① $f(t) = e^{-1/t}$; $t > 0$

$= 0$; $t < 0$

To show that —

$$\lim_{t \to 0+} \frac{d^n f}{dt^n} = 0 \quad , \forall n$$

We know,

$$\lim_{x \to \infty} \frac{p(x)}{e^{ax}} = 0$$ where $p(x)$ is a polynomial in x of finite degree.

$$\frac{d}{dt} e^{-1/t} = \frac{1}{t^2} e^{-1/t}$$

$$\Rightarrow \frac{d^2}{dt^2} e^{-1/t} = -\frac{2}{t^3} e^{-1/t} + \frac{1}{t^4} e^{-1/t}$$

$$= \left(-\frac{2}{t^3} + \frac{1}{t^4}\right) e^{-1/t}$$

$\vdots$

$$\frac{d^n}{dt^n} e^{-1/t} = p(1/t)\, e^{-1/t}$$ where $p(1/t)$ is a polynomial in $(1/t)$ of finite deg.

$$\frac{d^n}{dt^n} e^{-1/t} = \frac{p(1/t)}{e^{1/t}}$$

$$\lim_{1/t \to \infty} \frac{d^n e^{-1/t}}{dt^n} = \lim_{t \to 0} \frac{d^n e^{-1/t}}{dt^n} = \lim_{1/t \to \infty} \frac{p(1/t)}{e^{1/t}} = 0.$$

2.

1.) $f(t) = f_1(t) + f_2(t)$, $f_1(t) \in C^\infty$

$f_2(t) \in C^\infty$

$\therefore f(t) \in C^\infty$

2) $f(t) = a f_1(t)$, $f_1(t) \in C^\infty$

$\therefore a f(t) \in C^\infty$

5a) If you differentiate a C^∞ function n no. of times it will again be C^∞.

Hence, any polynomial operation of the form

$$p\left(\frac{d}{dt}\right) = a_0 + a_1 \frac{d}{dt} + \cdots + a_n \frac{d}{dt}$$

when acts on C^∞ function say f. then it will also be C^∞.

i.e. $y = p\left(\frac{d}{dt}\right)(f)$, where $f \in C^\infty$

$$= a_0 f + a_1 \frac{d}{dt} f + \cdots + a_n \frac{d^n}{dt^n} f$$

$\therefore y \in C^\infty$.

3, 4) For convolution and multiplicational convolution, as it is closed under addition and multiplication, it will also be closed under convolution and convolutional multiplication.

~~3) y(t) = \int x(\tau) y(t~~

(3) $y(t) = x_1(t) * x_2(t)$

$$y(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau.$$

(a) Let, $y(t) = x_1(t) * \left(x_2(t) + x_3(t)\right)$

$$= \int_{-\infty}^{+\infty} x_1(\tau) \left(x_2(t-\tau) + x_3(t-\tau) \right) d\tau.$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau) + \int_{-\infty}^{+\infty} x_1(\tau)\, x_3(t-\tau)\, d\tau$$

$$\Rightarrow y(t) = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

$\Rightarrow$ Convolution distributes over addition

(b) Let $y(t) = x_1(t) * \left(x_2(t) * x_3(t)\right)$

To show that, $y(t) = \left(x_1(t) * x_2(t)\right) * x_3(t)$

$$y(t) = x_1(t) * \left(x_2(t) * x_3(t)\right)$$

$$= x_1(t) * \underbrace{\left[\int_{-\infty}^{+\infty} x_2(\tau)\, x_3(t-\tau)\, d\tau. \right]}_{z(t)\ (\text{say})}$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\, z(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} \left[x_1(\tau) \int_{-\infty}^{+\infty} x_2(s)\, x_3(t-\tau-s)\, ds \right] d\tau$$

$$y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)\, x_2(s)\, x_3(t-\tau-s)\, ds\, d\tau.$$

A change of variable is performed by letting $s = z - \tau$,

$ds = dz$. Hence, $z \to \infty$ as $s \to \infty$ and $z \to -\infty$, as $s \to -\infty$

$$y(t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} x_1(\tau)\, x_2(z-\tau)\, x_3(t-z)\, dz\, d\tau$$

Changing the order of integration,

$$y(t) = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x_1(\tau)\, x_2(z-\tau)\, d\tau\right] x_3(t-z)\, dz$$

$$= (x_1 * x_2) * x_3.$$

(c) Let, $y(t) = x_1(t) * x_2(t)$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \quad \text{by definition}$$

performing a change of variables by letting $(t-\tau) = \alpha$,

then, $\tau = t - \alpha$, $d\tau = -d\alpha$ as $\tau \to \infty$, $\alpha \to -\infty$

and as $\tau \to -\infty$, $\alpha \to \infty$

$$\therefore x_1(t) * x_2(t) = -\int_{+\infty}^{-\infty} x_1(t-\alpha)\, x_2(\alpha)\, d\alpha = \int_{-\infty}^{+\infty} x_2(\alpha)\, x_1(t-\alpha)\, d\alpha$$

$$= x_2(t) * x_1(t)$$

(4) Given

$$f(t) \xrightarrow{F} F(j\omega)$$

$$f(t) = F^{-1}\left(F(j\omega)\right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

Differentiating both sides —

$$\frac{df}{dt} = \frac{1}{2\pi} (j\omega) \int_{-\infty}^{+\infty} F(j\omega)\, e^{j\omega t}\, d\omega$$

$$= (j\omega)\, F^{-1}\left(F(j\omega)\right)$$

$$\therefore \quad \frac{df}{dt} \xrightarrow{F} (j\omega)\, F(j\omega)$$

5. 1st part

$h' * x = h * x'$ where '$'$' is $\frac{d}{dt}$ operator

$$F(h' * x) = j\omega\, H(j\omega) \cdot X(j\omega)$$

$$= H(j\omega) \cdot (j\omega\, X(j\omega))$$

$$\therefore (h' * x) = F^{-1}\left(H(j\omega) \cdot (j\omega\, X(j\omega))\right)$$

$$= h(t) * x'(t).$$