9.1 PROPERTIES AND TEST OF POSITIVE-REAL FUNCTIONS

In Chapter 3 we established the fact that the driving-point immittance functions of *RLCM* networks (that is, networks entirely consisting of *R*, *L*, *C*, and *M*) are positive-real functions. That is, a function $H(s)$ must be a positive-real function to be physically realizable as a driving-point impedance *or* admittance (immittance) of a positive *RLCM* network. We can always check whether a given function $H(s)$ is positive-real by applying the definition of a positive-real function. This checking process, however, is *not* always practically feasible, particularly when the function is of higher order. Furthermore, it is preferable (actually it is essential, as would become apparent soon) to break down the definition of a positive-real function into a number of simple conditions so that they can be used as a set of design criteria, and this is the motivation of this section.

9.1.1 Properties of positive-real functions

A large number of analytical properties of a positive-real function can be derived from its definition. Among them, the most basic and significant ones are first summarized in Table 9.1.1, and then the proof and practical implementation of each property will be in order.

Table 9.1.1 *PROPERTIES OF POSITIVE-REAL FUNCTIONS*

$$\text{Positive-real function } Z(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}$$

1. $Z(s)$ is real when s is real.
2. $Re Z(s) \geq 0$ for $Re s \geq 0$.
3. All the coefficients must be real and non-negative.
4. $1/Z(s)$ is also positive-real.
5. No poles or zeros must be in the right half (in which $\sigma > 0$ for $s = \sigma + j\omega$) of the s-plane.
6. Poles of $Z(s)$ and poles of $1/Z(s)$ on the finite imaginary axis must be simple and have real and positive residues.
7. $|n - m| \leq 1$; poles and zeros at infinity must be simple.
8. $|\arg Z(s)| \leq \pi/2$ for $|\arg s| \leq \pi/2$.
9. A sum of positive-real functions is positive-real.
10. $F[Z(s)]$ is positive-real if both $F(s)$ and $Z(s)$ are positive-real.

The first two properties are nothing but the definition of a positive-real function, and the third follows directly from the second and sixth. The fourth property was already established in Chapter 3.

To prove the fifth property, assume that function $Z(s)$ has a pole of multiplicity m at $s = \lambda$ in the right half of the s-plane, where λ is a complex number with positive-real part. Then, in the neighborhood of the pole, $Z(s)$ can be expanded into a Laurent series of the form

$$Z(s) = \frac{A_m}{(s-\lambda)^m} + \frac{A_{m-1}}{(s-\lambda)^{m-1}} + \cdots + A_1(s-\lambda) + Z'(s) \qquad (9.1.1)$$

where $Z'(s)$ is the remainder and the A's are complex constants. Now, if we choose a point in the s-plane sufficiently near to λ, $Z(s)$ behaves in very much the same way as the first term in (9.1.1). Therefore we have

$$Z(s) \approx \frac{A_m}{(s-\lambda)^m} \qquad (9.1.2)$$

Consider a small circle of radius d drawn about the point λ as shown in Fig. 9.1.1. Then for values of s on this circle,

$$s - \lambda = de^{j\theta} \qquad (9.1.3)$$

We also rewrite A_m of (9.1.2) in the polar form

$$A_m = |A_m|e^{j\psi} \qquad (9.1.4)$$

Using (9.1.3) and (9.1.4), rewrite (9.1.2) as

$$Z(s) \approx \frac{|A_m|\, e^{j\psi}}{d^m e^{jm\theta}} = \frac{|A_m|}{d^m} e^{j(\psi - m\theta)} \qquad (9.1.5)$$

Thus

$$\text{Re}\, Z(s) \approx \frac{|A_m|}{d^m} \cos(\psi - m\theta) \qquad (9.1.6)$$

But

$$\cos(\psi - m\theta) < 0 \quad \text{for} \quad \frac{\pi}{2} < (\psi - m\theta) < \frac{3\pi}{2} \qquad (9.1.7)$$

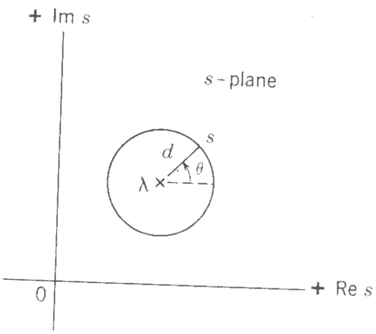

Fig. 9.1.1

and this contradicts the assumption that $Z(s)$ is positive-real. Similarly, it can be proved that $1/Z(s)$ has no poles [that is, zeros of $Z(s)$] in the right half of the s-plane since it is positive-real. Hence the property.

The evaluation of poles of a high order is not practically feasible, and a simple testing method for this property is therefore desirable. This will be discussed in the next section.

In order to prove property 6, assume that $Z(s)$ has a pole at $s = s_i$ of multiplicity n and that $s_i = j\omega_i$. In the neighborhood of the pole, we can approximate $Z(s)$ by taking the first terms of its Laurent expansion as

$$Z(s) \approx \frac{A_n}{(s - j\omega_i)^n} \tag{9.1.8}$$

If we draw a small circle of radius r about the pole $j\omega_i$ on the imaginary axis as shown in Fig. 9.1.2, we have

$$s - j\omega_i = re^{j\theta} \tag{9.1.9a}$$

$$A_n = |A_n|e^{j\phi} \tag{9.1.9b}$$

Therefore, rewrite (9.1.8) as

$$Z(s) \approx \frac{|A_n|}{r^n} e^{j(\phi - n\theta)} = \frac{|A_n|}{r^n}[\cos(\phi - n\theta) + j\sin(\phi - n\theta)] \tag{9.1.10}$$

