Math. model of pendulum 1 $\ddot{y} + y = u$

$\uparrow$

Force.

DC gain. 2 $\cancel{y \theta}$

$G(0)$ $y \leftrightarrow \theta$

$\frac{1}{s}$

unit step

EE113: other closely related topics

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4

High gain $\overset{?}{\Longrightarrow}$ instability

22nd Feb 2022

3

RHP pole
zero.

This lecture: a glimpse of inter-relations between

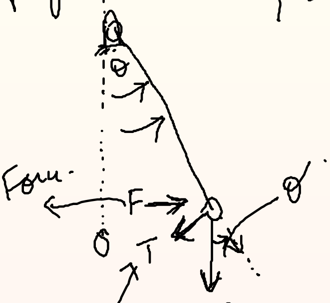
- # Kirchhoff's current laws, # Kirchhoff's voltage laws (for planar/more-general circuits?)
- Tellegen's theorem: topology of circuit (already) implies conservation of power
- Laplacian matrix L: window to a world of graph theory
- Hairy ball theorem: (in)ability to 'smoothly comb' hair on a fully haired ball
- Euler's formula (convex polyhedron): #Faces - #Edges + #Vertices = 2 (or more generally (for toruses)?)
- Types of equilibrium points (and their index) (on a sphere/other 'manifolds')

~~Inverted~~ pendulum: linearized model

$\ddot{y} + y = F$

$-y$ ← unst

Regular



torque

$T = mg\sin\theta \times L.$

F = external force along θ

For small θ, $\sin\theta = \theta$.

$I\ddot{\theta} = -mgL\sin\theta + F\cdot L$

nonlin eqns $\boxed{I\ddot{\theta} + mgL\sin\theta = FL}$

$\boxed{I\ddot{\theta} + mgL\theta = FL}$

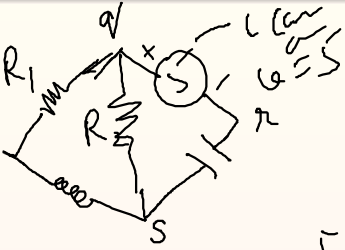
regular

inverted → $I\ddot{\theta} - mgL\theta = F$

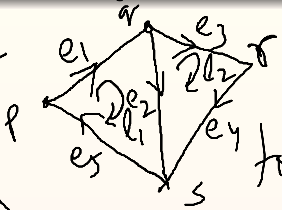
Circuit graph

planar graph.

$R_1 > 0$
$R_2 > 0$



$i \ (a$ $v = 5$



topology

ℓ_3

$$i = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_5 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 = v_p - v_q \\ \\ \\ \\ \end{bmatrix}$$

5 i
5 v

KVL, KCL

5
5 de

Circuit topology

https://www.tutorialspoint.com/network%5Ftheory/network%5Ftheory%5Ftopology%5Fmatrices.htm

- Consider the graph: assign edge directions (in any way)
- Assign currents as positive along the edge
- Assign voltage-drops as positive along the current
- If an edge e_n is a resistor, then $v_n i_n > 0$: thus 'load convention'
- Construct 'node edge incidence matrix' A and loop edge '~~comprisal~~' matrix B (which loop contains which edges)

occurrence

Two situations

For the same topology (and same orientations of the edges):

Situation 1: put for each edge: any device: measure currents: vector i

Situation 2: put each edge any device: measure voltages: vector v

Then: (Tellegen's weak form): $v^T i = 0$ $= v_1 i_1 + v_2 i_2 + \cdots v_5 i_5$

Strong form: dimension of all possible v's + all possible i's = number of edges

rank, independence (linear)

(Voltage-vectors 'complement' the current vectors):

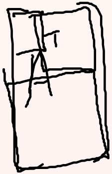
after removing (shrinking) 'redundant' rows within A, and

after removing (shrinking) 'redundant' rows within B

$\begin{bmatrix} A \\ B \end{bmatrix}$ is nonsingular, and

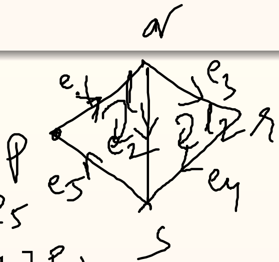
$AB^T = 0$

v



Easier to see in an example

Graph with 5 edges



Node-edge incidence ~~matrix~~ (for KCL) A
(Inwards : +) and outwards: -)

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array}$$

$$A = \left[\begin{array}{ccccc} -1 & 0 & 0 & 0 & 1 \\ +1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{array}\right]\begin{array}{c} p \\ q \\ r \\ s \end{array} \Bigg\} \text{ nodes}$$

