writing test.

EE113: High-gain feedback and control: proportional and PD Control

Madhu N. Belur

Control & Computing group,
EE, IITB

21st Feb 2022

Recap

Saw close links between excites

- Impulse as input (δ), and the output: impulse response $h(t)$
- Laplace transform of signals: $L(f(t)) = F(s)$
- In particular, $L(\delta) = 1$ in time domain
- Laplace transform converts convolution to multiplication $\Rightarrow$ in s-domain
- Any system: any Linear Time-Invariant (LTI) system output y for input u is just:
  $u(t)$ convolved with impulse response $h(t)$

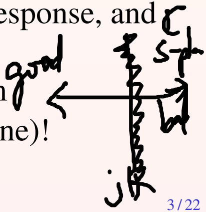
Transfer function

Transfer function, Laplace transforms, impulse response, convolution: closely linked!

$T \approx 20^\circ C$

$y(t) = (T - 20)$

- Input u is also called 'forcing function'
- Output: variable y to be 'regulated' (measure deviation from set-point)
- Transfer function from input u to output y: say $G(s) = \frac{n(s)}{d(s)}$ for polynomials $n(s)$ and $d(s)$ with real coeff.
- Roots of numerator $n(s)$ are called 'zeros' of the system, of $G(s)$
- Roots of denominator $d(s)$ are called 'poles' of the system, of $G(s)$
- Poles show-up within **exponents** (e^{pt}) in: impulse response, and also 'unforced response'
- Hence: called 'natural response': intrinsic to system



Worrisome if **poles** are in RHP (Right Half Complex plane)!

Why study **zeros** of $G(s)$? Today

Initial Value Theorem, Final Value theorem

- Recall: s and t are opposite dimensions: s is units of $1/\text{time}$

  $\downarrow t$

- Initial-time: $\equiv s \to \infty$
- Final-time: $\equiv s \to 0$
- Remember to multiply by s (since $Y(s)$ has dimensions of time (in multiplication to $y(t)$))

  $y(t)$ = voltage

  $Y(s)$ voltage·time $\equiv$

  $$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt$$

- **Provided** the values exist:

  $y(0^+)$: initial value = $\lim_{s\to\infty} sY(s)$

  $y(\infty)$: final value = $\lim_{s\to 0} sY(s)$

  find

  finite

- Suits best for step response, since **unit step input** has Laplace transform: $1/s$

  step $\xrightarrow{\mathcal{L}} \left(\frac{1}{s}\right)$

Transfer function $G(s)$: unit step response: initial/final values

$e^{s_0 t}$ $G(s_0)e^{s_0 t}$ $G(2) = G(s)\big|_{s=2}$ $G(s)\big|_{s=\infty}$

Provided the initial/final values are finite: $G(\infty) = \lim_{s\to\infty} G(s)$

- Step response initial value: $G(\infty)$: proper/strictly-proper/improper

  deg n < deg d. $\frac{s^2+3s+8}{s+5}$ $\frac{9}{5} \in \mathbb{Q}$

- Step response 'final' value: $G(0)$: DC-gain

  $s \to 0$ $G(0)$

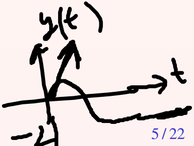
Examples: $G(s) = \frac{s+1}{s^2-3s+2}$: final value: unbounded!