The real part of $Z(s)$ is then found as

$$\text{Re } Z(s) = \frac{|A_n|}{r^n}\cos(\phi - n\theta) \tag{9.1.11}$$

Now, in order to assure that the real part of $Z(s)$ is always positive, we need

$$\cos(\phi - n\theta) \geqq 0 \tag{9.1.12a}$$

or

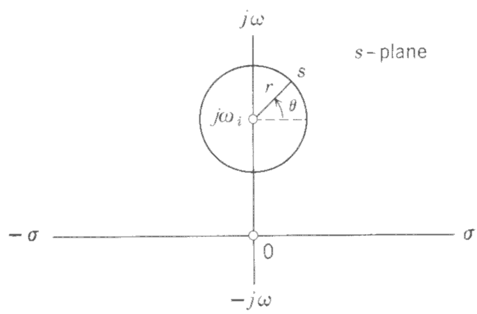

Fig. 9.1.2

$$-\frac{\pi}{2} \leqq \phi - n\theta \leqq \frac{\pi}{2} \tag{9.1.12b}$$

However, the argument θ stays in the range of

$$-\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2} \tag{9.1.13}$$

To have both conditions (9.1.12b) and (9.1.13) consistent, it is necessary that

$$\phi = 0 \qquad \text{and} \qquad n = 1 \tag{9.1.14}$$

In other words, the pole on the imaginary axis of $Z(s)$ must be simple and its residue must be real and positive (that is, A_n is real and positive). If $Z(s)$ has zeros on the imaginary axis, then $1/Z(s)$ has poles at the same points. Since $1/Z(s)$ is also a positive-real function, these poles [that is, zeros of $1/Z(s)$] must satisfy the conditions required by property 6. Hence the sixth property. The seventh property is a special case of property 6.

Example 9.1.1. Consider the partial fraction expansion of $Z(s)$ given by (9.1.15):

$$Z(s) = \frac{s^4 + 3s^2 + 1}{s(s^2 + a)} = \frac{1}{s} + \frac{\frac{1}{2}}{s + j} + \frac{\frac{1}{2}}{s - j} + s \tag{9.1.15}$$

The residues 1, $\frac{1}{2}$, $\frac{1}{2}$, and 1 are real and positive, and thus satisfy property 6. Actually $Z(s)$ given in (9.1.15) satisfies all the other properties and therefore is a positive-real function. We now combine the second and the third terms in the expansion and obtain

$$Z(s) = \frac{1}{s} + \frac{s}{s^2 + 1} + s \tag{9.1.16}$$

If we apply the definition of a positive-real function to each term in the right-hand expression of (9.1.16), we can easily check that each of them is again a positive-real function. It is therefore possible to represent each term as the impedance or as the admittance of a simple network. If we regard $Z(s)$ as a driving-point impedance, then we obtain the realization shown in Fig. 9.1.3. The reader should realize $Z(s)$ as a driving-point admittance and compare the realization with the one shown in Fig. 9.1.3.

The last three properties[1] directly follow from the definition of a positive-real function (see Problems 9.1 and 9.2).

Each of the properties listed in Table 9.1.1 is necessary for a function to be positive-real. However, by combining some of these properties we may con-

[1]Property 8 is actually identical to Property 2.

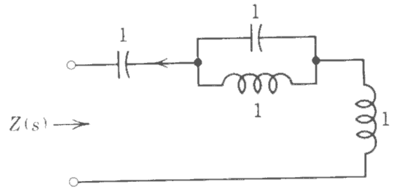

Fig. 9.1.3 A realization of $Z(s)$ given by (9.1.16) as driving-point impedance.

struct a test which will be necessary and sufficient for a function to be positive-real; one of them is given below.

Theorem 9.1.1. *Given a real rational function $Z(s)$ such that*

(i) *$Z(s)$ has no poles in the right-half of the s-plane;*

(ii) *If $Z(s)$ has poles on the imaginary axis or at infinity, the poles must be simple with real and positive residue;*

(iii) Re $Z(j\omega) \geqq 0$ *for all* ω.

Then, $Z(s)$ is positive-real.

It is clear that these conditions are necessary for $Z(s)$ to be positive-real from the properties listed in Table 9.1.1. Therefore we are going to prove only the sufficiency; that is, if a function $Z(s)$ satisfies these conditions, then $Z(s)$ must be positive-real. Let us define a closed region (or contour) going around the imaginary poles in s-plane as shown in Fig. 9.1.4.

In the region indicated, we let $r \to 0$ and $R \to \infty$; then it will cover the entire right half of the s-plane, including the imaginary axis, that is, the $j\omega$-axis, except the poles of $Z(s)$ on the imaginary axis. Let us choose points

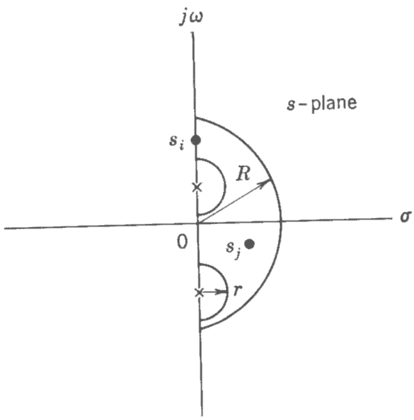

Fig. 9.1.4 A closed region in which $Z(s)$ is analytic.

s_i and s_j such that s_i is on the boundary and s_j is inside the region. Then, since $Z(s)$ is analytic throughout the right half of the plane except at any poles on the imaginary axis, the *minimum real part theorem*,[2] yields the relationship that

$$\text{Re}\,[Z(s_j)] \geqq \text{Re}\,[Z(s_i)] \tag{9.1.17}$$

where s_i is a point located on the boundary and s_j is any point inside the region in which $Z(s)$ is analytic including the boundary. Note here that $s_i = j\omega_i$ and $s_j = \sigma_j + j\omega_j$ and $\sigma_j > 0$. Moreover, we have already assumed that $\text{Re}\,[Z(j\omega)] \geqq 0$. Therefore, $\text{Re}\,[Z(s)] \geqq 0$ for $\sigma \geqq 0$ and $Z(s)$ is already a real function. Hence the theorem.

