A/D conversion of bandlimited fields in additive independent Gaussian noise

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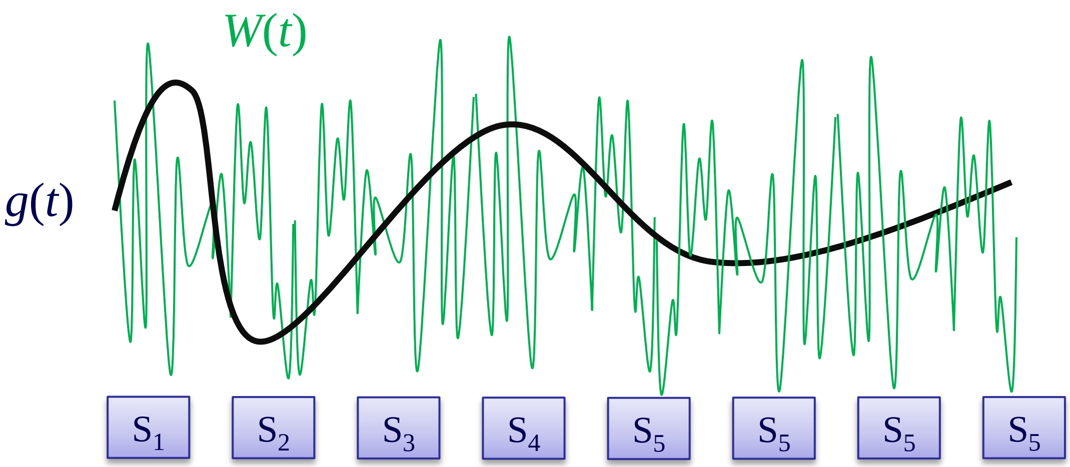
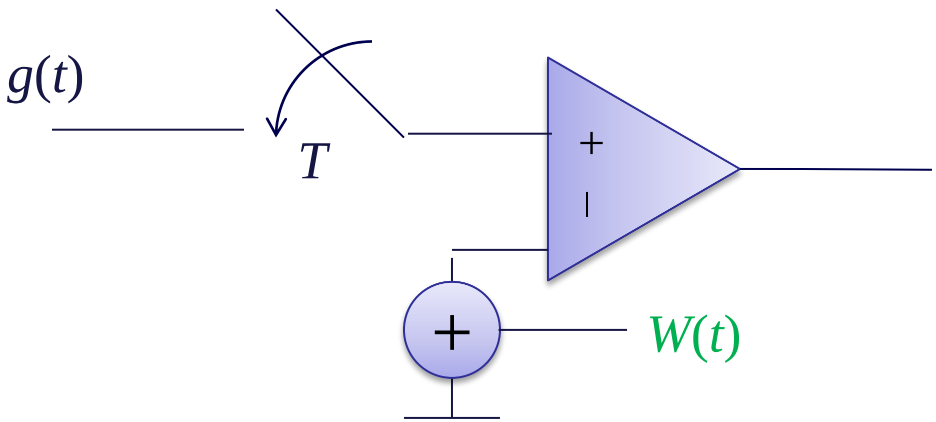
Sampling of a field in finite-variance noise



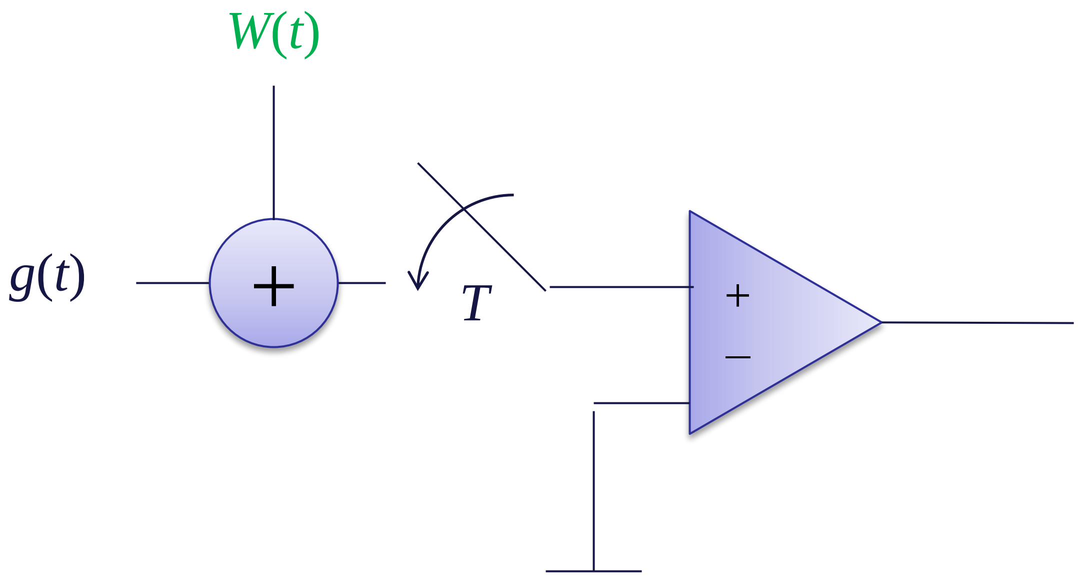
Consider an array of sensors sampling a bandlimited field in additive and independent noise process. The sampling is distributed and the noise variance is finite

Sampling and quantization with noisy ADCs



Consider the sampling of bandlimited signal with ADCs (comparators) which have independent offset voltages

Sampling with noisy ADCs

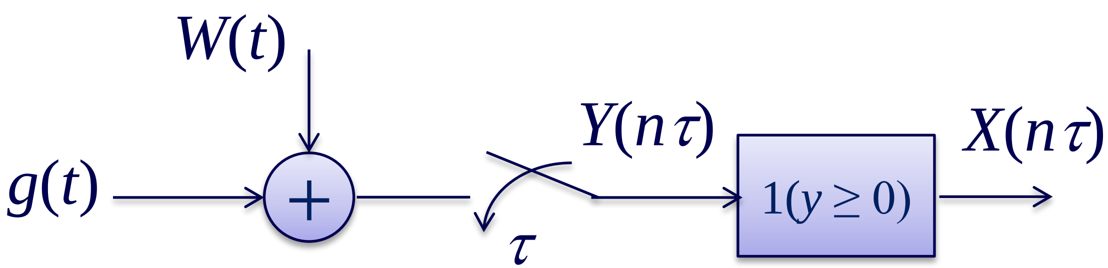


Consider the sampling of a bandlimited signal in wideband noise, where bandwidth of noise $>> 1/T$

