properties of (2.4). Hence any rational, positive real function is acceptable as either an impedance or an admittance.

PROBLEMS

2.1. Prove that a pole at $s = 0$ of a driving-point impedance must be simple and have a positive residue.

2.2. Which of the following functions are realizable as impedances or admittances?

$$(a)\ \frac{s^2 + s + 3}{s^2 + s + 1}. \qquad (b)\ \frac{s^2 + s + 5}{s^2 + s + 1}. \qquad (c)\ \frac{s^3 + 3s^2 + 6s + 5}{(s + 1)^3}.$$

2.3. What are the conditions on the constants a, b, α, β, γ for the following functions to be realizable as impedances or admittances?

$$(a)\ \frac{as + b}{\alpha s^2 + \beta s + \gamma}. \qquad (b)\ \frac{bs^2 + as}{\gamma s^2 + \beta s + \alpha}.$$

Can you find any way of providing the conditions for (*b*) from the results of (*a*)?

2.4. Prove that the impedance

$$Z(s) = H \frac{s^2 + \alpha s + \beta}{s^2 + as + b}$$

is realizable if H, α, β, a, b are positive and

$$\sqrt{\alpha a} \geqq |\sqrt{\beta} - \sqrt{b}|$$

What property does the impedance possess if the last expression is an equality?

CHAPTER 3

FOSTER'S REACTANCE SYNTHESIS

As a prelude to the synthesis of general impedances, it is worthwhile first to consider the synthesis of a circuit containing only two types of elements. In this chapter, the problem of synthesizing an impedance containing only inductances and capacitances, i.e., a pure reactance, will be considered. The next chapter will treat the case where only resistances and inductances, or resistances and capacitances, are present.

The first synthesis of a reactance is due to Foster.[1] Basically, his procedure was to observe that a reactance, $Z(j\omega) = jX(\omega)$, can be written in the form

$$X(\omega) = H\omega \frac{(\omega_1{}^2 - \omega^2)(\omega_3{}^2 - \omega^2) \cdots}{(\omega_0{}^2 - \omega^2)(\omega_2{}^2 - \omega^2) \cdots} \tag{3.1}$$

where ω_{2n+1} are the zeros of $X(\omega)$, ω_{2n} the poles, and H is a positive constant. He then proceeds to prove that

$$\frac{d}{d\omega} X(\omega) \geqq 0 \tag{3.2}$$

This fact [Eq. (3.2)] in connection with (3.1) implies the property of "separation of poles and zeros." In other words, the poles and zeros of (3.1) can be ordered in such a fashion that

$$0 \leqq \omega_0 < \omega_1 < \omega_2 < \omega_3 \cdots \tag{3.3}$$

This is Foster's theorem, and his synthesis follows directly from it. Hence (3.1) and (3.2) constitute necessary and sufficient conditions for a reactance to be realizable.

Instead of taking Foster's approach, the synthesis of reactances will be treated here as a direct consequence of the realizability conditions presented in the previous chapter. To begin, a reactance is defined as an impedance whose resistance is zero; i.e.,

$$Z(j\omega) = jX(\omega) \qquad X(\omega) \text{ real} \tag{3.4}$$

The first realizability condition (2.4*a*) states that an impedance is a

[1] R. M. Foster, A Reactance Theorem, *Bell System Tech. J.*, **3**:259 (1924).

ratio of two polynomials, and real for s real. This fact, taken together with (3.4), implies that, for a reactance, $Z(s)$ must be of the form

$$Z(s) = Hs\,\frac{(s^2 + \omega_1^2)(s^2 + \omega_3^2) \cdots}{(s^2 + \omega_0^2)(s^2 + \omega_2^2) \cdots} \tag{3.5}$$

The constants ω_0, ω_1, ω_2 must be real since, if they were not, $Z(s)$ would have poles and zeros in the right half of the s plane. A partial fraction expansion of $Z(s)$ can now be performed, i.e.,

$$Z(s) = \sum_{k=0}^{N} \left(\frac{A_{2k}}{s - j\omega_{2k}} + \frac{A_{2k}^*}{s + j\omega_{2k}} \right) + A_\infty s \tag{3.6}$$