Practical implementation of this theorem, however, requires the evaluation of poles as well as of residues at the imaginary poles of a function to be tested. We shall therefore introduce in the following section an equivalent test which has a simpler checking procedure.

9.1.2 Hurwitz-polynomial test

As illustrated through several examples in the foregoing section, each property can be checked directly on a function under consideration. However, the practical implementation of some of them becomes very laborious when the function is of higher order. These include the determination of locations of poles and zeros, and the evaluation of residues of imaginary poles. Therefore, alternative but simpler testing methods will be briefly introduced in this and the following sections.

A *Hurwitz polynomial* is a polynomial all of whose roots have zero or negative real parts. More specifically, we may further break down Hurwitz polynomials into two subclasses: "*strict*" and *modified* Hurwitz polynomials. The former includes no roots with zero real parts (that is, pure imaginary zeros) whereas the latter does. We now introduce a *necessary* and *sufficient* test for a Hurwitz polynomial.

Consider a real polynomial $K(s)$ such that

$$K(s) = m(s) + n(s) \tag{9.1.18}$$

where $m(s)$ and $n(s)$ are the even and odd parts of the polynomial, respectively. Then, form a ratio $H(s)$ such that

$$H(s) = \frac{m(s)}{n(s)} \tag{9.1.19}$$

[2]Minimum real part theorem: If $F(s)$ is analytic in a closed region of the complex plane and also on the boundary of the region, then the real part of $F(s)$ is minimum on the boundary and not in the interior of the region.

and expand $H(s)$ into a finite *Stieltjes continued fraction* by synthetic division or, as it is sometimes called, the *Euclid algorithm*, as

$$H(s) = \alpha_1 s + \cfrac{1}{\alpha_2 s + \cfrac{1}{\ddots + \cfrac{1}{\alpha_k s}}} \tag{9.1.20}$$

where k is the order of $m(s)$ and it is assumed that the order of $m(s)$ is higher than that of $n(s)$ by unity; otherwise, form a ratio such that $H(s) = n(s)/m(s)$. If all α's in the expansion, called *Hurwitz coefficients*, are positive, $K(s)$ is a (strict) Hurwitz polynomial. This is a necessary and sufficient condition for a real polynomial $K(s)$ to be Hurwitz. If $K(s)$ contains pure imaginary zeros (that is, a modified Hurwitz polynomial), the number of α's is smaller than the order of $m(s)$ or $n(s)$ (whichever is higher). That is, the expansion terminates prematurely. This will be illustrated by the following examples.

Example 9.1.2. Consider three real polynomials $K_1(s)$, $K_2(s)$, and $K_3(s)$ defined by

$$\begin{aligned} K_1(s) &= 2s^4 + s^3 + 7s^2 + s + 1 \\ K_2(s) &= s^4 + 2s^3 + 2s^2 + 2s + 1 \\ K_3(s) &= s^3 + s^2 + s + 2 \end{aligned} \tag{9.1.21}$$

Then the Hurwitz test yields

$$H_1(s) = \frac{2s^4 + 7s^2 + 1}{s^3 + s} = 2s + \cfrac{1}{\frac{1}{5}s + \cfrac{1}{(25/4)s + \cfrac{1}{(4/5)s}}} \tag{9.1.22}$$

and thus $K_1(s)$ is (strict) Hurwitz. On the other hand, the continued fraction expansion for $K_2(s)$ gives

$$H_2(s) = \frac{s^4 + 2s^2 + 1}{2s^3 + 2s} = \frac{1}{2}s + \frac{1}{2s} \tag{9.1.23}$$

That is, we obtain

$$\begin{array}{rl} 2s^3 + 2s\,)\,s^4 + 2s^2 + 1\,\Big(\dfrac{s}{2} & \\ \underline{s^4 + s^2} & \\ s^2 + 1\,)\,2s^2 + 2s\,(2s & \\ \underline{2s^2 + 2s} & \\ 0 & \end{array} \tag{9.1.24}$$

Thus, $K_2(s)$ is a modified Hurwitz polynomial having zeros at $s = \pm j$. Finally, for $K_3(s)$ we obtain

$$H_3(s) = \frac{s^3 + s}{s^2 + 2} = s + \frac{1}{-s + (1/-\frac{1}{2}s)} \tag{9.1.25}$$

Therefore, $K_3(s)$ is not Hurwitz and has zeros in the right half-plane.

In view of the test just described for Hurwitz polynomials, Theorem 9.1.1 can be replaced by the following theorem[3] which eliminates the necessity of the evaluation of residues.

Theorem 9.1.2. *For a rational function $Z(s)$, if*

(i) *$Z(s)$ is a real function of s;*
(ii) *$P(s) + Q(s)$ is strict Hurwitz;*
(iii) *$\text{Re } Z(j\omega) \geqq 0$ for all ω*

then $Z(s)$ is a positive-real function.

