BRIAN D. O. ANDERSON
SUMETH VONGPANITLERD

Network Analysis and Synthesis

A MODERN SYSTEMS THEORY APPROACH

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NETWORK ANALYSIS AND SYNTHESIS

A Modern Systems Theory Approach

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Preface

For many years, network theory has been one of the more mathematically developed of the electrical engineering fields. In more recent times, it has shared this distinction with fields such as control theory and communication systems theory, and is now viewed, together with these fields, as a subdiscipline of modern system theory. However, one of the key concepts of modern system theory, the notion of state, has received very little attention in network synthesis, while in network analysis the emphasis has come only in recent times. The aim of this book is to counteract what is seen to be an unfortunate situation, by embedding network theory, both analysis and synthesis, squarely within the framework of modern system theory. This is done by emphasizing the state-variable approach to analysis and synthesis.

Aside from the fact that there is a gap in the literature, we see several important reasons justifying presentation of the material found in this book. First, in solving network problems with a computer, experience has shown that very frequently programs based on state-variable methods can be more easily written and used than programs based on, for example, Laplace transform methods. Second, the state concept is one that emphasizes the internal structure of a system. As such, it is the obvious tool for solving a problem such as finding all networks synthesizing a prescribed impedance; this and many other problems of internal structure are beyond the power of classical network theory. Third, the state-space description of passivity, dealt with at some length in this book, applies in such diverse areas as Popov stability, inverse optimal control problems, sensitivity reduction in control systems, and Kalman–Bucy or Wiener filtering (a topic of such rich application surely deserves treatment within a textbook framework). Fourth, the graduate major in systems science is better served by a common approach to different disciplines of systems science than by a multiplicity of different approaches with no common ground; a move injecting the state variable into network analysis and synthesis is therefore as welcome from the pedagogical viewpoint as recent moves which have injected state variables into communications systems.

The book has been written with a greater emphasis on passive than on active networks. This is in part a reflection of the authors' personal interests, and in part a reflection of their view that passive system theory is a topic about which no graduate systems major should be ignorant. Nevertheless, the inclusion of starred material which can be omitted with no loss of continuity offers the instructor a great deal of freedom in setting the emphasis in a course based on the book. A course could, if the instructor desires, de-emphasize the networks aspect to a great degree, and concentrate mostly on general systems theory, including the theory of passive systems. Again, a course could emphasize the active networks aspect by excluding much material on passive network synthesis.

The book is aimed at the first year graduate student, though it could certainly be used in a class containing advanced undergraduates, or later year graduate students. The background required is an introductory course on linear systems, the usual elementary undergraduate networks material, and ability to handle matrices. Proceeding at a fair pace, the entire book could be completed in a semester, while omission of starred material would allow a more leisurely coverage in a semester. In a course of two quarters length, the book could be comfortably completed, while in one quarter, particularly with students of strong backgrounds, a judicious selection of material could build a unified course.

ACKNOWLEDGMENTS

Much of the research reported in this book, as well as the writing of it, was supported by the Australian Research Grants Committee, the Fulbright Grant scheme and the Colombo Plan Fellowship scheme. There are also many individuals to whom we owe acknowledgment, and we gratefully record the names of those to whom we are particularly indebted: Robert Newcomb, in his dual capacities as a Prentice-Hall editor and an academic mentor; Andrew Sage, whose hospitality helped create the right environment at Southern Methodist University for a sabbatical leave period to be profitably spent; Rudy Kalman, for an introduction to the Positive Real Lemma; Lorraine Pogonoski, who did a sterling and untiring job in typing the manuscript; Peter McLauchlan, for excellent drafting work of the figures; Peter Moylan, for extensive and helpful criticisms that have added immeasurably to the end result; Roger Brockett, Len Silverman, Tom Kailath and colleagues John Moore and Tom Fortmann who have all, in varying degrees, contributed ideas which have found their way into the text; and Dianne, Elizabeth, and Philippa Anderson for criticisms and, more importantly, for their patience and ability to generate inspiration and enthusiasm.