From the realizability condition (2.4c), all the poles must be simple and the residues A_{2k} positive. Hence the partial fraction expansion (3.6) can be written in the form

$$Z(s) = \sum_{k=0}^{N} \frac{2A_{2k}s}{s^2 + \omega_{2k}^2} + A_\infty s \tag{3.7}$$

The synthesis of the reactance $Z(s)$ is now apparent as a series connection of antiresonant circuits and is given in Fig. 3.1. A pole at $s = 0$ is

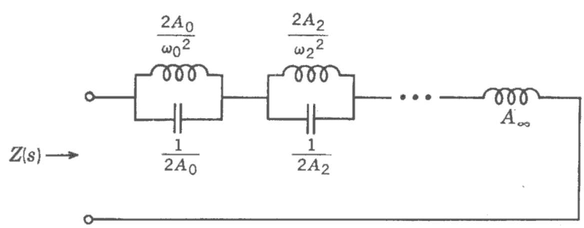

FIG. 3.1. Foster I synthesis of reactances.

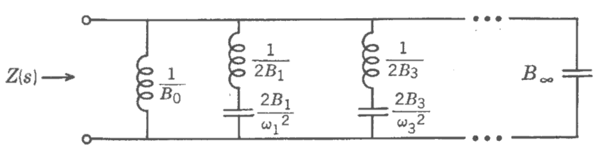

FIG. 3.2. Foster II synthesis of reactances.

included by taking $\omega_0 = 0$, in which case the inductance is absent in the first parallel LC circuit. A circuit of this form (Fig. 3.1) is known as the *Foster I type.*

Foster II synthesis is identical to Foster I synthesis except that the admittance instead of the impedance is dealt with. Thus,

$$Y(s) = \frac{1}{Z(s)} = \frac{(s^2 + \omega_0^2)(s^2 + \omega_2^2) \cdots}{Hs(s^2 + \omega_1^2)(s^2 + \omega_3^2) \cdots}$$

$$Y(s) = \frac{B_0}{s} + \sum_{0}^{M} \left(\frac{B_{2k+1}}{s - j\omega_{2k+1}} + \frac{B^*_{2k+1}}{s + j\omega_{2k+1}} \right) + B_\infty s$$

$$= \frac{B_0}{s} + \sum_{0}^{M} \frac{2sB_{2k+1}}{s^2 + \omega^2_{2k+1}} + B_\infty s \tag{3.8}$$

where, again, if the admittance is to be realizable, the B_{2k+1} must be real and positive. The circuit follows directly as a parallel connection of resonant circuits from (3.8) and is given in Fig. 3.2. The leading shunt inductance is not present if $\omega_0 = 0$.

Setting $s = j\omega$ in (3.7), with $Z(j\omega) = jX(\omega)$, it is seen that

$$X(\omega) = \sum_{k=0}^{N} \frac{2A_{2k}\omega}{\omega_{2k}{}^2 - \omega^2} + A_\infty \omega \tag{3.9}$$

Differentiating (3.9),

$$\frac{d}{d\omega}[X(\omega)] = \sum_{k=0}^{N} \frac{2A_{2k}(\omega^2 + \omega_{2k}{}^2)}{(\omega_{2k}{}^2 - \omega^2)^2} + A_\infty \geqq 0 \tag{3.10}$$

Thus, Foster's original condition (3.2) is verified as being necessary. Dividing (3.9) by ω and adding or subtracting it from (3.10), the somewhat stronger inequality

$$\frac{d}{d\omega}[X(\omega)] \geqq \left| \frac{X(\omega)}{\omega} \right| \tag{3.11}$$

is obtained.