Proof. Let us transform the right half of the Z-plane into a circle of unit radius in the w-plane, by using the bilinear transformation of the form

$$w(s) = \frac{Z(s) - 1}{Z(s) + 1} = \frac{P(s) - Q(s)}{P(s) + Q(s)} \tag{9.1.26}$$

or

$$w(s) = \frac{\text{Re } Z(s) - 1 + j \text{ Im } Z(s)}{\text{Re } Z(s) + 1 + j \text{ Im } Z(s)} \tag{9.1.27}$$

From (9.1.27), it is apparent that

$$\begin{aligned} &\text{if and only if } \text{Re } Z(s) = 0, \quad \text{then } |w(s)| = 1 \\ &\text{if and only if } \text{Re } Z(s) > 0, \quad \text{then } |w(s)| < 1 \\ &\text{if and only if } \text{Re } Z(s) < 0, \quad \text{then } |w(s)| > 1 \end{aligned} \tag{9.1.28}$$

and these are illustrated in Fig. 9.1.5. We therefore restate the definition of a positive-real function $Z(s)$:

$$\begin{aligned} &\text{(a)} \quad Z(s) \text{ is a real function of } s \\ &\text{(b)} \quad \text{Re } Z(s) \geqq 0 \quad \text{for Re } s \geqq 0 \end{aligned} \tag{9.1.29}$$

by the following statement in terms of $w(s)$:

$$\begin{aligned} &\text{(a)} \quad w(s) \text{ is a real function of } s \\ &\text{(b)} \quad |w(s)| \leqq 1 \quad \text{for Re } s \geqq 0 \end{aligned} \tag{9.1.30}$$

[3] A. Talbot, "A new method of synthesis of reactance networks," Monograph No. 77, *Proc. IEE* (London), Part IV, **101**, 73–90 (1954).

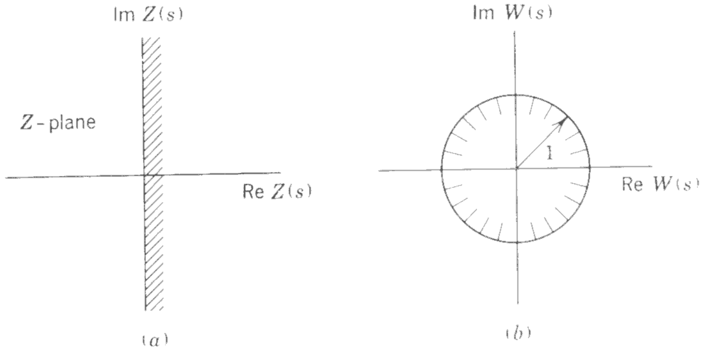

Fig. 9.1.5 (*a*) The right half of the *Z*-plane, (*b*) transformed into a circle of unit radius about the origin in the *w*-plane.

The first statement of (9.1.30) is obvious if $Z(s)$ is real for real s. The second condition requires the magnitude of $w(s)$ to attain the bounded maximum value on the boundary of the circle of unit radius in the *w-plane*. In order for $w(s)$ to achieve the maximum magnitude on the boundary of the circle, $w(s)$ must not contain any poles in the interior of the circle,[4] which alternately implies that the denominator of $w(s)$, $P(s)$ plus $Q(s)$, must not contain any zeros in the right half of the s-plane including the imaginary axis. In a simple statement, we would say that the denominator polynomial, $P(s) + Q(s)$, must be a Hurwitz polynomial. We therefore rewrite the statements in (9.1.29) and (9.1.30):

(a″) $Z(s) = \dfrac{P(s)}{Q(s)}$ is real for real s

(b″) $P(s) + Q(s)$ is a Hurwitz polynomial

(c″) $\text{Re}\, Z(j\omega) \geqq 0$ for all ω

Hence the theorem.

Example 9.1.3. Prove that if $Z(s) = P(s)/Q(s)$ is a positive-real function, so is $Z_1(s) = P_1(s)/Q_1(s)$, where $P(s) = m_1(s) + n_1(s)$, $Q(s) = m_2(s) + n_2(s)$, $P_1(s) = m_1(s) + n_2(s)$, and $Q_1(s) = m_2(s) + n_1(s)$. It is assumed here that all $m(s)$ and $n(s)$ are different from zero.

First, we know that $Z_1(s)$ is real when s is real. Second, $P_1(s) + Q_1(s)$ is a Hurwitz polynominal, since $P_1(s) + Q_1(s) = P(s) + Q(s)$. Furthermore, in

[4] Because of the maximum modulus theorem which is dual to the minimum real part theorem used previously, and the unbounded behavior of $w(s)$ at a pole.

$$\text{Re } Z_1(j\omega) = \frac{m_1(j\omega)m_2(j\omega) + n_1(j\omega)n_2(j\omega)}{m_2^2(j\omega) - n_1^2(j\omega)}$$

the denominator is always positive, and the numerator is identical to that of $\text{Re}[Z(j\omega)]$. Therefore, $\text{Re}[Z_1(j\omega)] \geqq 0$ for all ω. Thus, from Theorem 9.1.2, $Z_1(s)$ is positive-real.

This particular result is a basis of a popular two-port synthesis technique which is in use.[5]

Example 9.1.4. Expand $Z(s)$, considered in Example 9.1.1, into a continued fraction expansion, and obtain

$$Z(s) = \frac{s^4 + 3s^2 + 1}{s(s^2 + 1)} = s + \cfrac{1}{\frac{1}{2}s + \cfrac{1}{4s + 1/(s/2)}} \tag{9.1.32}$$

All the coefficients in the expansion are real and positive because $Z(s)$ is positive-real and the numerator and denominator polynomials form a Hurwitz polynomial. If we again regard $Z(s)$ as a driving-point impedance, the expansion gives the realization of Fig. 9.1.6.

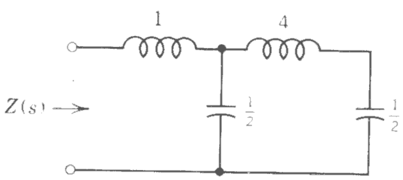

Fig. 9.1.6 A realization of $Z(s)$ given by (9.1.32).

9.1.3 Sturm test

A test required for the property of $\text{Re } Z(j\omega) \geqq 0$ for all ω can be simplified greatly if the method called *Sturm test* is employed.