Part I

INTRODUCTION

What is the modern system theory approach to network analysis and synthesis? In this part we begin answering this question.

1

Introduction

1.1 ANALYSIS AND SYNTHESIS

Two complementary functions of the engineer are analysis and synthesis. In analysis problems, one is usually given a description of a physical system, i.e., a statement of what its components are, how they are assembled, and what the laws of physics are that govern the behavior of each component. One is generally required to perform some computations to predict the behavior of the system, such as predicting its response to a prescribed input. The synthesis problem is the reverse of this. One is generally told of a behavioral characteristic that a system should have, and asked to devise a system with this prescribed behavioral characteristic.

In this book we shall be concerned with *network* analysis and synthesis. More precisely, the networks will be electrical networks, which *always* will be assumed to be (1) linear, (2) time invariant, and (3) comprised of a finite number of lumped elements. Usually, the networks will also be assumed to be passive. We presume the reader knows what it means for a network to be linear and time invariant. In Chapter 2 we shall define more precisely the allowed element classes and the notion of passivity.

In the context of this book the analysis problem becomes one of knowing the set of elements comprising a network, the way they are interconnected, the set of initial conditions associated with the energy storage elements, and the set of excitations provided by externally connected voltage and current generators. The problem is to find the response of the network, i.e., the resultant set of voltages and currents associated with the elements of the network.

In contrast, tackling the synthesis problem requires us to start with a statement of what responses should result from what excitations, and we are required to determine a network, or a set of network elements and a scheme for their interconnection, that will provide the desired excitation-response characteristic. Naturally, we shall state both the synthesis problem and the analysis problem in more explicit terms at the appropriate time; the reader should regard the above explanations as being rough, but temporarily satisfactory.

1.2 CLASSICAL AND MODERN APPROACHES

Control and network theory have historically been closely linked, because the methods used to study problems in both fields have often been very similar, or identical. Recent developments in control theory have, however, tended to outpace those in network theory. The magnitude and importance of the developments in control theory have even led to the attachment of the labels "classical" and "modern" to large bodies of knowledge within the discipline. It is the development of network theory paralleling modern control theory that has been lacking, and it is with such network theory that this book is concerned.

The distinction between modern and classical control theory is at points blurred. Yet while the dividing line cannot be accurately drawn, it would be reasonable to say that frequency-domain, including Laplace-transform, methods belong to classical control theory, while state-space methods belong to modern control theory. Given this division within the field of control, it seems reasonable to translate it to the field of network theory. Classical network theory therefore will be deemed to include frequency-domain methods, and modern network theory to include state-space methods.

In this book we aim to emphasize application of the notions of state, and system description via state-variable equations, to the study of networks. In so doing, we shall consider problems of both analysis and synthesis.

As already noted, modern theory looks at state-space descriptions while classical theory tends to look at Laplace-transform descriptions. What are the consequences of this difference? They are numerous; some are as follows.

First and foremost, the state-variable description of a network or system emphasizes the *internal structure* of that system, as well as its input–output performance. This is in contrast to the Laplace-transform description of a network or system involving transfer functions and the like, which emphasizes the input–output performance alone. The internal structure of a network must be considered in dealing with a number of important questions. For example, minimizing the number of reactive elements in a network synthesizing a prescribed excitation-response pair is a problem involving examination of the details of the internal structure of the network. Other pertinent examples include minimizing the total resistance of a network, examining the effect of nonzero initial conditions of energy storage or reactive elements on the externally observable behavior of the network, and examining the sensitivity of the externally observable behavior of the circuit to variations in the component values. It would be quite illogical to attempt to solve all these problems with the tools of classical network theory (though to be sure some progress could be made on some of them). It would, on the other hand, be natural to study these problems with the aid of modern network theory and the state-variable approach.

