Fourier Transform of a Gaussian

By a "Gaussian" signal, we mean one of the form e^{-Ct^2} for some constant C. We will show that the Fourier transform of a Guassian is also a Gaussian. Three different proofs are given, for variety. The first uses complex analysis, the second uses integration by parts, and the third uses Taylor series.

Theorem 0.1. *The Fourier transform of $f(t) = e^{-Ct^2}$ is $F(\omega) = e^{-\omega^2/4C} \sqrt{\pi/C}$.*

First a lemma that will be used in the first two proofs. Its proof uses a trick - the desired integral is squared and then converted into a double integral for which polar coordinates work well.

Lemma 0.2. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

Proof.

$$\text{Let } I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$\therefore I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy$$

$$(\text{now let } x = r\cos\theta, \ y = r\sin\theta, \ dx\,dy = r\,dr\,d\theta)$$

$$= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r\, dr\, d\theta$$

$$= 2\pi \int_{0}^{\infty} e^{-r^2} r\, dr$$

$$= -\pi e^{-r^2} \Big|_{0}^{\infty}$$

$$= \pi$$

$$\therefore I = \sqrt{\pi} \qquad \blacksquare$$

The first proof makes use of Lemma 0.3 below, which is based on complex analysis. You would normally learn this material, for example, at UCSD in Math 120A. If you haven't learned this yet, you can skip to the other two proofs. Specifically, Cauchy's theorem says that the contour integral of a complex function around some closed path is 2π times the sum of the residues of the function. If the function is analytic, then there are no residues and the contour integral is zero. We use a path which is a rectangle and then let its width become infinite in both directions.

Lemma 0.3. *If $b > 0$, then $\int_{-\infty}^{\infty} e^{-(x-bj)^2} dx = \sqrt{\pi}$.*

Proof. The function e^{-z^2} is analytic in the complex plane, so its integral around the rectangle shown below is zero by Cauchy's theorem.
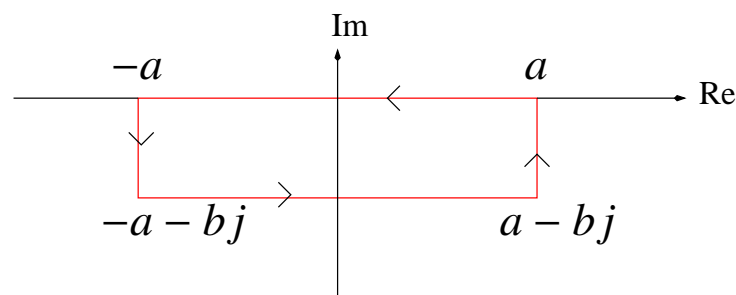
That is,

$$0 = \oint e^{-z^2} dz$$

$$= \int_{-a}^{a} e^{-(x-bj)^2} dx + \int_{-b}^{0} e^{-(a+yj)^2} dy + \int_{a}^{-a} e^{-x^2} dx + \int_{0}^{-b} e^{-(-a+yj)^2} dy$$

$$\therefore \left| \int_{-a}^{a} e^{-(x-bj)^2} dx - \int_{-a}^{a} e^{-x^2} dx \right| = \left| \int_{-b}^{0} e^{-(-a+yj)^2} dy - \int_{-b}^{0} e^{-(a+yj)^2} dy \right|$$

$$\leq \int_{-b}^{0} \left| e^{-(-a+yj)^2} \right| dy + \int_{-b}^{0} \left| e^{-(a+yj)^2} \right| dy$$

$$= 2e^{-a^2} \int_{-b}^{0} e^{y^2} dy$$

$$\leq 2e^{-a^2} b e^{b^2} \quad \to 0 \quad \text{as} \quad a \to \infty$$

$$\therefore \int_{-\infty}^{\infty} e^{-(x-bj)^2} dx = \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$= \sqrt{\pi} \quad \text{by Lemma 0.2} \qquad \blacksquare$$

Proof #1 of Theorem 0.1.

$$F(\omega) = \int_{-\infty}^{\infty} e^{-Ct^2} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-C(t^2 + (j\omega/C))} dt$$

$$= \int_{-\infty}^{\infty} e^{-C(t + (j\omega/2C))^2 - \omega^2/4C} dt$$

$$= e^{-\omega^2/4C} \int_{-\infty}^{\infty} e^{-C(t - (j\omega/2C))^2} dt$$

$$\left(\text{now let } \tau = t\sqrt{C}, \quad d\tau = \sqrt{C} dt \right)$$

$$= \frac{e^{-\omega^2/4C}}{\sqrt{C}} \int_{-\infty}^{\infty} e^{-(\tau - (j\omega/2\sqrt{C}))^2} d\tau$$

$$= e^{-\omega^2/4C} \sqrt{\pi/C} \quad \text{by Lemma 0.3} \qquad \blacksquare$$

The second proof uses integration by parts on the Fourier transform. This results in a first order seperable differential equation, which can easily be solved by integration and then determination of the constant of integration. Such differential equations are standard material in a high school Calculus BC course or at UCSD in Math 20D, and probably also earlier such as in Math 20ABC.