The real part of $Z(j\omega)$ is given by

$$\text{Re } Z(j\omega) = \frac{m_1(j\omega)m_2(j\omega) + n_1(j\omega)n_2(j\omega)}{m_2^2(j\omega) - n_2^2(j\omega)} = \frac{A(\omega^2)}{B(\omega^2)} \tag{9.1.33}$$

Here, it is apparent that $B(\omega^2) \geqq 0$ for all ω. Therefore the only test which would be required is to check whether $A(\omega^2)$ ever becomes negative for any value of ω. We let $A(\omega^2)$ be expressed generally by

[5] S. Darlington, "Synthesis of reactance 4-poles which produce prescribed insertion loss characteristics," *Jour. Math. and Phys.*, **18**, 257–353 (1939).

$$\begin{aligned} A(\omega^2) &= a_n\omega^{2n} + a_{n-1}\omega^{2(n-1)} + \cdots + a_0 \\ &= (\omega^2 - \alpha_1)^{n_1}(\omega^2 - \alpha_2)^{n_2} \cdots (\omega^2 - \alpha_k)^{n_k} \end{aligned} \tag{9.1.34}$$

Then, from (9.1.34) we note that, if $\alpha_i > 0$,

$$(\omega^2 - \alpha_i)^{n_i} \begin{cases} \geqq 0 & \text{for all } \omega & \text{if } n_i = \text{even number} \\ \geqq 0 & \text{for } \omega^2 \geqq \alpha_i & \text{if } n_i = \text{odd number} \\ < 0 & \text{for } \omega^2 < \alpha_i & \text{if } n_i = \text{odd number} \end{cases} \tag{9.1.35}$$

It is thus clear that, if $A(\omega^2)$ contains *real positive* zeros of *odd* multiplicity, it will become negative for a range of some values of ω. That is, $A(\omega^2)$ must have *no* real positive zeros of odd multiplicity. To test the odd multiplicity of zeros of $A(\omega^2)$ we employ the Sturm theorem.

Replace ω^2 in (9.1.34) by x, and obtain

$$A(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0 \tag{9.1.36}$$

where it is assumed (1) that $a_n \neq 0$ and (2) that there are no repeated zeros. Then, form the *Sturm sequence* defined by

$$\begin{aligned} A(x) &= \text{given} \\ A_1(x) &= \frac{d}{dx} A(x) = A'(x) \\ A_2(x) &= A_1(x)(k_1x + k_2) - A(x) \\ A_3(x) &= A_2(x)(k_3x + k_4) - A_1(x) \\ &\vdots \\ A_n &= \text{a constant} \end{aligned}$$

where all k's are real numbers. If we assume a range of values of x, say $b_1 \leqq x \leqq b_2$, then we have the following relationship:

$$|S_{b_1} - S_{b_2}| = \text{the number of real zeros of } A(x) \text{ in the range of } n_1 \leqq x \leqq b_2 \tag{9.1.38}$$

where S_{b_1} = the number of sign variations of the Sturm sequence with x replaced by b_1, that is, signs of $A(b_1), A_1(b_1), \ldots, A_n$

S_{b_2} = the number of sign variations of the Sturm sequence with x replaced by b_2, that is, signs of $A(b_2), A_1(b_2), \ldots A_n$

Example 9.1.5. Consider

$$A(\omega^2) = \omega^6 - 3\omega^4 + 6\omega^2 - 1$$

By replacing ω^2 by x, we obtain the Sturm sequence:

$$A(x) = x^3 - 3x^2 + 6x - 1$$

$$A_1(x) = 3x^2 - 6x + 6$$

$$A_2(x) = -(2x + 3)$$

$$A_3(x) = 39/4$$

In the range of x for $0 \leqq x \leqq \infty$, we find that

$$S_0 = 2, S_\infty = 2;$$

$$\text{thus } |S - S_\infty| = 0$$

Therefore, $A(\omega^2)$ has no real zero in the range of $0 \leqq \omega^2 \leqq \infty$. Thus, $A(\omega^2)$ never becomes negative.

Note here that the Sturm test described so far is based on the assumption that $A(x)$ has no repeated zeros. If it does, the Sturm process terminates prematurely because of the fact that the Sturm sequence is constructed by the Euclid algorithm. We thus summarize the testing procedure of the real-part requirement:

(a) If all a's are nonnegative in (9.1.34), $A(\omega^2) > 0$.
(b) If the Sturm sequence does *not* terminate prematurely, $A(\omega^2)$ has all of its zeros of order 1.
(c) If $A(\omega^2)$ has repeated zeros, say at $\omega^2 = \alpha_i$, then the Sturm process terminates prematurely. Divide $A(\omega^2)$ by a factor $(\omega^2 - \alpha_i)^{n_i}$ for $n_i \geqq 2$ and then proceed with the Sturm process.

The reader should work out the problems given at the end of this chapter to familiarize himself with the Sturm test outlined above.

9.2 BASIC REALIZATION PRINCIPLES OF POSITIVE-REAL FUNCTIONS

Although a few examples of network realization were given in the preceding section, the basic principles underlying network synthesis techniques will be briefly discussed in this section, in order for the reader to establish a sufficient background to follow various techniques available in network synthesis problems. Again here, the reader is reminded of the aim of this book; that is, we are primarily concerned throughout this text with the network analysis rather than the network synthesis. Therefore a more thorough study of the latter should include a text on the subject of network synthesis.

Basically, the realization of positive-real functions is accomplished by repeated use of the following four processes:

(a) Removal of a pole at infinity.
(b) Removal of a pole at the origin.
(c) Removal of imaginary poles.
(d) Removal of a constant.

Each of these procedures will be briefly discussed in sequence.