Theoretical abstraction of the problems

These sampling problems can be abstracted into the following

$g(t) + W(t)$ is a signal or field to be sampled through precision-limited or single-bit quantizers (comparators)



We will assume that $g(t)$ is bandlimited in a finite bandwidth and $W(t)$ is an additive independent Gaussian noise process

Organization

- ◊ Introduction to the problem
- ◊ Field/signal model
- ◊ Insights into main result using degrees of freedom
- ◊ Analysis of proposed single-bit quantization scheme
- ◊ Conclusions

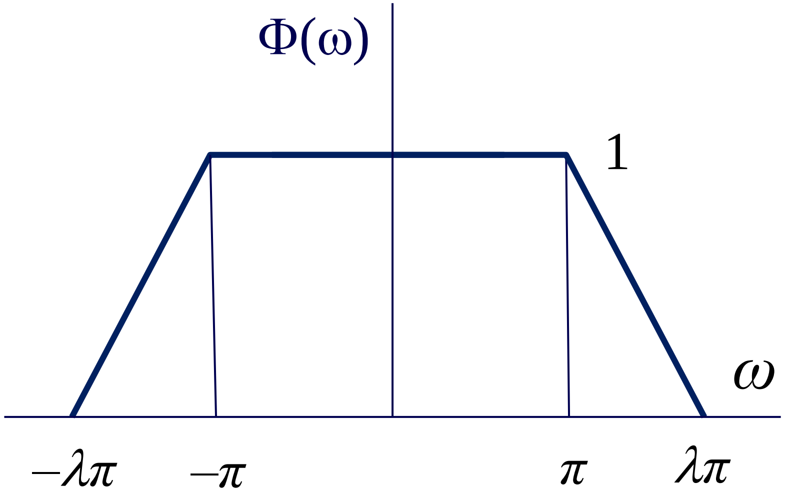
The bandlimited signal model

Define $\phi(t)$ as follows for $\lambda > 1$

and $a = (\lambda - 1)/2$

$$\phi(t) = \frac{1}{\pi a t^2} \sin((\pi + a)t) \sin(at)$$

$$\phi(0) = 1 + \frac{a}{\pi}$$

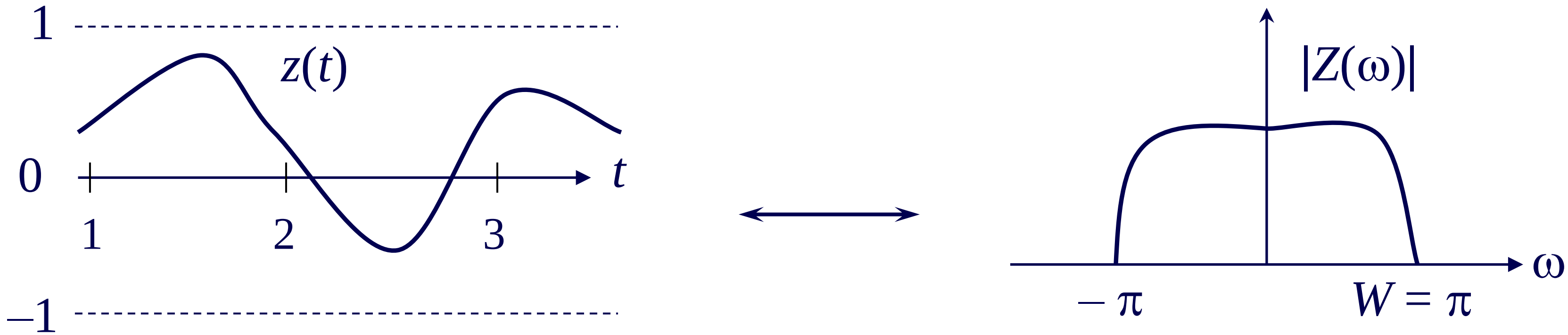


A subset of Zakai class of bandlimited signals is our signal model

$$BL = \{g(t) : |g(t)| \leq 1 \text{ and } g(t) \star \phi(t) = g(t), \text{ for all } t \text{ real}\}$$

The kernel $\phi(t)$ is square and absolutely integrable, which aids in worst-case or pointwise error analysis

$L^2(R)$ bandlimited $\subset$ Zakai bandlimited



Nyquist interval = $T_{NQ} = \pi / W = 1$

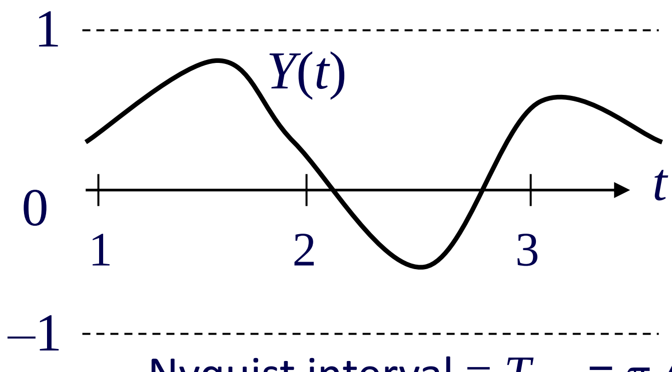
- 1-D real, continuous, and bounded: $|z(t)| < 1$ (normalized)
- Nyquist period, $T_{NQ} = 1$
- Finite energy in L^2 sense

Then $f(t)$ belongs to Zakai class of bandlimited signals since

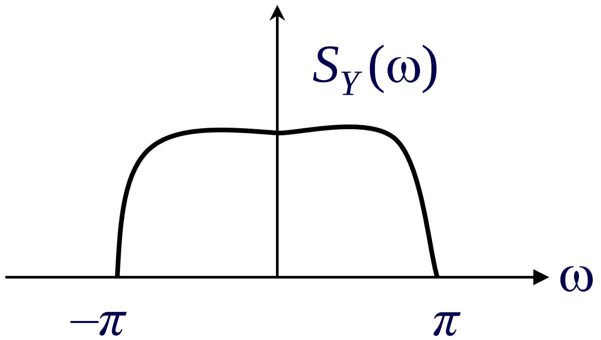
$$z(t) \star \phi(t) = \mathfrak{I}^{-1}[Z(\omega)\Phi(\omega)] = \mathfrak{I}^{-1}[Z(\omega)] = z(t), \text{ for all } t \text{ real}$$

Stationary bandlimited ⊂ Zakai bandlimited

Sample path



Nyquist interval = $T_{NQ} = \pi / W$



Power spectrum

◊ 1-D, real valued, wide-sense stationary signals with amplitude sample paths bounded by 1.