It is, perhaps, worthwhile to illustrate the Foster synthesis with a numerical example. Consider the impedance

$$Z(s) = \frac{8(s^2 + 1)(s^2 + 3)}{s(s^2 + 2)(s^2 + 4)} \tag{3.12}$$

Since this function has its poles and zeros located on the imaginary axis and possesses the "separation property," it is realizable as a reactance. The Foster I expansion is of the form

$$Z(s) = \frac{A_0}{s} + \frac{A_2 s}{s^2 + 2} + \frac{A_4 s}{s^2 + 4} \tag{3.13}$$

and the problem is to evaluate A_0, A_2, and A_4. This can be done most easily by dividing (3.13) by s and setting $x = s^2$. Then from (3.12) and (3.13),

$$F(x) = \frac{Z(\sqrt{x})}{\sqrt{x}} = \frac{8(x + 1)(x + 3)}{x(x + 2)(x + 4)} = \frac{A_0}{x} + \frac{A_2}{x + 2} + \frac{A_4}{x + 4} \tag{3.14}$$

Equation (3.14) is now a conventional partial fraction expansion. Hence,

$$A_0 = \lim_{x \to 0} [xF(x)] = \frac{8 \times 1 \times 3}{2 \times 4} = 3$$

$$A_2 = \lim_{x \to -2} [(x+2)F(x)] = \frac{8(-2+1)(-2+3)}{(-2)(-2+4)} = 2$$

$$A_4 = \lim_{x \to -4} [(x+4)F(x)] = \frac{8(-4+1)(-4+3)}{(-4)(-4+2)} = 3$$

and so $Z(s)$ [Eq. (3.12)] can be written as

$$Z(s) = \frac{3}{s} + \frac{2s}{s^2+2} + \frac{3s}{s^2+4}$$

with the corresponding circuit given by Fig. 3.3. The Foster II synthesis

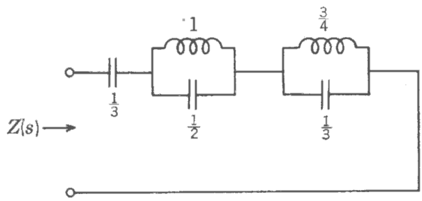

FIG. 3.3. Foster I synthesis of Eq. (3.12).

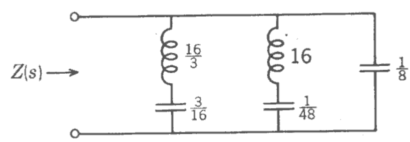

FIG. 3.4. Foster II synthesis of Eq. (3.12).

is equally direct. Thus,

$$Y(s) = \frac{1}{Z(s)} = \frac{s(s^2+2)(s^2+4)}{8(s^2+1)(s^2+3)}$$

$$= \frac{B_1 s}{s^2+1} + \frac{B_3 s}{s^2+3} + B_\infty s$$

Again, dividing by s and setting $s^2 = x$,

$$G(x) = \frac{(x+2)(x+4)}{8(x+1)(x+3)} = \frac{B_1}{x+1} + \frac{B_3}{x+3} + B_\infty$$

and hence

$$B_1 = \lim_{x \to -1} [(x+1)G(x)] = 3/16$$

$$B_3 = \lim_{x \to -3} [(x+3)G(x)] = 1/16$$

$$B_\infty = \lim_{x \to \infty} [G(x)] = 1/8$$

The resulting circuit is given in Fig. 3.4.

It is to be noted that the Foster synthesis procedure yields canonic circuits, i.e., circuits synthesized with the least number of elements. Thus in the numerical example above there are five parameters: two poles, two zeros, and the constant multiplier. The circuits of Figs. 3.3 and 3.4 each contain five elements.

It is instructive to view these Foster synthesis procedures as a process of pole removals. As an illustration, consider the numerical example dealt with previously. This impedance [Eq. (3.12)] can be written in the form

$$Z(s) = \frac{8(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)} = \frac{2s}{s^2+2} + Z_1(s) \qquad (3.15a)$$

where

$$Z_1(s) = Z(s) - \frac{2s}{s^2+2} = \frac{6(s^2+2)}{s(s^2+4)} \qquad (3.15b)$$

It is apparent from (3.15) that the poles at $\pm j\sqrt{2}$ have been removed from $Z(s)$. The circuit corresponding to (3.15a) is that of Fig. 3.5.

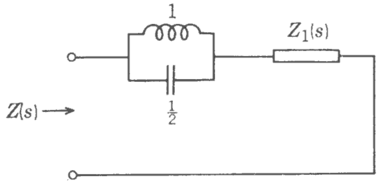

FIG. 3.5. Single-pole removal from $Z(s)$ [Eq. (3.15a)].