(A) Removal of a pole at infinity: For there to be such a pole, a positive real function $H(s)$ is required to be of the form that its numerator is of degree one higher than that of the denominator. Thus, we write $H(s)$ in general as

$$H(s) = \frac{P(s)}{Q(s)} = \frac{a_{n+1}s^{n+1} + a_n s^n + \cdots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}$$

$$= \frac{a_{n+1}}{b_n} s + H_1(s) \tag{9.2.1}$$

where a_{n+1}/b_n is the residue of the pole at infinity, and $H_1(s)$ is the remainder. Thus, it is clear that, if $H(s)$ is a driving-point impedance, an inductance of value (a_{n+1}/b_n) henrys connected in series with $H_1(s)$ represents the decomposition given by (9.2.1). This is shown in Fig. 9.2.1*a*. On the other hand, if $H(s)$ happens to be a driving-point admittance, the network of Fig. 9.2.1*b* represents (9.2.1). Note that all *a*'s and *b*'s in (9.2.1) are real and nonnegative and both a_{n+1} and b_n are greater than zero.

Now, the question is whether $H_1(s)$ is still positive real. However, the following are apparent:

(1) $H_1(s)$ is real for a real s,

(2) the residues of finite imaginary poles of $H_1(s)$ are identical to those of $H(s)$, ($H_1(s)$ has no longer a pole at infinity),

(3) Re $H_1(j\omega)$ = Re $H(j\omega) \geqq 0$ since the real part of $(a_{n+1}/b_n)j\omega$ is zero.

Thus, $H_1(s)$ is still positive-real. Therefore we can work on $H_1(s)$ again; $H_1(s)$ no longer has a pole at infinity, but it may have a zero at infinity. That is, $1/H_1(s)$ may have a pole at infinity, and this pole then can be removed again, leaving the remainder still positive real. [Note that, if $H_1(s)$ is

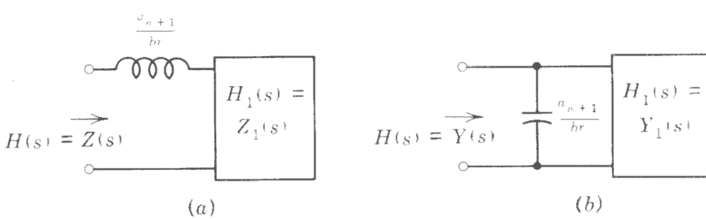

Fig. 9.2.1

positive real, so is $1/H_1(s)$.] This successive removal of poles at infinity results in a so-called *Cauer's first form* realization, which will be illustrated by the following example.

Example 9.2.1.Cauer's First Form Realization. Consider a driving-point impedance given by

$$Z(s) = \frac{2(s^2+1)(s^2+3)}{s(s^2+2)} \tag{9.2.2}$$

The pole at infinity is now removed from $Z(s)$. This results in

$$Z(s) = 2s + Z_1(s)$$

where

$$Z_1(s) = \frac{4s^2+6}{s(s^2+2)}$$

Since $1/Z_1(s)$ has a pole at infinity, this pole will be removed. Thus we have

$$\frac{1}{Z_1(s)} = \frac{s}{4} + Y_2(s)$$

where

$$Y_2(s) = \frac{s}{2(s^2+\frac{6}{4})}$$

Again, $1/Y_2(s)$ has a pole at infinity, and this will be removed. That is,

$$\frac{1}{Y_2(s)} = 8s + \frac{12}{s}$$

This successive removal process of poles at infinity is nothing but a step-by-step development of continued fraction expansion. When the numerator of $Z(s)$ is divided through (that is, synthetic division) by the denominator in descending order, we see that

$$Z(s) = \underset{L_1 s}{\underset{\downarrow}{2s}} + \cfrac{1}{\underset{C_1 s}{\underset{\downarrow}{\frac{s}{4}}} + \cfrac{1}{\underset{L_2 s}{\underset{\downarrow}{8s}} + \cfrac{1}{\underset{C_2 s}{\underset{\downarrow}{\frac{s}{12}}}}}} \tag{9.2.3}$$

and the corresponding realization is given in Fig. 9.2.2.

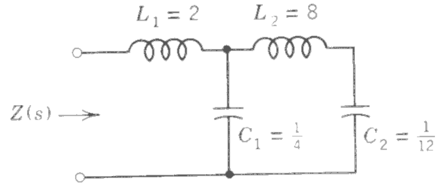

Fig. 9.2.2 Cauer's first form realization of $Z(s)$ of (9.2.2).

(B) Removal of a pole at $s = 0$. If a positive-real function $H(s)$ has a pole at $s = 0$, it can be generally written in the form of

$$H(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{b_m s^m + a_{m-1}s^{m-1} + \cdots + b_1 s} = \frac{a_0}{b_1 s} + H_1(s) \qquad (9.2.4)$$

where $m = n \pm 1$ or $m = n$, and $H_1(s)$ is the remainder. Moreover, (a_0/b_1) is the residue of the pole at the origin. This decomposition results in the realization shown in Fig. 9.2.3, where $H(s) = Z(s)$ or $H(s) = Y(s)$ as the case may be.

The remainder $H_1(s)$ is again positive-real for precisely the same reasons used for the case of removal of a pole at infinity. This process can be successively repeated if $1/H_1(s)$ has again a pole at the origin, resulting in a so-called "Cauer's second form" realization which will be illustrated by the following example.

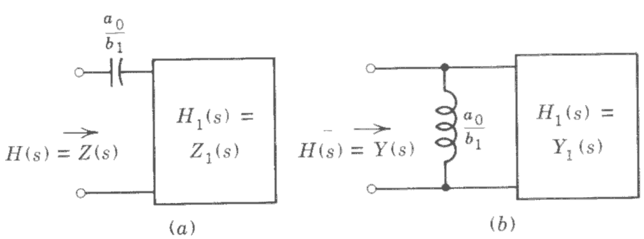

Fig. 9.2.3 Realization corresponding to the removal of a pole at $s = 0$.