◊ Autocorrelation function is finite-energy bandlimited with $T_{NQ} = 1$

◊ See [Zakai'65] and [Masry'76] ,

$$Y(t) = Y(t) \star \phi(t),$$

almost surely for WSS bandlimited $Y(t)$

Properties of Zakai sense bandlimited signals

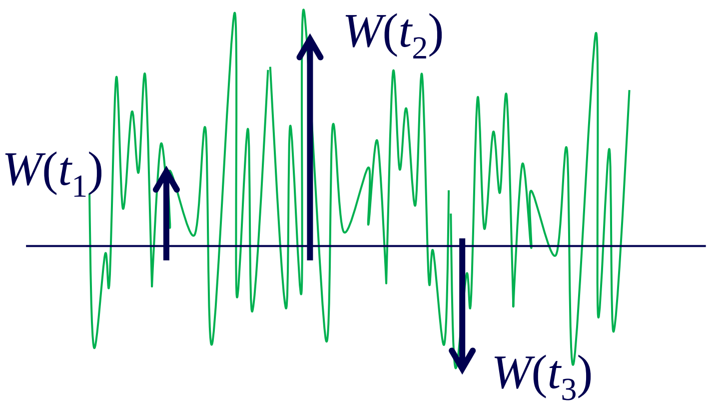
- Any result that applies to bounded-amplitude Zakai sense bandlimited signals will also apply to bounded-amplitude finite energy bandlimited signals as well as bounded stationary bandlimited signals
- Since $\phi(t)$ is smooth and absolutely integrable, $g(t) = g(t) \star \phi(t)$ can be used to establish smoothness of $g(t)$
- Zakai sense bandlimited signals admit a sampling theorem with a stability factor $\lambda > 1$. We have

$$g(t) = \lambda \sum_{n \in \mathbb{Z}} g\left(\frac{n}{\lambda}\right) \phi\left(t - \frac{n}{\lambda}\right)$$

Noise model and mean-squared distortion

The noise $W(t)$ is assumed to be an independent Gaussian process. That is,

$W(t_1), W(t_2), \ldots, W(t_n)$ are independent and identically distributed $N(0, \sigma^2)$



For example, $W(t_1)$, $W(t_2)$, $W(t_3)$ are independent for any t_1, t_2, t_3

Distortion considered for statistical signals is maximum mean-squared error

$$D_{\text{rec}} := \sup_{t \in \mathbb{R}} D_{\text{rec}}(t) = \sup_{t \in \mathbb{R}} \mathbb{E} \left| \widehat{G}_{\text{rec}}(t) - g(t) \right|^2$$

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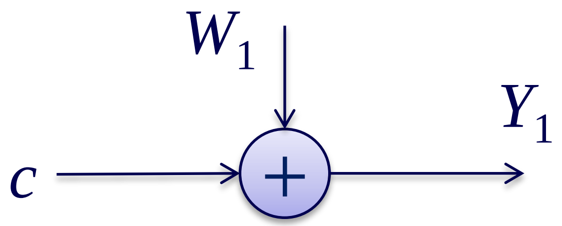
Estimation of a constant from one reading

Consider the problem of estimating a bounded constant (one degree of freedom) in one reading with additive Gaussian noise

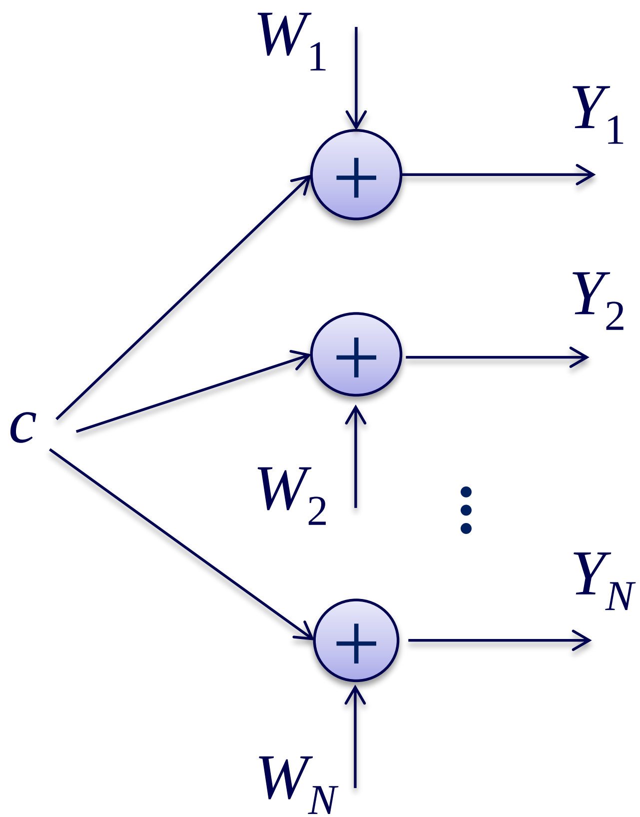


That is estimate c from $Y_1 = c + W_1$

There will be a mean-squared error $\geq \sigma^2$ regardless of the procedure adopted (see Cramer-Rao lower bound)