Some other important differences between the classical and modern approaches can be quickly summarized:

1. The classical approach to synthesis usually relies on the application of ingeniously contrived algorithms to achieve syntheses, with the number of variations on the basic synthesis structures often being severely circumscribed. The modern approach to synthesis, on the other hand, usually relies on solution, without the aid of too many tricks or clever technical artifices, of a well-motivated and easily formulated problem. At the same time, the modern approach frequently allows the straightforward generation of an infinity of solutions to the synthesis problem.
2. The modern approach to network analysis is ideally suited to implementation on a digital computer. Time-domain integration of state-space differential equations is generally more easily achieved than operations involving computation of Laplace transforms and inverse Laplace transforms.
3. The modern approach emphasizes the algebraic content of network descriptions and the solution of synthesis problems by matrix algebra. The classical approach is more concerned with using the tools of complex variable analysis.

The modern approach is not better than the classical approach in every way. For example, it can be argued that the intuitional pictures provided by Bode diagrams and pole-zero diagrams in the classical approach tend to be lost in the modern approach. Some, however, would challenge this argument on the grounds that the modern approach subsumes the classical approach. The modern approach too has yet to live up to all its promise. Above we listed problems to which the modern approach could logically be applied; some of these problems have yet to be solved. Accordingly, at this stage of development of the modern system theory approach to network analysis and synthesis, we believe the classical and modern approaches are best seen as being complementary. The fact that this book contains so little of the classical approach may then be queried; but the answer to this query is provided by the extensive array of books on network analysis and synthesis, e.g., [1–4], which, at least in the case of the synthesis books, are heavily committed to presenting the classical approach, sometimes to the exclusion of the modern approach.

The classical approach to network analysis and synthesis has been developed for many years. Virtually all the analysis problems have been solved, and a falling off in research on synthesis problems suggests that the majority of those synthesis problems that are solvable may have been solved. Much of practical benefit has been forthcoming, but there do remain practical problems that have yet to succumb. As we noted above, the modern approach has not yet solved all the problems that it might be expected to solve; particularly is it the case that it has solved few practical problems, although in isolated cases, as in the case of active synthesis discussed in Chapter 13, there have been spectacular results. We attribute the present lack of other tangible results to the fact that relatively little research has been devoted to the modern approach; compared with modern control theory, modern network theory is in its infancy, or, at latest, its early adolescence. We must wait to see the payoffs that maturity will bring.

1.3 OUTLINE OF THE BOOK

Besides the introductory Part I, the book falls into five parts. Part II is aimed at providing background in two areas, the first being m-port networks and means for describing them, and the second being state-space equations and their relation with transfer-function matrices. In Chapter 2, which discusses m-port networks, we deal with classes of circuit elements, such network properties as passivity, losslessness, and reciprocity, the immittance, hybrid and scattering-matrix descriptions of a network, as well as some important network interconnections. In Chapter 3 we discuss the description of lumped systems by state-space equations, solution of state-space equations, such properties as controllability, observability, and stability, and the relation of state descriptions to transfer-function-matrix descriptions.

Part III, consisting of one long chapter, discusses network analysis via state-space procedures. We discuss three procedures for analysis of passive networks, of increasing degree of complexity and generality, as well as analysis of active networks. This material is presented without significant use of network topology.

Part IV is concerned with translating into state-space terms the notions of passivity and reciprocity. Chapter 5 discusses a basic result of modern system theory, which we term the positive real lemma. It is of fundamental importance in areas such as nonlinear system stability and optimal control, as well as in network theory; Chapter 6 is concerned with developing procedures for solving equations that appear in the positive real lemma. Chapter 7 covers two matters; one is the bounded real lemma, a first cousin to the positive real lemma, and the other is the state-space description of the reciprocity property, first introduced in Chapter 2.