Example 9.2.2. Cauer's Second Form Realization. Consider the same function considered in Example 9.2.1, and remove from it a pole at the origin. That is,

$$Z(s) = \frac{2(s^2 + 1)(s^2 + 3)}{s(s^2 + 2)} = \frac{3}{s} + Z_1(s) \qquad (9.2.5)$$

where

$$Z_1(s) = \frac{s(2s^2 + 5)}{s^2 + 2}$$

Since $1/Z_1(s)$ has a pole at $s = 0$, the pole is removed. Thus

$$\frac{1}{Z_1(s)} = \frac{s^2 + 2}{s(2s^2 + 5)} = \frac{2}{5s} + Y_2(s)$$

where

$$Y_2(s) = \frac{\frac{3}{25}s}{\frac{2}{5}s^2 + 1}$$

Next, the pole at $s = 0$ is removed from $1/Y_2(s)$. That is,

$$\frac{1}{Y_2(s)} = \frac{25}{3s} + \frac{1}{(6/125s)}$$

These successive processes carried out are the synthetic division process in ascending order. We therefore have

$$Z(s) = \underset{\substack{\downarrow \\ 1/C_1 s}}{\frac{3}{s}} + \cfrac{1}{\underset{\substack{\downarrow \\ 1/L_1 s}}{\frac{2}{5s}} + \cfrac{1}{\underset{\substack{\downarrow \\ 1/C_2 s}}{\frac{25}{3s}} + \cfrac{1}{\underset{\substack{\downarrow \\ 1/L_2 s}}{\frac{6}{125s}}}}} \tag{9.2.6}$$

and the corresponding network realization is given in Fig. 9.2.4, where the units for the element values are farads and henrys.

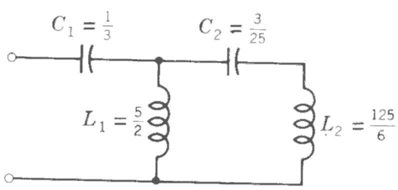

Fig. 9.2.4 Cauer's second form realization of $Z(s)$ of (9.2.2).

(C) Removal of imaginary poles. If a positive-real function $H(s)$ has poles on the imaginary axis, say poles at $s = \pm j\omega_1$, it can be generally of the form

$$H(s) = \frac{P(s)}{(s^2 + \omega_1^2)Q_1(s)} = \frac{K_1}{s - j\omega_1} + \frac{K_2}{s + j\omega_1} + H_1(s) \tag{9.2.7}$$

where K_1 and K_2 are residues of the poles at $s = j\omega_1$ and $s = -j\omega_1$, respectively, and $H_1(s)$ is the remainder.

Since $H(s)$ is a positive-real function, both K_1 and K_2 are real and positive and $K_1 = K_2$. We therefore rewrite (9.2.7) as

$$H(s) = \frac{2Ks}{s^2 + \omega_1^2} + H_1(s) \tag{9.2.8}$$

where $K_1 = K_2 = K$. Here, the remainder $H_1(s)$ is again positive-real for the reasons parallel to those used previously. Then the network configuration representing (9.2.8) is given in Fig. 9.2.5.

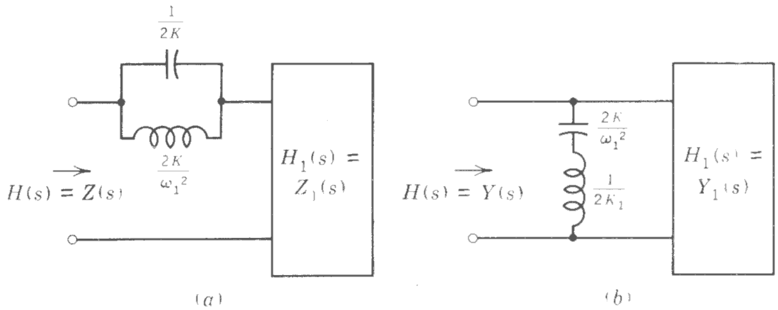

Fig. 9.2.5 Network representation for imaginary poles removed.

Example 9.2.3. Foster Realization Techniques. Consider the driving-point impedance $Z(s)$ given by (9.2.9a):

$$Z(s) = \frac{(s^2 + 1)(s^2 + 9)}{s(s^2 + 4)} \tag{9.2.9a}$$

We first remove the pole at infinity and then remove the poles at $s = \pm j2$. Thus

$$Z(s) = s + \frac{(15/4)s}{s^2 + 4} + \frac{9}{4s} \tag{9.2.9b}$$

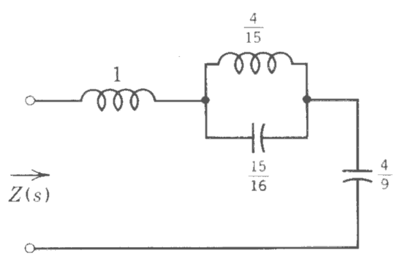

Fig. 9.2.6 Foster's first form realization of $Z(s)$ of (9.2.9a).

and the corresponding network representation is shown in Fig. 9.2.6; this configuration is called *Foster's first form.*

If we realize the same $Z(s)$ on admittance basis, we obtain

$$Y(s) = \frac{1}{Z(s)} = \frac{s(s^2+4)}{(s^2+1)(s^2+9)} = \frac{(3/8)s}{s^2+1} + \frac{(5/8)s}{s^2+9} \qquad (9.2.10)$$

resulting in a so-called *Foster's second form* realization as shown in Fig. 9.2.7.