Obviously, if the poles at 0, $\pm j2$ were removed from $Z_1(s)$, the resulting circuit would be identical with the Foster I synthesis discussed previously. However, $Z_1(s)$ can also be developed on an admittance basis, i.e., a Foster II expansion. Thus,

$$Y_1(s) = \frac{1}{Z_1(s)} = \frac{s(s^2+4)}{6(s^2+2)} = \frac{s}{6} + \frac{s/3}{s^2+2}$$

with the resulting circuit that of Fig. 3.6, which is different from those obtained previously (Figs. 3.3 and 3.4). It is now apparent that a large

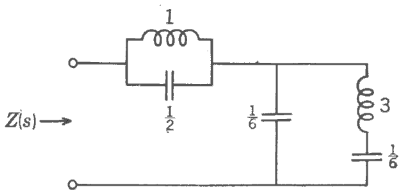

FIG. 3.6. Generalized Foster synthesis of $Z(s)$ [Eq. (3.12)].

number of distinct circuits, yielding the same reactance, can be obtained by removing poles from the impedance, then poles from the remaining admittance, then poles from the remaining impedance, etc. In Fig. 3.7, there are indicated nine more circuits which synthesize the reactance of (3.12). The notation below each circuit indicates the location of the pole and whether it was removed from an admittance or impedance. On the basis of this notation, the circuit of Fig. 3.3 would be labeled

0I, 2I, 4I; that of Fig. 3.4, 1A, 3A, ∞ A; and of Fig. 3.6, 2I, ∞ A, 2A. It is suggested that the reader derive some of the circuits of Fig. 3.7 for himself as exercises.

It is apparent that all 12 of these Foster-type circuits (Figs. 3.3, 3.4, and 3.7) are canonic. Lest the impression be given that such an array

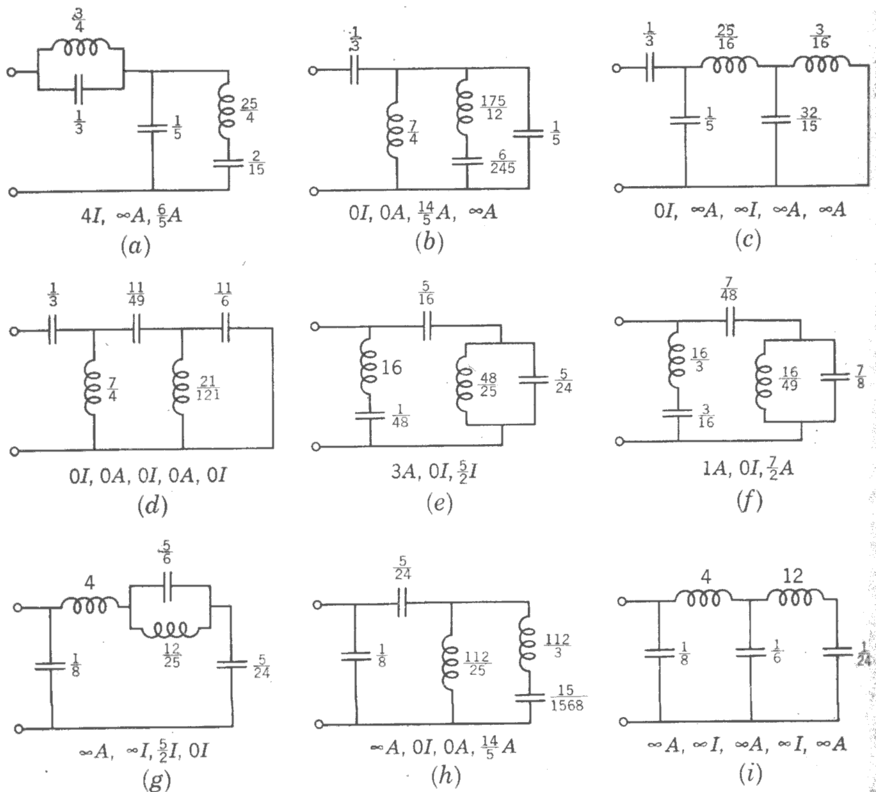

FIG. 3.7. Generalized Foster circuits for the reactance $Z(s) = \dfrac{8(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}$.