$(s-1)(s-2)$ poles $\in$ RHP

F.V. exist

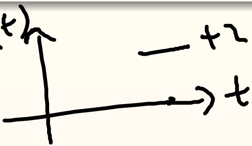
Examples: $G(s) = \frac{s+4}{s^2+3s+2}$: final value: finite $\lim_{t\to\infty} y(t) = -2$

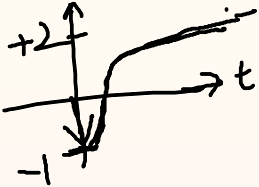
Initial value:



Unit step response

$u(t) =$ unit step $\frac{1}{s}$



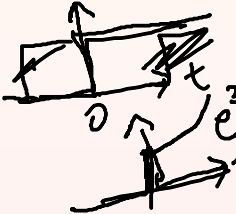


$G(s) = \frac{-s^2+4}{s^2+3s+2}$: final value: finite

Initial value:

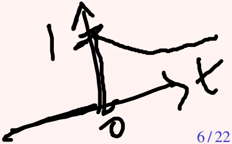
Look up table

$\frac{1}{s+8} \rightarrow e^{-8t}$ fn $t \geq 0$



$\leftarrow \frac{1}{s}$

$e^{3t} \leftarrow \frac{1}{s-3}$



Regular and inverted pendulum

unstable eq. pt.

$\leftarrow$ eq. pt.

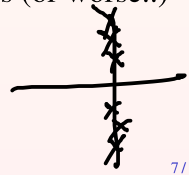
Regular pendulum: bob at the bottom.
Inverted pendulum: bob (and rod) upright: needs balancing control
Let input $u = F$ forcing function: the force, and
output $y = \theta$=angle: deviation from up/down position
(Scale appropriately)

mathematical modelling

- Regular: $\frac{d^2}{dt^2}y + y = u$
- Inverted pendulum: $\frac{d^2}{dt^2}y - y = u$
- Use $u = -k_p y$, with k_p as proportional control
- Whatever k_p is used, poles always on imaginary axis (or worse..)
-

$$\left[\frac{d^2}{dt^2} + (k_p - 1)\right] y(t) = 0$$

$k_p > 1$



Proportional k_p + Derivative k_d (PD) control

slope > 0

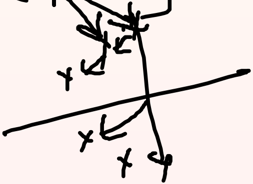
$$u = -(k_p y + k_d \tfrac{d}{dt} y)$$

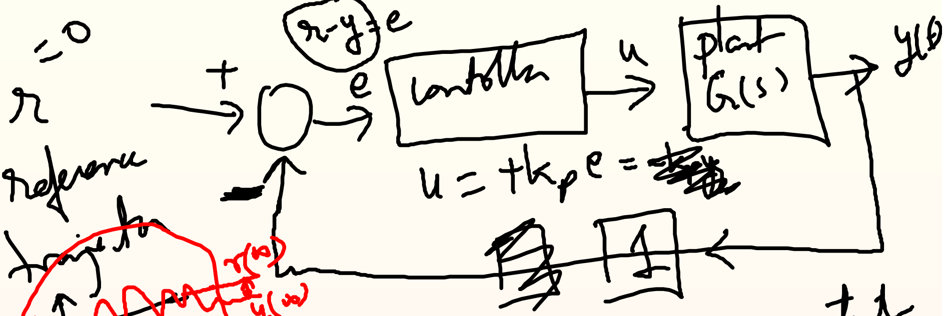
$F \propto -k_p \theta - k_d \dot{\theta}$ $\propto -k \int_0^t \theta \, d\tau$

