Brief solutions to Problems 8 and 9 of Tutorial sheet 5 (dated 11th Oct 2019)

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Problem 8: Check/obtain "steady state output" $y(t)$ for the differential equation $\dot{y}+3y=x$ for:
(a): $x(t)=e^{-t}$, (b): $x(t)=e^{-t}u(t)$, (c): $x(t)=e^{-3t}$, (d): $x(t)=e^{-3t}u(t)$

Brief solution 8: Use 'eigenfunction' argument when obtaining *an* output: $y(t)=cx(t)$ with c to be evaluated by *substituting* $y(t)$ in the differential equation: this method is for the case when $x(t)$ is an eigenfunction, i.e. an exponential function (with any exponent) *for all time* t. (So this argument is not usable for $x(t)=e^{at}u(t)$ due to the presence of the step function $u(t)$.) Relate c with transfer function evaluated at a.

$y(t)$ obtained as above is *a* solution because this is just a so-called particular solution, and the so-called homogenous solutions are yet to be added. This brings us to why the above method is 'steady state output'.

When $x(t)=e^{at}u(t)$, then Laplace transform $X(s)=\frac{1}{s-a}$ and obtaining $Y(s)$ from the above differential equation and performing partial fraction expansion of $Y(s)$ gives

$$Y(s)=\frac{b_1}{s-p_1}+\frac{c}{s-a}$$

as long as exponent a in the input $x(t)=e^{at}u(t)$ is not equal to the system pole p_1: in the above example $p_1=-3$. For this case, call $b_1e^{p_1t}u(t)$ as the 'transients' part and ce^{at} as the 'steady state part' of the output: this is because the transients is caused due to system's inherent poles, while the steady state part is caused due to the forcing function $x(t)=e^{at}u(t)$.

For this case (i.e., when $a\neq -3$), notice that c is same as obtained using eigenfunction argument.

Check that $y(t)=\frac{1}{2}e^{-t}$ is this steady state output for (a) and (b).

On the other hand, for the remaining cases: (c) and (d), due to the exponent a being equal to system pole -3, we need a slightly different argument. Substitute $y(t)=c_1te^{-3t}+c_0e^{-3t}$ and find value of c_1 and c_0. Call c_0e^{-3t} as homogenous solutions (and coming from system poles). In this case, for $x(t)=e^{-3t}u(t)$ gives (using Laplace transform): $y(t)=te^{-3t}u(t)$.

Problem 9: Consider 8-point DFT of vector $x[0]$, $x[1]$, $x[2]$, $x[3]$, $x[4]$, $x[5]$, $x[6]$, $x[7]$, to get its DFT $X[0]$, $X[1]$, $X[2]$, $X[3]$, $X[4]$, $X[5]$, $X[6]$, $X[7]$. Construct the matrix $\mathscr{F}\in\mathbb{C}^{8\times 8}$ such that

$$\begin{bmatrix}X[0]\\X[1]\\X[2]\\X[3]\\X[4]\\X[5]\\X[6]\\X[7]\end{bmatrix}=\mathscr{F}\begin{bmatrix}x[0]\\x[1]\\x[2]\\x[3]\\x[4]\\x[5]\\x[6]\\x[7]\end{bmatrix}.$$

Rearrange columns of $\mathscr{F}$ to get $\tilde{\mathscr{F}}$ which acts on even part and odd part of x: $\tilde{x}=\left[\begin{array}{c}x[0]\\x[2]\\x[4]\\x[6]\\\hline x[1]\\x[3]\\x[5]\\x[7]\end{array}\right]$. Rename the four blocks of $\tilde{\mathscr{F}} = \begin{bmatrix} A_{ee} & A_{eo} \\ A_{oe} & A_{oo} \end{bmatrix}$ appropriately and find a structure in them:

Brief solution 9: For 8-point DFT, using the definition of DFT as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-2\pi kn}{N}}$$

and defining $\omega = e^{\frac{-2\pi}{8}}$, we get $\mathscr{F} \in \mathbb{C}^{8\times 8}$ as follows

$$\mathscr{F} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ 1 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} \begin{array}{c} \text{using } \omega^8 = 1, \\ \text{we get} \end{array} \mathscr{F} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 & 1 & \omega^4 \\ 1 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega & \omega^6 & \omega^3 \\ 1 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{bmatrix}$$

and upon re-arranging the columns to get $\tilde{\mathscr{F}}$ that acts on x with even and odd elements re-arranged (with x and X as shown below), we get

$$\left[\begin{array}{c} X[0] \\ X[1] \\ X[2] \\ X[3] \\ \hline X[4] \\ X[5] \\ X[6] \\ X[7] \end{array}\right] = \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 & \omega^7 \\ 1 & \omega^4 & 1 & \omega^4 & \omega^2 & \omega^6 & \omega^2 & \omega^6 \\ 1 & \omega^6 & \omega^4 & \omega^2 & \omega^3 & \omega & \omega^7 & \omega^5 \\ \hline 1 & 1 & 1 & 1 & \omega^4 & \omega^4 & \omega^4 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^5 & \omega^7 & \omega & \omega^3 \\ 1 & \omega^4 & 1 & \omega^4 & \omega^6 & \omega^2 & \omega^6 & \omega^2 \\ 1 & \omega^6 & \omega^4 & \omega^2 & \omega^7 & \omega^5 & \omega^3 & \omega \end{array}\right] \left[\begin{array}{c} x[0] \\ x[2] \\ x[4] \\ x[6] \\ \hline x[1] \\ x[3] \\ x[5] \\ x[7] \end{array}\right]$$

Using the partitions as shown, define the four square 4×4 blocks of $\tilde{\mathscr{F}}$ as

$$\tilde{\mathscr{F}} = \begin{bmatrix} A_{ee} & A_{eo} \\ A_{oe} & A_{oo} \end{bmatrix}.$$

Verify that $A_{oo} = \omega^4 A_{eo}$ and $A_{ee} = A_{oe}$.

First notice that A_{ee} is nothing but the DFT matrix corresponding to 4-point DFT. Second, $A_{eo} = DA_{ee}$ with the diagonal matrix D having diagonal elements: $[1, \omega, \omega^2, \omega^3]$.

Conclusion: obtain 4-point DFT of $[x[0], x[2], x[4], x[6]]$ and $[x[1], x[3], x[5], x[7]]$ separately and add them after appropriate 'component-by-component' scaling of the DFT of $[x[1], x[3], x[5], x[7]]$.

Next, to compute the 4-point DFT of a vector, again separate its even components and odd-components, etc and find 2-point DFT. This recursive method is called FFT (Fast Fourier Transform): we will see later what is *fast* about it.

Problem 10: (new problem) Consider the 8×8 matrix $\mathscr{F}$ defined above. Scale the matrix and consider $\frac{1}{\sqrt{8}}\mathscr{F}$. Check that this scaled matrix is unitary. (A square matrix $Q \in \mathbb{C}^{n\times n}$ is called unitary if $U^{-1} = U^*$, with U^* denoting the complex conjugate transpose.) Check this for any DFT matrix (for N-point DFT) scaled by $\frac{1}{\sqrt{N}}$.

Brief solution 10: Denote columns i and j of the original (i.e. unscaled) F as v_i and v_j. Use $\omega^8 = 1$ to check that inner product $v_i^* v_j = 0$ if $i \neq j$, and . $v_i^* v_i = N$. (Here $N = 8$.) Next use the fact that for a square matrix, if any pair of columns are mutually perpendicular (i.e. inner product is zero) and each column length equals 1, then that matrix is unitary.