of networks is exhaustive, it must be pointed out that they are all of the form of generalized ladders, i.e., of the form of Fig. 3.8. If other types of configurations are used, many more canonic circuits are usually possible. Thus, the impedance of (3.12) might be synthesized in one (or more) ways with networks of the form of Fig. 3.9. However, finding values of the elements in such an array is likely to be very difficult. For many network topologies, this is impossible without resorting to numerical techniques. It is an unsolved problem as to how many different canonic networks there are which synthesize a given reactance.

Consider now the networks i and d of Fig. 3.7 which have particularly simple ladder structures. They are termed *Cauer I and II*, respectively, after their originator.[1] Since the driving-point impedance of a ladder structure such as Fig. 3.8 is given by the continued fraction

$$Z = Z_1 + \cfrac{1}{\cfrac{1}{Z_2} + \cfrac{1}{Z_3 + \cfrac{1}{\cfrac{1}{Z_4} + \cfrac{1}{Z_5 + \cdots}}}}$$

a standard synthetic-division procedure can be used to determine the coefficients in this continued fraction. Hence the element values of

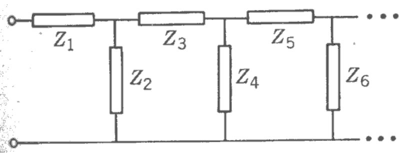

Fig. 3.8. Ladder network.

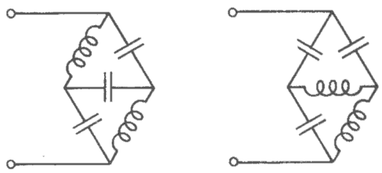

Fig. 3.9. Other possible network topologies for the reactance [Eq. (3.12)].

the networks (Figs. 3.7*i* and 3.7*d*) can be obtained in this manner. Thus, performing a synthetic division on (3.12) yields

$$\begin{array}{l}
8s^4 + 32s^2 + 24 \;\overline{\big)\, s^5 + 6s^3 + 8s}\quad s/8 \\
\qquad\qquad\qquad\; \underline{s^5 + 4s^3 + 3s} \\
\qquad\qquad\qquad\qquad 2s^3 + 5s \;\overline{\big)\, 8s^4 + 32s^2 + 24}\quad 4s \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{8s^4 + 20s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad 12s^2 + 24 \;\overline{\big)\, 2s^3 + 5s}\quad s/6 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{2s^3 + 4s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \;\overline{\big)\, 12s^2 + 24}\quad 12s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{12s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 24 \;\overline{\big)\, s}\quad s/24 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0
\end{array}$$

and the continued-fraction expansion is

$$Z(s) = \cfrac{1}{\cfrac{s}{8} + \cfrac{1}{4s + \cfrac{1}{\cfrac{s}{6} + \cfrac{1}{12s + \cfrac{1}{s/24}}}}}$$

[1] W. Cauer, *Arch. Elektrotech.*, **17**:355 (1927).

from which the element values of the circuit of Fig. 3.7i are obtained directly. Similarly, doing this synthetic division in terms of $1/s$ yields

$$
\begin{array}{l}
\frac{8}{s^4}+\frac{6}{s^2}+1 \;\Big)\; \frac{24}{s^5}+\frac{32}{s^3}+\frac{8}{s} \;\Big(\; 3/s \\
\qquad\qquad\qquad\;\; \frac{24}{s^5}+\frac{18}{s^3}+\frac{3}{s} \\
\qquad\qquad\qquad\;\; \overline{\qquad\;\; \frac{14}{s^3}+\frac{5}{s}} \;\Big)\; \frac{8}{s^4}+\frac{6}{s^2}+1 \;\Big(\; 4/7s \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{8}{s^4}+\frac{20}{7s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad \frac{22}{7s^2}+1} \;\Big)\; \frac{14}{s^3}+\frac{5}{s} \;\Big(\; 49/11s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{14}{s^3}+\frac{49}{11s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\qquad\quad \frac{6}{11s}} \;\Big)\; \frac{22}{7s^2}+1 \;\Big(\; 121/21s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \frac{22}{7s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{\qquad\quad 1} \;\Big)\; \frac{6}{11s} \;\Big(\; 6/11s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{6}{11s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \overline{\;\;0\;\;}
\end{array}
$$