Oversampling is needed to reduce the mean-squared error

Estimation of a constant with oversampling



◊ Now consider the problem of estimating a **bounded** constant (one degree of freedom) in additive independent (i.i.d.) Gaussian noise

$$Y_1 = c + W_1, \ldots, Y_N = c + W_N$$

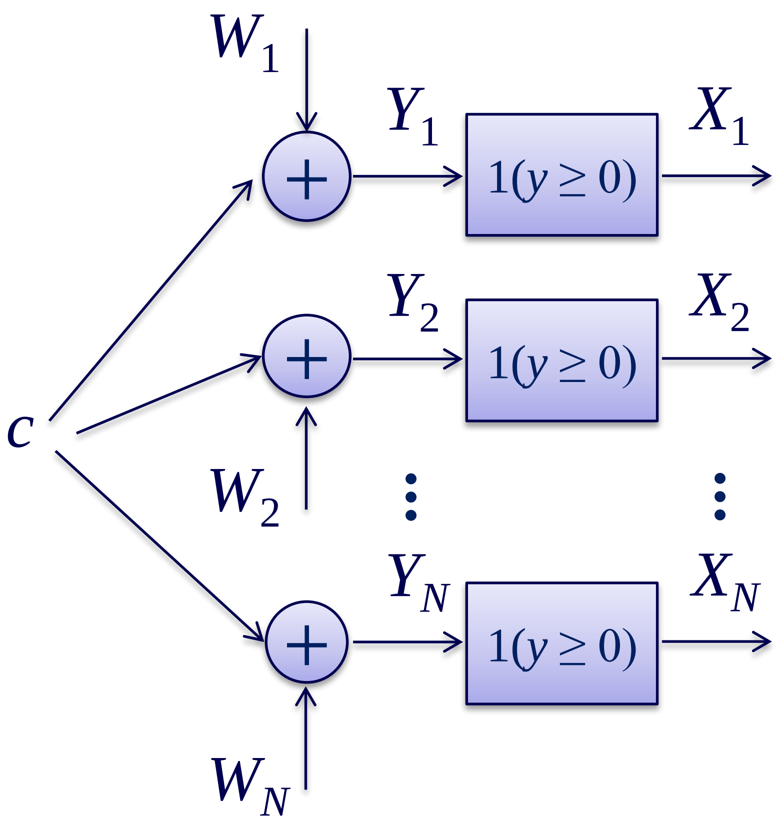
◊ It is known that the best mean-squared error estimate is the avg of $Y_1, Y_2, \ldots, Y_N$

◊ And mean-squared error between $(Y_1 + \ldots Y_N)/N$ and c is $O(1/N)$

Unquantized samples, oversampling by N, and mean-squared error is $O(1/N)$

Estimation and quantization (nonlinearity)

Now consider the same problem in the presence of single-bit quantization



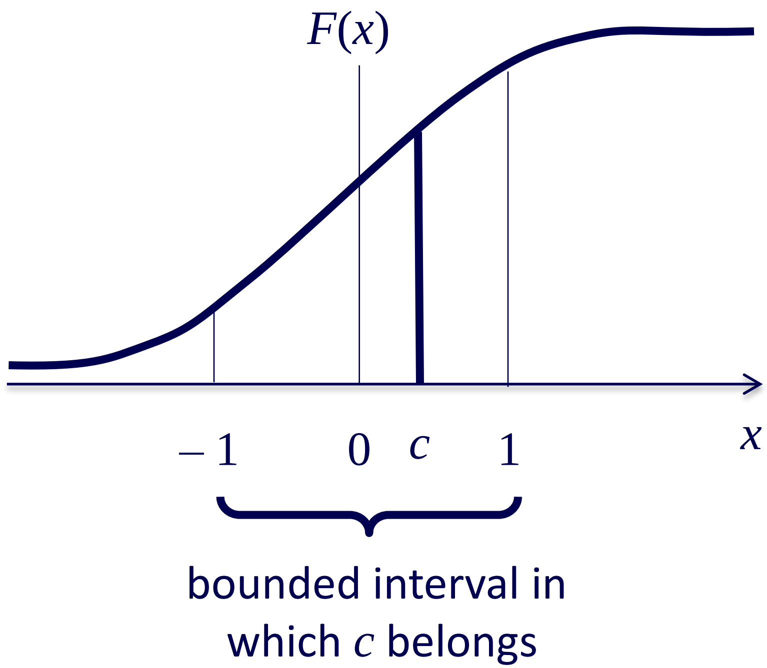
$X_1 = 1(c + W_1 \geq 0)$, $X_2 = 1(c + W_2 \geq 0)$, $\ldots$, $X_N = 1(c + W_N \geq 0)$

◊ These variables are i.i.d. Bernoulli($F(c)$), where $F(x)$ is the cumulative distribution function of W

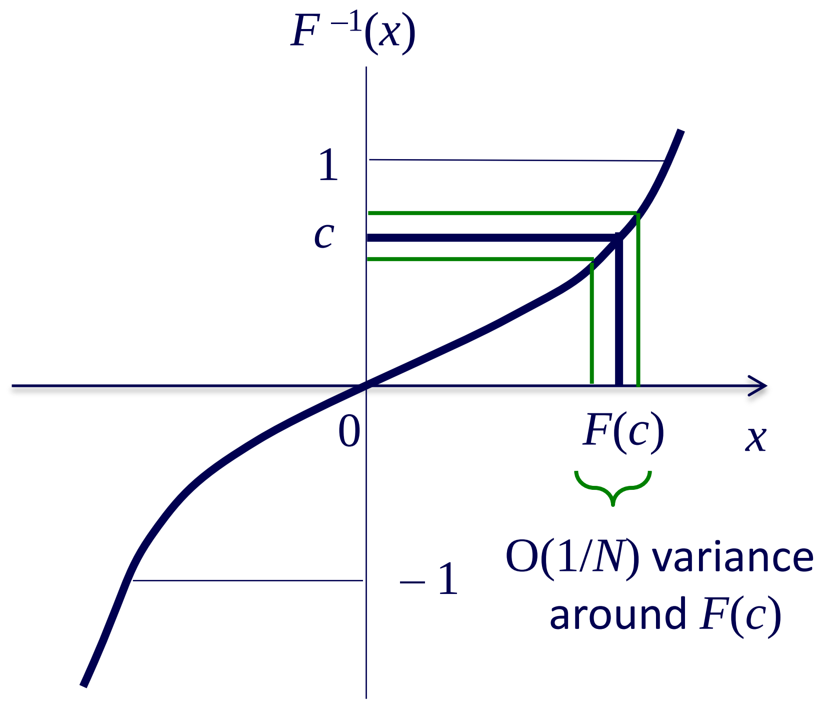
◊ It is known that the average of X_1, X_2, $\ldots$, X_N converges with variance $O(1/N)$ to $F(c)$

