantization part

aliasing part



[Kumar-Ishwar-Ramchandran'04]

For exponential spectral decay:

◊ The aliasing term is $\int_{\omega > W} |U(\omega)|\, d\omega \propto \exp(-\pi a/T)$

◊ The pseudo Nyquist interval T is found by balancing error terms: $\exp(-\pi a/T) = 2^{-k}$

◊ This balancing of errors is the key step in finding achievable reconstruction distortion

◊ Solving these equations we get, $R(D) = O(|\log D|^2)$ or $D = O(\exp(-\text{sqrt}\{\beta R\}))$

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Summary: bandlimited signals

◊ There is a fundamental asymptotic trade-off between quantizer-precision and sample-density (bit-conservation principle)

◊ For BL-signals the following tradeoff equations were obtained irrespective of the quantizer-precision:

◊ b-bit sample density	=	2^{k-b+1} (per Nyquist interval)
◊ Distortion	=	$O(2^{-k})$
◊ Bit-budget per Nyquist	=	k bits

Summary: smooth non-bandlimited signals

◊ Non-bandlimited signals have spectrum dependent distortion-rate characteristics.

◊ The reconstruction bandwidth should be chosen to balance aliasing and quantization errors

◊ Flexibility in trading off sample-density with quantizer-precision for similar asymptotic distortion

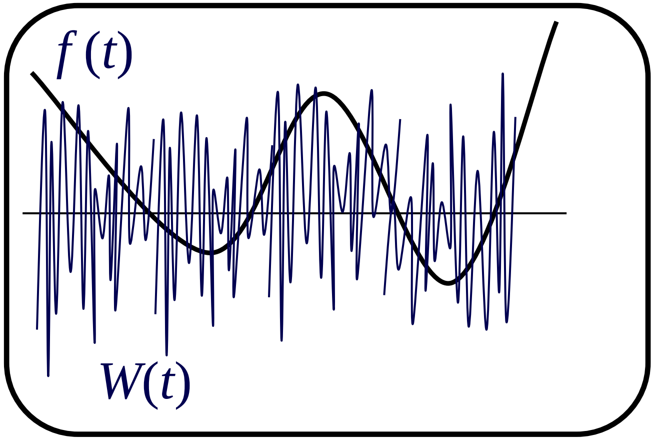
For smooth non-bandlimited signal with exponential spectral decay:

◊ Target distortion $= D$

◊ Sample density $= O(|\log D|/D)$ (per unit time)

◊ Bit-rate per unit time $= O(|\log D|^2)$

Future research



◊ Practically, noise is ubiquitous in acquisition

◊ Distortion-rate tradeoffs and the bit-conservation principle needs to be revisited with sensing noise included in the analysis

Publications on distributed sampling

1. "High-resolution distributed sampling of bandlimited fields with low-precision sensors," **A. Kumar,** P. Ishwar, and K. Ramchandran, Submitted to the IEEE Transactions on Information Theory.
2. "Dithered A/D conversion of smooth non-bandlimited signals," **A. Kumar**, P. Ishwar, and K. Ramchandran, Accepted for publication in the IEEE Transactions on Signal Processing.
3. "On distributed sampling of bandlimited and non-bandlimited sensor fields," **A. Kumar**, P. Ishwar, and K. Ramchandran, ***Invited paper***, ICASSP, Montreal, Canada, May 2004.
4. "On distributed sampling of smooth non-bandlimited fields," **A. Kumar**, P. Ishwar, and K. Ramchandran, IPSN 2004, Berkeley, CA, April 2004.
5. "On distributed sampling in dense sensor networks: a bit-conservation principle," P. Ishwar, **A. Kumar**, and K. Ramchandran, ***Invited Paper***, 41st Annual Allerton Conference on Communication, Control, and Computing, UIUC, IL 2003.
6. "Distributed sampling in dense sensor networks: a ''bit-conservation'' principle," P. Ishwar, **A. Kumar**, and K. Ramchandran, IPSN 2003, Palo Alto, CA, April 2003.