EE 113 - Control Systems module: LTI, Laplace transform

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Outline

- LTI systems
- Convolution
- Laplace transforms
- Exponential functions as ‘eigenfunctions’
- Transfer functions as: gain (scaling) and also ratio of Laplace transforms of output to input

LTI systems and exponential signals

- Linear Time Invariant (LTI) systems:
  systems governed by linear, constant coefficient,
  ordinary differential equations
- Signals: exponential functions e^{at}: function of time t
- a need not be real: can consider complex-valued functions of
  real-variable time t

real (time)

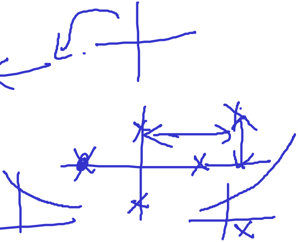
t

e^{at}, $a \in \mathbb{C}$

$f(t) = c_1 e^{at}$

$a \in \mathbb{R}$

$c_1 \in \mathbb{C}$



$e^{j\omega_0 t}$

$a = \sigma + j\omega$

$\sigma > 0$, $\sigma < 0$

LTI systems

$$\dot{u} + 3u = \ddot{y} + 3\dot{y} + 2y$$

LTI

$\cup$

subset

$$\underbrace{\begin{bmatrix} 3 & 1 & -2 & -3 & -1 \end{bmatrix}}_{\text{(constant) coefficients}} \begin{bmatrix} u \\ \dot{u} \\ y \\ \dot{y} \\ \ddot{y} \end{bmatrix} = 0$$

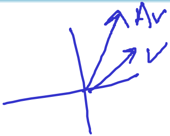
$\subset$ LTI

subset

pure delay.

Exponential signals are ‘eigenfunctions’

eigenvector
eigenvalue



- For (square) matrices, we speak of eigenvectors
- Consider: $A = \begin{bmatrix} 7 & -10 \\ 5 & -8 \end{bmatrix}$

$v \xrightarrow{A} Av$

$u \to \boxed{\text{sys}} \to y$

LTI

- Vectors: $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- For some real square matrices, need to allow eigenvectors to be complex

$v_1 \neq 0$

$v_1 \to \alpha_1 v_1$

$c_1 v_1 \to \alpha_1 (c_1 v_1)$

$c_1 \neq 0$

(eigenfunct)
exponentials
LTI

LTI systems and exponential signals

$$\frac{d}{dt}u + 3u = \left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right)y$$

$$u \to \boxed{} \to y$$

(s^2+3s+2)

$$y_h = c_1 e^{-2t} + c_2 e^{-t} \qquad u(t) \equiv 0$$

- LTI systems have all exponential signals as eigen-'functions'
- Scaling: depends on the system. 'Transfer function'
- Transfer function also rings a bell (resonates?) with 'resonance'!
- Poles and zeros for input to output map
- Convolution is how input gets mapped to output

e^{at}

α is a fnctn "a"

$$u(t) = 3e^{at} \rightarrow c_3 \alpha e^{at} \rightarrow \text{solve for } \alpha \in \mathbb{R}.$$

$$y(t) = y_p(t) = \frac{c_3 \alpha e^{at}}{}$$

$\alpha = ?$

$$G(s)\Big|_{s=a} = \frac{s+3}{s^2+3s+2}$$

LTI systems and sinusoids signals

$G(s)$, $s = j\omega$ $G(j\omega)$

$a = \sigma + j\omega$, $\sigma = 0$

$e^{j\omega t}$ like $\leftarrow \sin \omega t$

$G(j\omega)$

- For system with transfer function $G(s)$
- Sinusoids are also exponentials: real part zero
- Transfer function: complex (when evaluated at $j\omega$): magnitude and angle depends on ω: 'phasor' analysis
- If input is $\sin(\omega t)$, then output is $|G(j\omega)| \sin(\omega t + \angle G(j\omega))$
- Bode plot: plot $|G(j\omega)|$ versus ω (in log-log scale)
- and also: $\angle G(j\omega))$ versus ω (in normal-log scale)
- Low-pass filters/high-pass filters

Capacitor $Z(s)$ $Z(j\omega)$ $\dfrac{1}{Cj\omega}$.