Quantized samples, oversampling by N, and mean-squared error in $F(c)$ is $O(1/N)$

The delta-method



An estimate for c is desired, but an estimate for $F(c)$ is present with small variance of $O(1/N)$



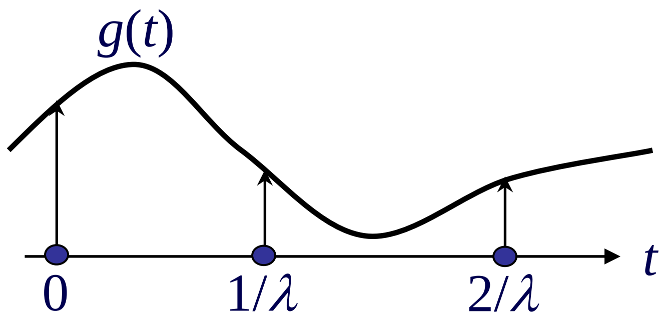
If $F^{-1}(x)$ has a finite slope, then

$F^{-1}[(X_1 + \ldots + X_N)/N]$ converges to

$F^{-1}[F(c)] = c$ with a variance of $O(1/N)$

Thus "**precision indifference**" holds while estimating one degree of freedom in Gaussian noise

Noisy samples and bandlimited signals



$$g(t) = \lambda \sum_{n \in \mathbb{Z}} g\left(\frac{n}{\lambda}\right) \phi\left(t - \frac{n}{\lambda}\right)$$

◊ Samples are uniformly spaced slightly closer than the Nyquist points ($\lambda > 1$)

◊ Thus, there is one degree of freedom every Nyquist interval in $g(t)$

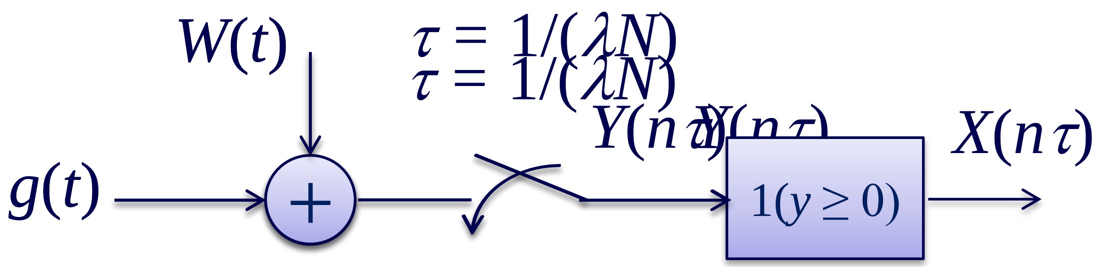
◊ If we oversample $g(t) + W(\mathrm{t})$ by a factor of N, there are N noisy reading for each degree of freedom on an average

◊ Thus we <u>expect</u> optimal distortion to be $O(1/N)$ with perfect samples!

Key result that will be shown next

A distortion of $O(1/N)$ is achievable with single-bit quantized samples!

This improves the previously known bound of $O(1/N^{2/3})$ **[Masry 1981]**



The Zakai class of bandlimited signals will be the signal model

$BL = \{g(t){:}\ |g(t)| \leq 1$ and $g(t) \star \phi(t) = g(t)$, for all t real$\}$

The results will apply to finite energy bandlimited signals as well as stationary bandlimited signals

Related work

- **[Masry'1981]** Single-bit quantization of smooth signals, with additive noise as a dither. His analysis results in an mean-squared bound of $O(1/N^{2/3})$
- **[Wang-Ishwar'2009]** and **[Masry-Ishwar'2009]** Acquisition of a finite-support field in bounded noise. Mean-squared error bounds were established

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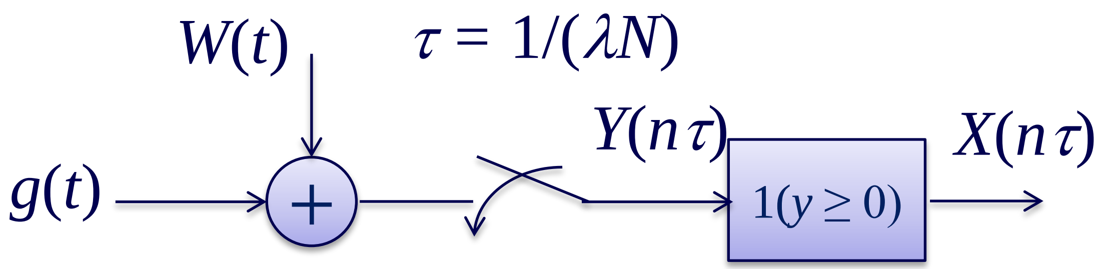
Interpolation of quantized samples



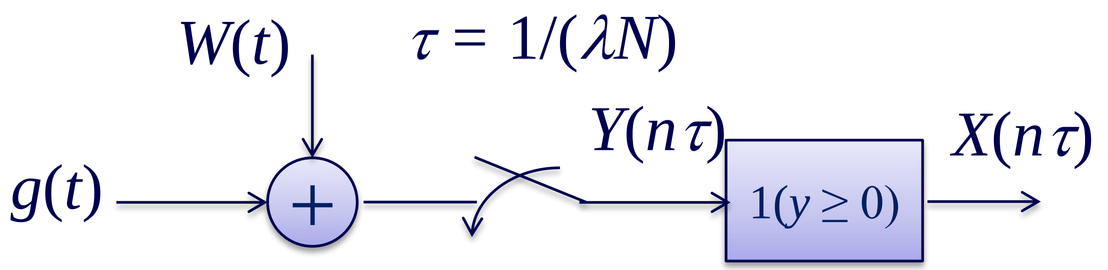
Define an interpolation from the quantized single-bit samples $X(n\tau)$

$$H_N(t) = \tau \sum_{l=-\infty}^{\infty} (X(n\tau) - ½)\phi(t - n\tau)$$

Note that $E[X(t)] = F(g(t))$

$$\tau \sum_{l=-\infty}^{\infty} (X(n\tau) - ½)\phi(t - n\tau) \xrightarrow{\text{LLN}} \tau \sum_{l=-\infty}^{\infty} (F(g(n\tau)) - ½)\phi(t - n\tau) \xrightarrow{\text{Smooth-ness}} (F(g(t)) - ½) \star \phi(t)$$