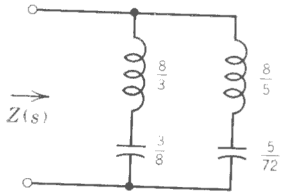

Fig. 9.2.7 Foster's second form realization of $Z(s)$ given by (9.2.9*a*).

(D) Removal of a constant. Since the real part of a positive-real function $H(s)$ never becomes negative as long as the real part of s is nonnegative, a typical characteristic of Re $H(j\omega)$ may take a form as the one shown in Fig. 9.2.8. Then it is clear that the smallest value of the real part which happens to be at $\omega = \omega_1$ can be subtracted from $H(s)$ without destroying its positive-real characteristic. Thus the remainder $H_1(s)$ given by

$$H_1(s) = H(s) - R_1 \qquad (9.2.11)$$

is still positive-real. This is one of the most fundamental processes in network synthesis techniques.

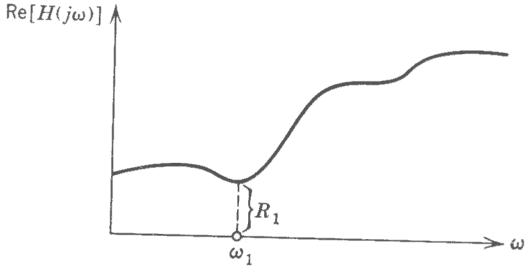

Fig. 9.2.8 Typical characteristic of the real part of a positive-real function.

Example 9.2.4. Consider a positive-real function $H(s)$ given by

$$H(s) = \frac{(s+1)(s+3)}{s(s+2)} \qquad (9.2.12)$$

First remove the pole at $s = 0$, resulting in

$$H(s) = \frac{3}{2s} + H_1(s)$$

where

$$H_1(s) = \frac{2s + 5}{2(s + 2)}$$

If we find the smallest value of Re $H_1(j\omega)$, it turns out to occur at $\omega = \infty$. We therefore know that the smallest value of Re$[1/H_1(j\omega)]$ occurs at $\omega = 0$. Actually,

$$\min \text{Re}\left[\frac{1}{H_1(j\omega)}\right] = \frac{4}{5} \qquad \text{at } \omega = 0$$

Thus we have

$$H(s) = \frac{3}{2s} + \frac{1}{(4/5) + Y_2(s)}$$

where

$$Y_2(s) = \frac{2s}{5(2s + 5)}$$

Since $1/Y_2(s)$ has a pole at the origin, it can be removed again. Therefore we have

$$H(s) = \frac{3}{2s} + \cfrac{1}{\cfrac{4}{5} + \cfrac{1}{\cfrac{25}{2s} + \cfrac{1}{\frac{1}{5}}}} \tag{9.2.13}$$

The corresponding network representations of (9.2.13) are given in Fig. 9.2.9, where the first considers $H(s)$ as an impedance and the other treats it as an admittance.

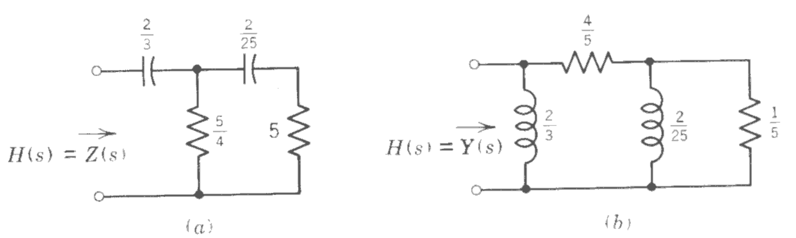

Fig. 9.2.9 Realizations of $H(s)$ of (9.2.12).

There is a large class of positive-real functions to which none of these removal processes is directly applicable. However, these will be considered elsewhere.

9.3 ANALYTICAL PROPERTIES OF TWO-PORT FUNCTIONS

Through Chapters 5 and 8, and the first section of this chapter, the following properties have been established for z-parameters of an *RLCM two-port*.

(1) $z_{11}(s)$ and $z_{22}(s)$ are driving-point impedances; thus they are positive-real functions.
(2) $z_{12}(s)$ and $z_{21}(s)$ are real function of s (that is, all the coefficients are real), but are not generally positive-real.
(3) The real parts of the parameters are related by $r_{11}r_{22} - r_{12}^2 \geqq 0$, where r_{ij} is the real part of $z_{ij}(s)$, and $r_{12} = r_{21}$, for Re $s \geq 0$.

Note that the same are true for y-parameters. These results are, however, insufficient for us to develop a systematic realization procedure for two-port networks. We shall therefore derive some more in this section.

Let us assume that $Z(s)$, $z_{11}(s)$, $z_{12}(s)$, and $z_{22}(s)$, have a simple pole at $s = j\omega_i$. Therefore we write

$$
\begin{aligned}
Z(s) &= Z'(s) + \frac{k}{s - j\omega_i} \\
z_{11}(s) &= z'_{11}(s) + \frac{k_{11}}{s - j\omega_i} \\
z_{22}(s) &= z'_{22}(s) + \frac{k_{22}}{s - j\omega_i} \\
z_{12}(s) &= z'_{12}(s) + \frac{k_{12}}{s - j\omega_i}
\end{aligned}
\tag{9.3.1}
$$

where each parameter with the prime superscript denotes the remainder of the parameter after the pole is removed. Since $Z(s)$, $z_{11}(s)$, and $z_{22}(s)$ are positive-real functions, the residues k, k_{11}, and k_{22} are real and positive. It is then clear that, in the close vicinity of this pole, each parameter can be approximated by

$$
\begin{aligned}
Z(s)|_{s \to j\omega i} &\approx \frac{k}{s - j\omega_i} \\
z_{11}(s)|_{s \to j\omega i} &\approx \frac{k_{11}}{s - j\omega_i}
\end{aligned}
\tag{9.3.2a}
$$