$s = j\omega$

Particular solution and homogenous solution

$$\frac{Y(s)}{U(s)} =: G(s) \qquad \alpha = f(a) \quad G(a)$$

$$\frac{s-2}{s^2+3s+2}$$

- Consider $\frac{d}{dt}u - u = \frac{d^2}{dt^2}y + 3\frac{d}{dt}y + 2y$
- Let $u(t) \equiv 0$, then any $y_h(t) = c_1 e^{-2t} + c_2 e^{-t}$ solves
- When $u(t) = e^{at}$, then ? $\quad y(t) = \alpha e^{at}$
- If homogenous solution goes to zero, then particular solution worth studying!
- Exponents in homogenous solution: 'poles' of the system
- 'Zeros' of the system: those exponents where gain $G(s)$ is zero
- Stability: all poles are in the LHP

$$y(t) = y_p(t) + y_h(t)$$

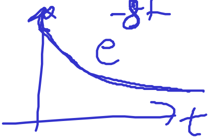
$$u(t) = 3e^{2t} \rightarrow y(t) \equiv 0 \checkmark \quad y_p(t) = 0.$$

$s = 0$

$$F(0) = \int_0^\infty f(t)\,dt$$

? $\int f(t)$

$s = 8$



$$F(8) = \int_0^\infty f(t)\,e^{-8t}\,dt$$

weighing

$F(s)$

$s \in \mathbb{C}$

What is Laplace transform?

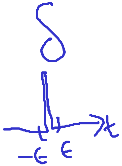
- Some similarities with Fourier transform, but not as much 'duality' between time/frequency domains
- Laplace transform of $f(t)$ gives $F(s)$: this is also called 'frequency' domain $(\sigma + j\omega)$
- $s = j\omega$: many Fourier transform properties get recovered
- $F(s) := \int_0^{+\infty} f(t)e^{-st}dt$
- View (for $s = +5$ (say)) e^{-5t} as decaying-weighting before area under $f(t)$ is calculated.
- If s is kept a (complex) variable, then $F(s)$ has 'full' info about $f(t)$:
  can recover $f(t)$ from $F(s)$.
- Technical detail: integral from 0 to ∞: integral might not exist!
  $e^{(t^2)}$
  $f(t)$ ought not 'grow too quickly': don't worry now. All practical signals have Laplace transforms!

Laplace transform, definition and properties

Function $f, f_1, f_2, g : [0, \infty) \to \mathbb{R}$: piecewise continuous

$\mathcal{L}(f(t))$

$$F(s) = \mathfrak{L}(f)(s), \text{ with } F(s) := \int_{0^-}^{\infty} f(t)e^{-st}dt$$



with real$(s) > \sigma_0$ large-enough, and inverse[1] defined using σ_0

- Linearity: $\mathfrak{L}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_2 F_1(s) + \alpha_2 F_2(s)$ for any real/complex constants α_1 and α_2
- Delayed f: $\mathfrak{L}(\sigma_T(f)) = e^{-sT}F(s)$ (with $T \geqslant 0$ and f-'zeroed'). $(\sigma_T(f)(t) := f(t-T))$.
- Derivative of f: $\mathfrak{L}(\frac{d}{dt}f) = sF(s) - f(0^-)$ and
- Integral of f: $\mathfrak{L}(\int_0^t f(\tau)d\tau) = \frac{F(s)}{s}$

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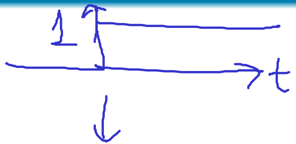
[1]
$$f(t) = \mathfrak{L}^{-1}(F)(t), \text{ with } f(t) := \frac{1}{2\pi j} \lim_{\omega_0 \to \infty} \int_{\sigma_0 - j\omega_0}^{\sigma_0 + j\omega_0} F(s)e^{st}dt$$