Interpolation of quantized samples



Remember that

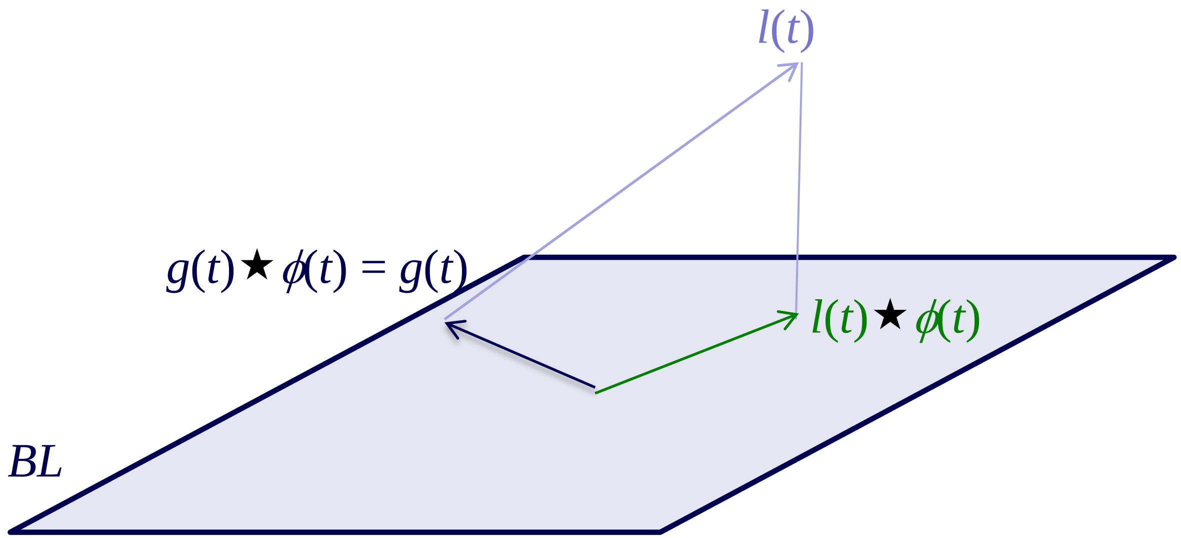
$$H_N(t) = \tau \sum_{l=-\infty}^{\infty} (X(n\tau) - ½)\phi(t - n\tau)$$

Proposition: Let $l(t) = (F(g(t)) - ½)$ and $H_N(t)$ be as defined above. Then,

$$\sup_{t \in \mathbb{R}} \mathbb{E}(H_N(t) - l(t) \star \phi(t))^2 \leq \frac{C_2}{N}$$

where C_2 does not depend on $g(t)$

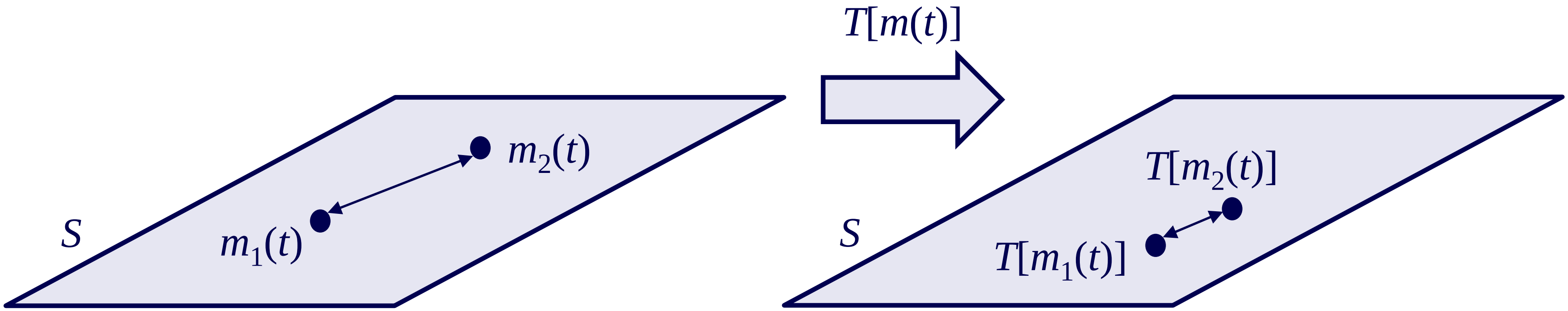
Invertibility of the limit



For the set BL the signal $l(t) \star \phi(t)$ is invertible and $g(t)$ can be obtained uniquely from it (in a pointwise or L^∞ sense). The proof follows using Banach's contraction theorem

$L^2(R)$ version of this problem has been considered by Landau and Miranker'1961. It cannot be used directly since statistical noise is not in L^2