Part V is concerned with passive network synthesis and relies heavily on the positive real lemma material of Part IV. Chapter 8 introduces the general approaches to synthesis and disposes of some essential preliminaries. Chapters 9 and 10 cover impedance synthesis and reciprocal impedance synthesis, respectively. Chapter 11 deals with scattering-matrix synthesis, and Chapter 12 with transfer-function synthesis.

Part VI comprises one chapter and deals with active *RC* synthesis, i.e., synthesis using active elements, resistors, and capacitors. As with the earlier part of the book, state-space methods alone are considered.

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Part II

BACKGROUND INFORMATION— NETWORKS AND STATE-SPACE EQUATIONS

In this part our main concern is to lay groundwork for the real meat of the book, which occurs in later parts. In particular, we introduce the notion of multiport networks, and we review the notion of state-space equations and their connection with transfer-function matrices. Almost certainly, the reader will have had exposure to many of the concepts touched upon in this part, and, accordingly, the material is presented in a reasonably terse fashion.

2

m-Port Networks and their Port Descriptions

2.1 INTRODUCTION

The main aim of this chapter is to define what is meant by a network and especially to define the subclass of networks that we shall be interested in from the viewpoint of synthesis. This requires us to list the the types of permitted circuit elements that can appear in the networks of interest, and to note the existence of various network descriptions, principally port descriptions by transfer-function matrices.

We shall define the important notions of *passivity*, *losslessness*, and *reciprocity* for circuit elements and for networks. At the same time, we shall relate these notions to properties of port descriptions of networks.

Section 2.2 of the chapter is devoted to giving basic definitions of such notions as m-port networks, circuit elements, sources, passivity, and losslessness. An axiomatic introduction to these concepts may be found in [1]. Section 2.3 states Tellegen's theorem, which proves a useful device in interpreting what effect constraints on circuit elements (such as passivity and, later, reciprocity) have on the behavior of the network as observed at the ports of the network. This material can be found in various texts, e.g., [2, 3]. In Section 2.4 we consider various port descriptions of networks using transfer-function matrices, and we exhibit various interrelations among them where applicable. Next, some commonly encountered methods of connecting networks to form a more complex network are discussed in Section 2.5. In Sections 2.6 and 2.7 we introduce the definitions of the *bounded real* and *positive real* properties of transfer-function matrices, and we relate these properties to port descriptions of networks. Specializations of the bounded real and positive real properties are discussed, viz., the *lossless bounded real* and *lossless positive real* properties, and the subclass of networks whose transfer-function matrices have these properties is stated. In Section 2.8 we define the notion of *reciprocity* and consider the property possessed by port descriptions of a network when the network is composed entirely of reciprocal circuit elements. Much of the material from Section 2.4 on will be found in one of [1–3].

We might summarize what we plan to do in this chapter in two statements:

1. We shall offer various port descriptions of a network as alternatives to a network description consisting of a list of elements and a scheme for interconnecting the elements.
2. We shall translate constraints on individual components of a network into constraints on the port descriptions of the network.

2.2 m-PORT NETWORKS AND CIRCUIT ELEMENTS

Multiport Networks

An *m-port network* is a physical device consisting of a collection of circuit elements or components that are connected according to some scheme. Associated with the m-port network are access points called terminals, which are paired to form ports. At each port of the network it is possible to connect other circuit elements, the port of another network, or some kind of exciting device, itself possessing two terminals. In general, there will be a voltage across each terminal pair, and a current leaving one terminal of the pair making up a port must equal the current entering the other terminal of the pair. Thus if the jth port is defined by terminals T_j and T'_j, then for each j, two variables, v_j and i_j, may be assigned to represent the voltage of T_j with respect to T'_j and the current entering T_j and leaving T'_j, respectively, as illustrated in Fig. 2.2.1a. A simplified representation is shown in Fig. 2.2.1b. Thus associated with the m-port network are two vector functions of time, the (vector) port voltage $v = [v_1 \quad v_2 \cdots v_m]'$ and the (vector) port current $i = [i_1 \quad i_2 \cdots i_m]'$. The physical structure of the m port will generally constrain, often in a very general way, the two vector variables v and i, and conversely the constraints on v and i serve to completely describe the externally observable behavior of the m port.