- Convolution and product:
  $(f * g)(t) := \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$, $\mathfrak{L}(f * g) = F(s)G(s)$

  $g(t) = \delta$
- Dirac delta: $\delta * f = f$ and $\mathfrak{L}(\delta) = 1$
- IVT: $f(0^+) = \lim_{t \downarrow 0} f(t) = \lim_{s \to \infty} sF(s)$
  (provided LHS exists, i.e. no impulses/their derivatives at $t = 0$.)
- FVT: $f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$
  (provided LHS exists, i.e. f neither diverges, nor oscillates)
- Time multiplication $\mathfrak{L}(tf(t)) = -\frac{d}{ds}F(s)$
- Complex shift: $\mathfrak{L}(e^{at} f(t)) = F(s-a)$
- Time scaling: $\mathfrak{L}(f(\frac{t}{a})) = aF(as)$ (for $a > 0$)

Polynomials/exponentials/sinusoids



unit step
Heaviside distribution

- $\mathfrak{L}(1) = \frac{1}{s}$ (note: functions are only on $[0, \infty)$)

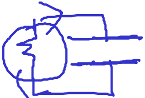
  $t > 0$

- $\mathfrak{L}(t) = \frac{1}{s^2}$
- $\mathfrak{L}(e^{at}) = \dfrac{1}{s-a}$
- $\mathfrak{L}(\sin(\omega t)) = \dfrac{\omega}{s^2+\omega^2}$ and $\mathfrak{L}(\cos(\omega t)) = \dfrac{s}{s^2+\omega^2}$
  (Use IVT to be sure of which is of which.)
- $\mathfrak{L}(e^{at}\sin(\omega t)) = \dfrac{\omega}{(s-a)^2+\omega^2}$

Positive/negative feedback

$u = i$

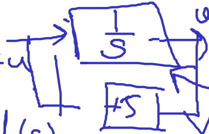
$R > 0$



$v = x$

$C = 1$

$i = u$



$u(t) = \int_0^t y(\tau)\,d\tau$

$s X(s) = U(s)$

$(x(0) = 0)$

$\frac{X(s)}{U(s)} = \frac{1}{s}$

- $\frac{d}{dt}x = u$
- Suppose $u = +5x$ (positive feedback)
- Suppose $u = -5x$ (negative feedback)
- Solve both differential equations (exponentially growing/decaying)

Convolution

f_i

For now: consider functions f_i satisfying $f_i(t) = 0$ for $t < 0$
(Linked to causality and input: zero-in-the-past)
Then $y(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau = \int_0^t f(t-\tau)h(\tau)d\tau$

"impulse response"

$h(t) = 0 \quad t < 0$

"causal"

Laplace transform advantages

- Linear equations in variables u, v and their derivatives can be rewritten in terms of their Laplace transforms $U(s)$ and $V(s)$
- KCL/KVL equations, impedances are simpler (with/without initial conditions)

  $Z(s) \rightarrow \frac{V(s)}{I(s)}$

- Take Laplace transform of 'both' sides in a differential equation

  $\frac{Y(s)}{U(s)} =: G(s)$

- Transfer function: defined as 'ratio' of Laplace transforms (output/input)

  initial conditions

  $\frac{d}{dt}u - 2u = \left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right)y$

- Transfer function: 'gain' at that 'frequency' (complex frequency)

  $\sigma + j\omega$

- Note: laplace transform: taken for *signals*
- Transfer function: ratio of Laplace transforms (of signals)
- Transfer function: also Laplace transform of some signal?

Transfer function: Laplace transform of which signal?

$$G(s) = \frac{Y(s)}{U(s)}$$

$U(s) \doteq 1$

$u(t) = \delta(t)$ unit

δ_c (continuous) delta.

$G(s) = Y(s)$ $y(t)$

The impulse response

Output = Convolution of: input and impulse-response (only for LTI systems)

$$y(t) = u(t) * h(t)$$

$$Y(s) = U(s) \cdot \boxed{G(s)}$$