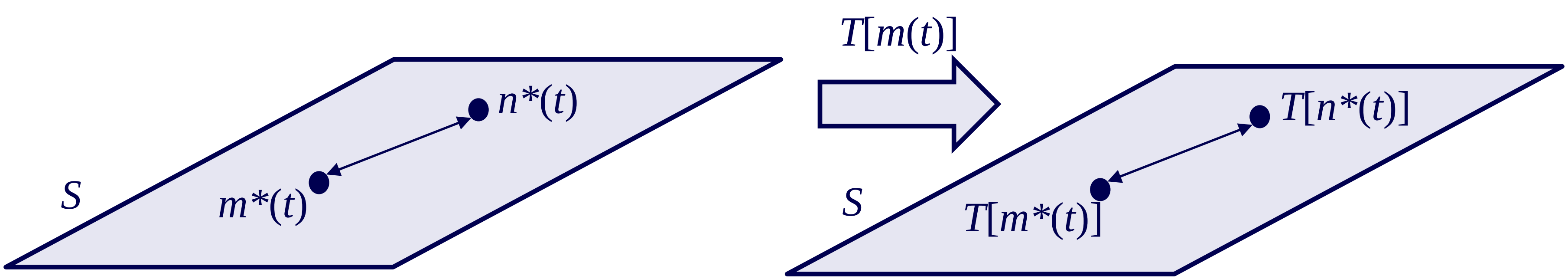
Banach's contraction theorem



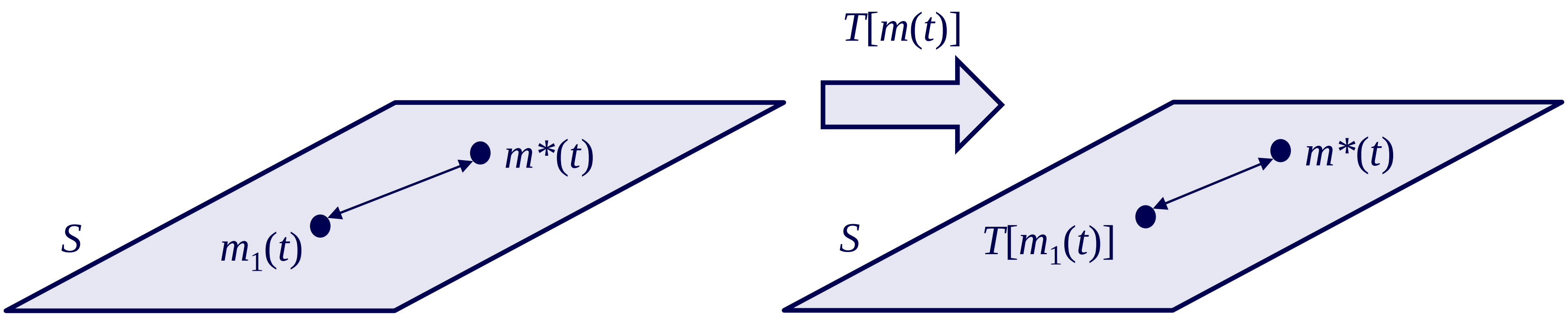
Ingredients: Banach's contraction theorem needs a closed set S, a map T, a distance metric $d(m_1, m_2)$ and a contraction property for any $m_1(t)$, $m_2(t)$ in S

$$d(T[m_1], T[m_2]) \leq \alpha\, d(m_1, m_2) \text{ with } 0 < \alpha < 1$$

Result: Then $T[m(t)] = m(t)$ has a unique solution $m^*(t)$ in S.



Banach's contraction theorem (contd.)



Fact: $m_k(t) = T[m_{k-1}(t)]$ sets up a recursive procedure to approach $m^*(t)$

And $m^*(t) = m_0(t) + [m_1(t) - m_0(t)] + [m_2(t) - m_1(t)] + [m_3(t) - m_2(t)] + \ldots$

$$= m_0(t) + [m_1(t) - m_0(t)] + T[m_1(t) - m_0(t)] + T^2[m_1(t) - m_0(t)] + \ldots$$

Landau and Miranker's contraction formula

Consider L^2-BL = $\{g(t){:}\ G(\omega)$ with support on $[-\pi, \pi]\}$

For this class of signal, the following fixed-point and recursion formula leads to $g(t)$ from $h(t) = [F(g(\mathrm{t})) - ½] \star \mathrm{sinc}(t)$

$$g(t) = \beta\, h(t) + (g(t) - \beta\, l(t)) \star \mathrm{sinc}(t)$$

Fixed-point equation

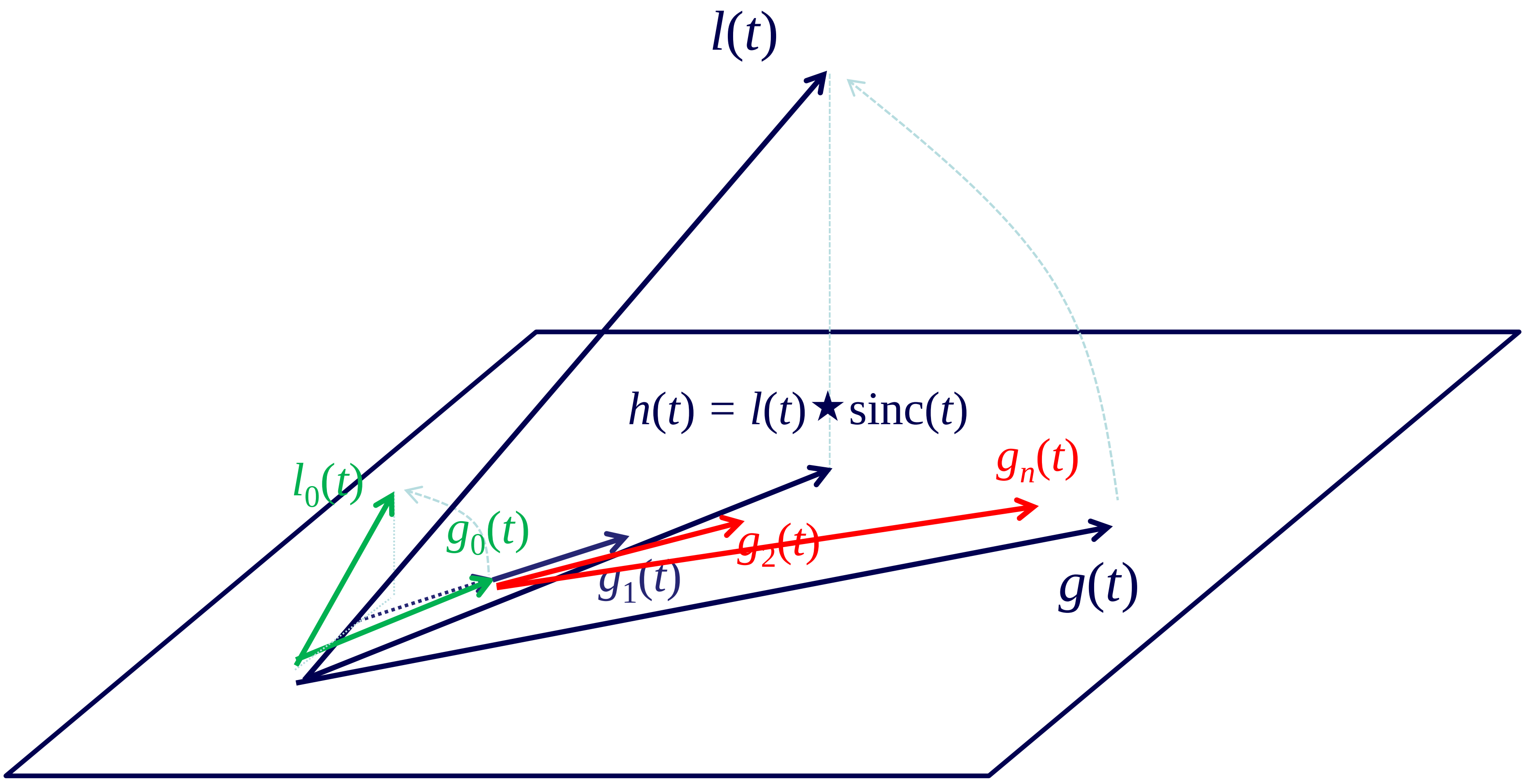
$$g_{n+1}(t) = \beta\, h(t) + (g_n(t) - \beta\, l_n(t)) \star \mathrm{sinc}(t)$$