and the continued-fraction expansion is

$$
Z(s) = \frac{3}{s} + \cfrac{1}{\cfrac{4}{7s} + \cfrac{1}{\cfrac{49}{11s} + \cfrac{1}{\cfrac{121}{21s} + \cfrac{1}{\cfrac{6}{11s}}}}}
$$

and hence the circuit of Fig. 3.7d.

Now, the question obviously arises, if one were actually building a reactance, which one of the many Foster-type circuits should be chosen? Considering the point of view of economy, as well as the fact that inductances are more lossy than capacitors, the circuit containing the least total inductance is probably the best choice. On this basis for the impedance of (3.12), the Foster I form (Fig. 3.3) and the network of Fig. 3.7c are the best, with the Cauer II form (Fig. 3.7d) running a close third. Since the Cauer forms are easiest to derive, and usually one of them has nearly the minimum inductance possible, they are normally chosen as the network to synthesize a given reactance.

In conclusion, it is desirable to point out that the Cauer forms serve another very useful purpose. In later chapters, it will be necessary to decide whether a given function is realizable as a reactance. The Cauer development provides a simple and direct way of doing this. Thus, for example, suppose it were desired to find out whether the function

$$Z(s) = \frac{s^9 + 4s^7 + 3s^5 + 2s^3 + 2s}{s^8 + s^6 + s^4 + s^2 + 1}$$

is realizable as a reactance. The process of locating its poles and zeros is a tedious job. However, if it is a reactance, it has a Cauer development. Thus, one can perform the synthetic division for its continued-fraction expansion; i.e.,

$$\begin{array}{r|l} & s \\ s^8 + s^6 + s^4 + s^2 + 1 & s^9 + 4s^7 + 3s^5 + 2s^3 + 2s \\ & s^9 + s^7 + s^5 + s^3 + s \\ \hline & 3s^7 + 2s^5 + s^3 + s \end{array}$$

$$\begin{array}{r|l} & s/3 \\ 3s^7 + 2s^5 + s^3 + s & s^8 + s^6 + s^4 + s^2 + 1 \\ & s^8 + \frac{2}{3}s^6 + \frac{s^4}{3} + \frac{s^2}{3} \\ \hline & \frac{s^6}{3} + \frac{2s^4}{3} + \frac{2s^2}{3} + 1 \end{array}$$

$$\begin{array}{r|l} & 9s \\ \frac{s^6}{3} + \frac{2s^4}{3} + \frac{2s^2}{3} + 1 & 3s^7 + 2s^5 + s^3 + s \\ & 3s^7 + 6s^5 + 6s^3 + 9s \\ \hline & -4s^5 - 5s^3 - 8s \end{array}$$

Since negative numbers occur, the process can at once be terminated as this function obviously cannot be a reactance.

PROBLEMS

3.1. Prove that the conditions (3.1) and (3.2) are sufficient for a function to be realizable as a reactance.

3.2. For a reactance synthesized by a canonic network, prove that the number of inductances and capacitances cannot differ by more than 1. On the basis of the dependence of the reactance at $s = 0$ and $s = \infty$, give conditions for deciding whether the number of inductances exceeds, is equal to, or is less than the number of capacitances.

3.3. Synthesize the following reactances in at least four ways:

(*a*) $Z(s) = \dfrac{s(s^2 + 2)}{(s^2 + 1)(s^2 + 3)}$. (*b*) $Z(s) = \dfrac{(s^2 + 1)(s^2 + 100)}{s(s^2 + 99)}$.