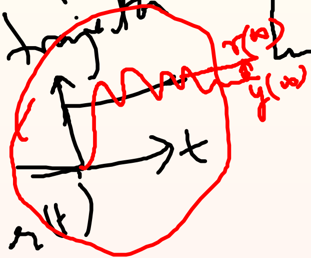
expedites

$$\left[\frac{d^2}{dt^2} + k_d \frac{d}{dt} + (k_p + 1) \right] y(t) \to 0$$

$k_d > 0$







$u = +k_p e$ proportional

$u = -k_p y$

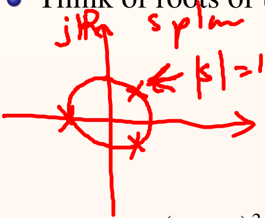
$r = 0$

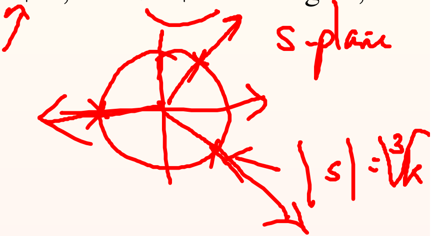
High-gain feedback: often helpful

- Fact: For a large family of systems (but not all): high gain in feedback helps in **reduced** steady state error in step response (even small error (at steady state) is made to result in adequate corrective action)
  Integrator: accumulate small error to generate corrective action: results in steady state error = 0!
- For **unit** step input (for $r(t)$), the output $y(t)$ is closer to **one** as $t \to \infty$
  Required
  desired -
  (provided $y(t)$ does not go unbounded!)
- But also many times: high-gain causes closed loop instability!
- Two easy-situations examples: 3 poles (and no zero): smells trouble for large-gain
- Right half plane zero:
  step response 'starts' opposite to final value!

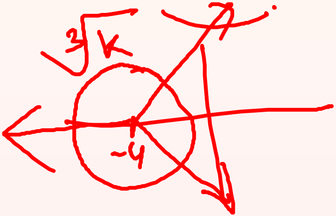
Cubic polynomial

- Think of roots of unity: consider $s^3 + 1$, then $s^3 + k$ for large k,





- Consider $(s + 4)^3 + k$ for large k,



$\frac{12}{3} = 4$

$(s^3 + 12s^2 + 48s + 64) +$

- Cubic polynomial: $s^3 + 12s^2 +$ lower order powers $+ k$
- Asymptotes diverge for large k: as if diverging from real-axis point: -4

$k = ?$

$s^3 + 6s^2 + 8s + k \to$

$\frac{6}{3} = 2$

$s^3 + 6s^2 + 18s + 27$

$(s+2)^3$

$s^3 + k$

$$\left(s^3 + 6s^2 + 8s + k\right) = \left(s^2 + w^2\right)(s + p)$$

$$s^3 + ps^2 + w^2 s + pw^2$$

$$k = 6 \times 8$$

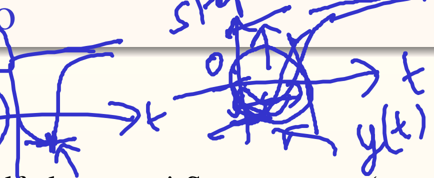
$$k = 48$$

$$\left(s^2 + 8\right)(s + 6)$$

RHP zero

non minimum phase

step resp



- Right half plane zero! Step response 'starts' opposite to 'eventual value'!!
  (misleading: for practitioners and also for high-gain controller)

stable

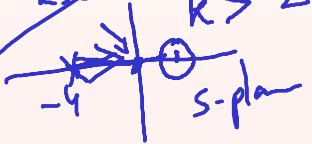
- Example: $G(s) = \frac{s-2}{s+4}$ for system $\frac{d}{dt}y + 4y = \frac{d}{dt}u - 2u$
  (for feedback $u = -ky$)

$$\left[(1+k)\frac{d}{dt} + (4-2k)\right]y = 0$$

- Check: for $k >$?
- More generally: closed loop poles are roots of : $d(s) + kn(s)$, i.e. roots of $\frac{1}{k}d(s) + n(s)$

k<2

k > 2



closed loop characteristic poly.

Next lecture: a glimpse of inter-relations between

KCL KVL

- # Kirchhoff's current laws, # Kircchoff's voltage laws (for planar/more-general circuits?)
- Tellegen's theorem: topology of circuit (already) implies conservation of power
- Hairy ball theorem: (in)ability to 'smoothly comb' hair on a fully haired ball
- Euler's formula (convex polyhedron): #Faces - #Edges + #Vertices = 2 (or more generally (for toruses)?)
- Types of equilibrium points (and their index) (on a sphere/other 'manifolds')