Vector of through (edge) currents $i = [i_1, i_2, i_3, i_4, i_5]^T$

$$i = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_5 \end{bmatrix}$$

(Laplacian matrix of the (undirected) graph) $L = AA^T$)
L gives number of (spanning) trees: Kirchhoff's matrix tree theorem

Loop-edge ‘occurrence’ matrix (KVL)

(except outer)

Loops: choose clockwise (for convenience)

Does an edge ‘occur’ in that loop? (aligned along loop : + and opposite: -)

$$\begin{array}{cc} & \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ B = & \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ -1 & 0 & -1 & -1 & -1 \end{bmatrix} \begin{array}{c} l_1 \\ l_2 \\ l_3 \end{array} \end{array}$$

Vector of voltage drops (along edges) $v = [v_1, v_2, v_3, v_4, v_5]^T$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_5 \end{bmatrix}$$

Where are the KCL and KVL?

$= 0$

$v_l = v_p - v_q$

KCL: $Ai = 0$

KVL: $Bv = 0$

In fact: $v = A^T v_{\text{nodes}}$

And: $i = B^T i_{\text{loop-curr}}$

And $AB^T = 0$ (KVL and KVL are orthogonal laws): Tellegen's weak form

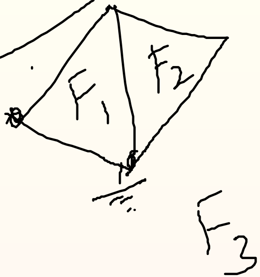
Further, $\begin{bmatrix} A \\ B \end{bmatrix}$ has rank = # edges (Tellegen's strong form: 'complementary')

5x5

nonsing

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix} = v_p - v_q = \begin{bmatrix} A \end{bmatrix}^T \begin{bmatrix} v_p \\ v_q \\ v_r \\ v_s \end{bmatrix}$$