3.4. Are either of the following functions realizable as reactances?

(*a*) $\dfrac{2s^5 + 5s^3 + 5s}{2s^6 + 19s^4 + 4s^2 + 3}$. (*b*) $\dfrac{s^7 + 18s^5 + 83s^3 + 66s}{2s^6 + 27s^4 + 85s^2 + 36}$.

CHAPTER 7

HURWITZ POLYNOMIALS

In the succeeding chapter on Darlington's synthesis procedure, and in later chapters dealing with the synthesis of transfer impedances, some results will be needed concerning a class of polynomials known as *Hurwitz polynomials.* It is the purpose of this chapter to present proofs of these properties and some of their applications to impedances.

A Hurwitz polynomial $Q(s)$ is defined as a polynomial that possesses the following properties:

1. $Q(s)$ is real for s real.
2. $Q(s)$ has its zeros located in the left half plane or on the imaginary axis.

It is apparent from the definition above that an impedance is the ratio of two Hurwitz polynomials. An alternate definition, and a very useful property of Hurwitz polynomials, is expressed by the following theorem:

A necessary and sufficient condition for a polynomial to be Hurwitz is that the function defined by the ratio of its odd to even parts is realizable as a pure reactance.

Thus, if the Hurwitz polynomial $Q(s)$ is explicitly

$$Q(s) = \sum_{0}^{N} q_n s^n = q_e(s) + q_o(s)$$

its even and odd parts are

$$q_e(s) = \Sigma q_{2n} s^{2n} \qquad q_o(s) = \Sigma q_{2n+1} s^{2n+1}$$

The above theorem then states that the function $q_o(s)/q_e(s)$ is realizable as a reactance if, and only if, the polynomial $Q(s)$ is Hurwitz.

A proof of this property can be constructed as follows: Consider a function defined by

$$Y(s) = \frac{q_e(s)}{Q(s)} = \frac{q_e(s)}{q_e(s) + q_o(s)} \tag{7.1}$$

If $Q(s)$ is Hurwitz, then the function $Y(s)$ is positive real, and hence a realizable admittance, since

1. It is real for s real.

2. $\text{Re}\,[Y(j\omega)] = \dfrac{[q_e(j\omega)]^2}{[q_e(j\omega)]^2 - [q_o(j\omega)]^2} \geqq 0.$

3. Since $Q(s)$ is Hurwitz, its poles are confined to the left half plane. There are no poles on the imaginary axis as any imaginary zeros of $Q(s)$ are also zeros of $q_e(s)$.

Since $Y(s)$ is a positive real function, $Z(s) = 1/Y(s)$ can be synthesized as an impedance. Note that, for (7.1),

$$\text{Re}\,[Z(j\omega)] = \text{Re}\,\frac{q_e(j\omega) + q_o(j\omega)}{q_e(j\omega)} = 1 \tag{7.2}$$

Hence the impedance $Z_1(s)$, defined by

$$Z_1(s) = Z(s) - 1 = \frac{1}{Y(s)} - 1 = \frac{q_o(s)}{q_e(s)} \tag{7.3}$$

is realizable since it merely represents the removal of the minimum value of the resistance from $Z(s)$, in a fashion identical to the start of the Brune procedure. Thus, the function $Z_1(s)$ in (7.3) has been shown to be realizable; moreover, it is a pure reactance (its resistance is zero). Combining this result [Eq. (7.3)] with (7.1) demonstrates the fact that if $Q(s)$ is Hurwitz, $q_o(s)/q_e(s)$ is a realizable reactance. A proof of sufficiency can be obtained by reversing the procedure.

One consequence of the above proof is that the even and odd parts of Hurwitz polynomials have their zeros located on the imaginary axis. A direct demonstration of this without using the idea of positive real functions can be quite difficult. It also should be pointed out that the fact that $q_o(s)/q_e(s)$ is a realizable reactance if, and only if, $Q(s)$ is Hurwitz means that a simple test as to whether a given polynomial is Hurwitz can be made by carrying out the synthetic-division procedure discussed in connection with the Cauer reactance forms (Chap. 3). As discussed in connection with the Foster preamble, this synthetic division will also locate any imaginary zeros $Q(s)$ might have.[1]