Recursion formula (Picard iteration)

Landau & Miranker [1961]

The technique works since L^2-BL is a closed subset and there is a value of β for which the recursion formula is a contraction (Banach's contraction mapping theorem)

Landau and Miranker's recursion in pictures



Ingredients of contraction for our problem

Let BL_{bdd} be the set defined as

$$BL_{\mathrm{bdd}} = \{m(t)\colon |m(t)| \le C_\phi \text{ and } m(t) \star \psi(t) = m(t), \text{ for all } t \text{ real}\}$$

where $\psi(t) = \phi(\lambda t)/\lambda$ has slightly larger bandwidth than $\phi(t)$

The distance metric is the maximum pointwise difference (L^∞ error)

Define $\mathrm{Clip}[x] = \mathrm{sgn}(x)$ if $|x| > 1$ and $\mathrm{Clip}[x] = x$, otherwise

$$T[g(t)] = \mathrm{Clip}\{\mu\, l(t) \star \phi(t) + [g(\mathrm{t}) - \mu\, l(t)] \star \phi(t)\} \star \phi(t)$$

Fixed-point equation

Set $g_0(t) = 0$ and

$$g_{n+1}(t) = \mathrm{Clip}\{\mu\, l(t) \star \phi(t) + [g_n(\mathrm{t}) - \mu\, l_n(t)] \star \phi(t)\} \star \phi(t)$$

Recursion formula

It turns out that there is a value of μ such that T is a contraction on BL_{bdd}

So $g(t)$ can be obtained from $l(t)$ by using T and recursion on BL_{bdd}

Final step of the proof

But $H_N(t)$ an approximation of $l(t) = F(g(t)) - ½)$ is available! However, contraction is stable to perturbations

The modified recursive map is

$$T[G_k(t)] = \text{Clip}\{\mu\, H_N(t) \star \phi(t) + [G_{k-1}(\text{t}) - \mu\,(F(G_{k-1}(t)) - 1/2) \star \phi(t)\ ]\} \star \phi(t)$$

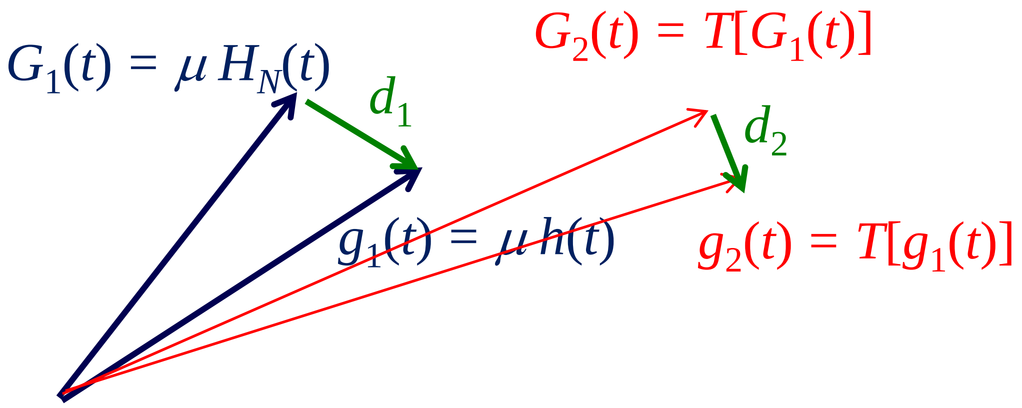
Theorem: Let $G_0(t) = 0$. Let $\hat{G}_{\text{1-bit}}(t)$ be the limit of $G_k(t)$. Then, the above recursion results in

$$D_{\text{1-bit}} := \sup_{t \in \mathbb{R}} \mathbb{E}(\widehat{G}_{\text{1-bit}}(t) - g(t))^2 = O(1/N)$$

which establishes the precision-indifference principle for bandlimited signals in additive independent Gaussian noise

Picture for the final step of proof

But $H_N(t)$ an approximation of $l(t) = F(g(t)) - ½)$ is available. Contraction is stable to perturbations



By contraction property, ... $d_3 \leq \alpha d_2 \leq \alpha^2 d_1$

Thus, by triangle inequality, the maximum error can be shown to be sum of all d_i's, or $d_1/(1-\alpha)$

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Summary

For bounded-amplitude bandlimited signals sampled in the presence of (additive) independent Gaussian process with oversampling N and mean-squared distortion

- ◊ Optimal distortion is expected to be

$$= O(1/N)$$

- ◊ Distortion achievable with single-bit quantized readings

$$= O(1/N)$$

Extensions or future work

◊ Our estimate is not minimum risk. Fast algorithms for finding Maximum likelihood estimates, which will also be accurate up to $O(1/N)$, will be useful

◊ Extension of these results to more classes of signals (FRI, finite-support, orthogonal spaces)

Further reading

1. Animesh Kumar and Vinod Prabhakaran, “Estimation of Bandlimited Signals in Additive Gaussian Noise: a "Precision Indifference" Principle”, arxiv preprint available at http://arxiv.org/abs/1211.6598
2. Animesh Kumar and Vinod Prabhakaran, “Estimation of bandlimited signals from the signs of noisy samples”, ICASSP 2013, Vancouver, BC Canada.