planar



graph
planar
outer most face

5 - edges
4 - vertices

$3 - 5 + 4 = 2$

KVL loops

vertices KCL

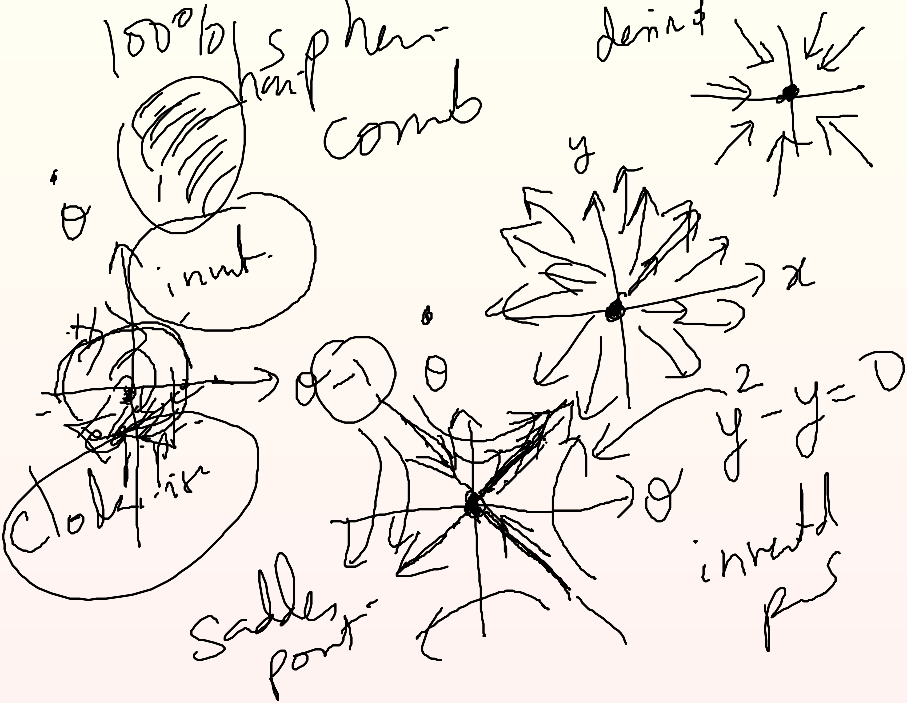
Types of equilibrium points

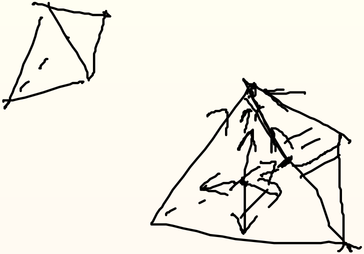
donut

$F - E + V = 0$

1 - torus

shrinking


100% sphere
hair comb
y
x
θ
$y^2 - y = 0$
Saddle point
inverted



4 Vertex ← inwards

6 edges ⟺ saddle

4 Faces ⟺ outward

$= 2$

$F - E + V = 0$

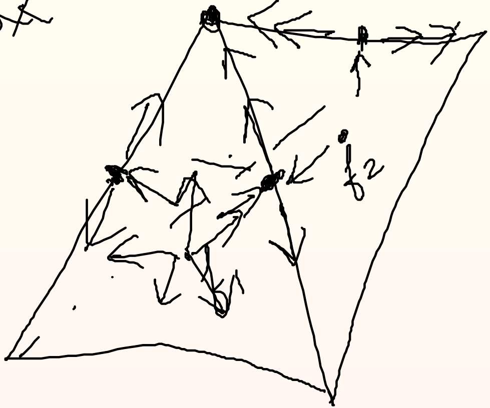
2 dim

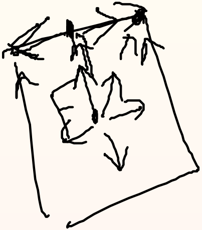
1 dim

0 dim

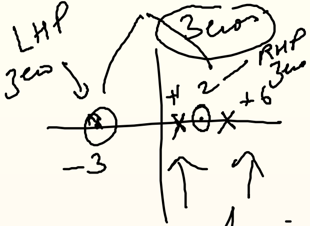
Saddle index $\rightarrow -1$

Cohomology





$$G(s) = \frac{(s-2)(s+3)}{s^2 - 7s + 6}$$



poles RHP

$\mathbb{C}$ s-plane

High gain $\Rightarrow$ closed loop instability

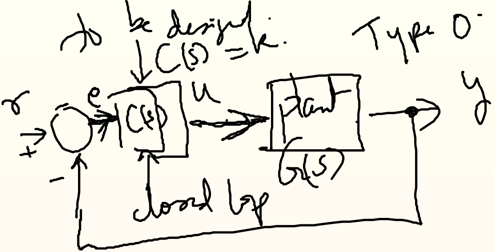
$$G(s) = \frac{s-2}{s+3}$$

$$\frac{d}{dt}y + 3y = \frac{d}{dt}u - 2u$$

$u = ky$.

$k >$ closed loop $\Rightarrow$ unst.

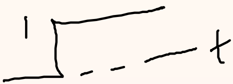
Hairy ball theorem



$$\lim_{t \to \infty} r(t) - y(t) = e(t)$$

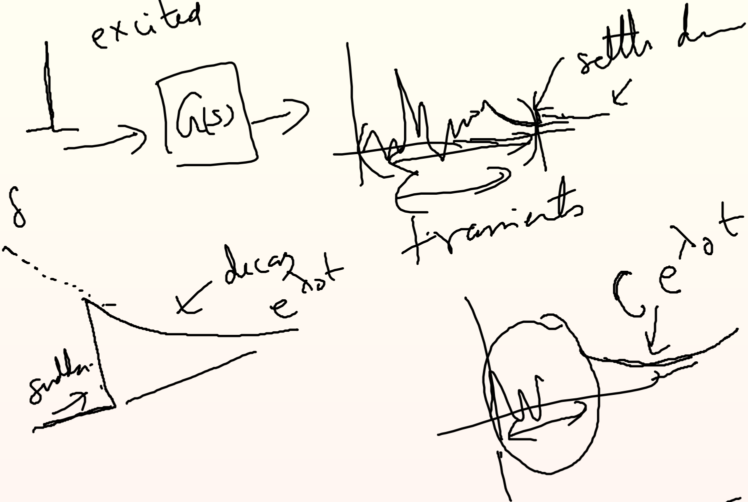
$\to$ ≃ steady state error

$$y(t) \to r(t)$$



$$y(\infty) \simeq 1.$$

$$r \longrightarrow y$$

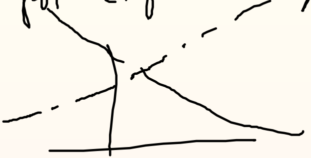

excited
G(s)
settle down
transients
δ
decay
$e^{\lambda_0 t}$
sudden
$C e^{\lambda_0 t}$

$$C = G(\lambda_0)$$

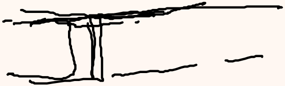
Tutorial on Thursday: will send problems today

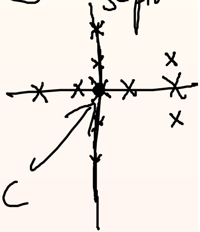
transfer fun ≡ gain (for exponential inp

(steady state)



unit step





$s = 0$ in $G(s)$

|||

DC.

$G(0) \equiv$ DC gain $s = j\omega$

(after transient (hopefully) settle down