Next, consider a realizable impedance

$$Z(s) = \frac{P(s)}{Q(s)} = \frac{p_e(s) + p_o(s)}{q_e(s) + q_o(s)} \tag{7.4a}$$

where the subscripts e and o denote the even and odd parts of the polynomials. It will now be shown that all four functions

$$\frac{p_o(s)}{p_e(s)} \qquad \frac{q_o(s)}{q_e(s)} \qquad \frac{p_o(s)}{q_e(s)} \qquad \frac{q_o(s)}{p_e(s)} \tag{7.4b}$$

[1] This process of determining whether a polynomial is Hurwitz, i.e., whether it has all its zeros located in the left half plane, is equivalent to the Routh test for stability of mechanical systems. (E. J. Routh, "Dynamics of a System of Rigid Bodies," 3d ed., Macmillan, London, 1877.)

are themselves realizable reactances. That the first two are realizable reactances are direct consequences of the fact that $P(s)$ and $Q(s)$ in (7.4a) must be Hurwitz polynomials as $Z(s)$ is a positive real function.

To prove that the function $p_o(s)/q_e(s)$ is a realizable reactance, it is necessary to show that its poles lie on the imaginary axis and have positive residues. That they lie on the imaginary axis is a consequence of the fact that $Q(s)$ is Hurwitz, and hence all the zeros of $q_e(s)$ are imaginary. To prove that the residues are positive, the original impedance of (7.4a) can be written in the form

$$Z(s) = \frac{p_e(s)/q_e(s) + p_o(s)/q_e(s)}{1 + q_o(s)/q_e(s)} \tag{7.5}$$

Let $j\omega_\nu$ be one of the poles of $p_o(s)/q_e(s)$. Evaluating (7.5) at $j\omega_\nu$ yields

$$\begin{aligned} Z(j\omega) &= \lim_{s \to j\omega_\nu} \frac{\dfrac{d}{ds}\left[\dfrac{p_e(s)}{q_e(s)} + \dfrac{p_o(s)}{q_e(s)}\right]}{\dfrac{d}{ds}\left[1 + \dfrac{q_o(s)}{q_e(s)}\right]} \\ &= \frac{p_e(j\omega_\nu)/q_e'(j\omega_\nu) + p_o(j\omega_\nu)/q_e'(j\omega_\nu)}{q_o(j\omega_\nu)/q_e'(j\omega_\nu)} \end{aligned} \tag{7.6}$$

where the primes denote differentiation with respect to s. It is to be noted in (7.6) that $p_e(j\omega_\nu)/q_e'(j\omega_\nu)$ is purely imaginary, while $p_o(j\omega_\nu)/q_e'(j\omega_\nu)$ and $q_o(j\omega_\nu)/q_e'(j\omega_\nu)$ are real. Taking the real part of (7.6) yields

$$\text{Re}\,[Z(j\omega_\nu)] = \frac{p_o(j\omega_\nu)/q_e'(j\omega_\nu)}{q_o(j\omega_\nu)/q_e'(j\omega_\nu)} \tag{7.7}$$

which must be positive since $Z(s)$ is realizable. It is to be noted that $q_o(j\omega_\nu)/q_e'(j\omega_\nu)$ is itself positive, since it is the residue of the pole at $j\omega$ of the realizable reactance $q_o(s)/q_e(s)$. Hence, from (7.6),

$$\frac{p_o(j\omega)}{q_e'(j\omega)} \geq 0$$

which states that the residue of $p_o(s)/q_e(s)$ at the pole $j\omega$ is positive. Hence $p_o(s)/q_e(s)$ is a realizable reactance. A similar proof can be carried out to show that $q_o(s)/p_e(s)$ is a realizable reactance.

In conclusion, it is worthwhile to consider how one might determine whether or not a given function $Z(s)$ is a realizable impedance. There are at least three possible methods, namely:

1. Determine whether or not $Z(s)$, or $1/Z(s)$, is a positive real function.
2. Determine whether or not $Z(s)$, or $1/Z(s)$, satisfies conditions (2.4).
3. Use a synthesis procedure to determine whether or not $Z(s)$, or $1/Z(s)$, can be synthesized.