Proof #2 of Theorem 0.1.

$$F(\omega) = \int_{-\infty}^{\infty} e^{-Ct^2} e^{-j\omega t} dt$$

$$F'(\omega) = \int_{-\infty}^{\infty} -jt e^{-Ct^2} e^{-j\omega t} dt$$

Now integrate by parts using:

$$u = -je^{-j\omega t} \qquad du = -\omega e^{-j\omega t} dt$$

$$v = -e^{-Ct^2}/(2C) \quad dv = te^{-Ct^2} dt$$

$$F'(\omega) = \underbrace{(j/2C)e^{-j\omega t} e^{-Ct^2}\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} (\omega/2C)e^{-Ct^2} e^{-j\omega t} dt$$

$$= -(\omega/2C)F(\omega)$$

$$\therefore \frac{F'(\omega)}{F(\omega)} = -\omega/2C.$$

Now integrate both sides with respect to ω to get

$$\ln(F(\omega)) = -\omega^2/4C + K$$

$$\therefore F(\omega) = e^{-\omega^2/4C} e^K.$$

Also, we have

$$F(0) = e^K$$

$$= \int_{-\infty}^{\infty} e^{-Ct^2} dt \qquad \text{(Plug } \omega = 0 \text{ into Fourier Transform definition)}$$

$$= \int_{-\infty}^{\infty} e^{-x^2} (dx/\sqrt{C}) \quad \text{(Substitute } x = t\sqrt{C})$$

$$= \sqrt{\pi/C} \qquad \text{(by Lemma 0.2).} \qquad \blacksquare$$

4

In the thrid proof, we first expand the complex exponential into its real and imaginary parts inside the integal. Then, since $\sin(\omega t)$ is an odd function and e^{-Ct^2} is an even function, their product is odd, so integrating the product on the whole real line gives zero. The Fourier transform then is the real integral of e^{-Ct^2} times $\cos(\omega t)$. We expand the cosine in its Taylor series about the origin and then integrate the product term by term (i.e. switching the order of summation and integration). Then we add up the result.

Proof #3 of Theorem 0.1.

$$\text{Let } g(C) = \int_{-\infty}^{\infty} e^{-Ct^2} dt = \sqrt{\pi/C} \quad \text{(by Lemma 0.2)} fg$$

$$g'(C) = \sqrt{\pi}(-1/2)C^{-3/2}$$

$$g''(C) = \sqrt{\pi}(-1/2)(-3/2)C^{-5/2}$$

$$g'''(C) = \sqrt{\pi}(-1/2)(-3/2)(-5/2)C^{-7/2}$$

$$\vdots$$

$$g^{(k)}(C) = \sqrt{\pi}\frac{(-1)^k(1 \cdot 3 \cdot 5 \ldots (2k-1))}{2^k}C^{-(2k+1)/2}$$

$$= \frac{\sqrt{\pi}(-1)^k C^{-(2k+1)/2}(2k-1)!!}{2^k}$$

$$= (-1)^k \int_{-\infty}^{\infty} t^{2k} e^{-Ct^2} dt$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-Ct^2} e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-Ct^2} \cos(\omega t) dt + j \underbrace{\int_{-\infty}^{\infty} e^{-Ct^2} \sin(\omega t)) dt}_{0}$$

$$= \int_{-\infty}^{\infty} e^{-Ct^2} \cos(\omega t) dt$$

$$= \int_{-\infty}^{\infty} e^{-Ct^2} \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{(-1)^{n/2}(\omega t)^n}{n!} dt$$

$$= \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{(-1)^{n/2}\omega^n}{n!} \int_{-\infty}^{\infty} e^{-Ct^2} t^n dt$$

$$= \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{\omega^n}{n!} \cdot g^{(n/2)}(C)$$

$$= \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{\omega^n}{n!} \cdot \frac{\sqrt{\pi}C^{-(n+1)/2}(-1)^{n/2}(n-1)!!}{2^{n/2}}$$

$$= \sqrt{\pi/C} \cdot \sum_{\substack{n=0 \\ \text{even}}}^{\infty} (-\omega^2/2C)^{n/2} \cdot \frac{(n-1)!!}{n!}$$

$$= \sqrt{\pi/C} \cdot \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{(-\omega^2/2C)^{n/2}}{2^{n/2}(n/2)!}$$

$$= \sqrt{\pi/C} \cdot \sum_{k=0}^{\infty} \frac{(-\omega^2/4C)^k}{k!}$$

$$= \sqrt{\pi/C} \cdot e^{-\omega^2/4C} \qquad \blacksquare$$