In following through the above remarks, the reader is asked to note two important points:

1. Given a network with a list of terminals rather than ports, it is not permissible to combine pairs of terminals and call the pairs ports *unless*

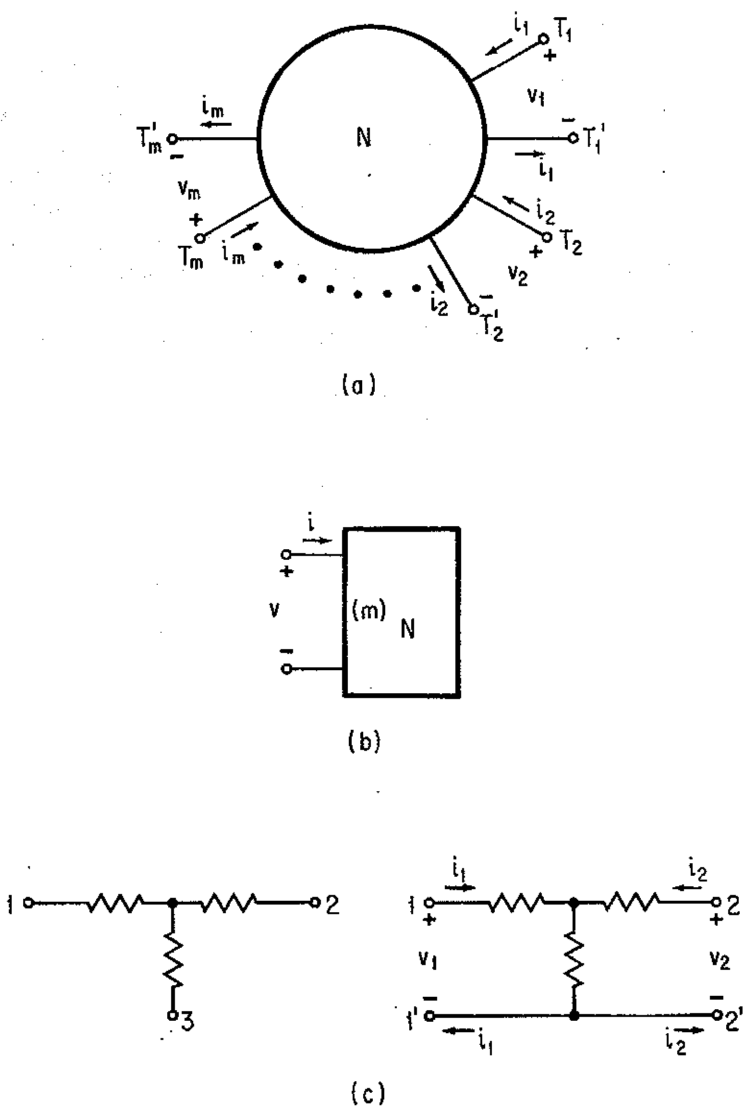


FIGURE 2.2.1. Networks with Associated Ports.

under all operating conditions the current entering one terminal of the pair equals that leaving the other terminal.

2. Given a network with a list of terminals rather than ports, if one properly constructs ports by selecting terminal pairs and appropriately constraining the excitations, it is quite permissible to include one terminal in two port pairs (see Fig. 2.2.1c for an example of a three-terminal network redrawn as a two-port network).

Elements of a Network

Circuit elements of interest here are listed in Fig. 2.2.2, where their precise definitions are given. (An additional circuit element, a generalization of the two-port transformer, will be defined shortly.) Interconnections of these elements provide the networks to which we shall devote most attention. Note that each element may also be viewed as a simple network in its own right. For example, the simple one-port network of Fig. 2.2.2a, the resistor, is described by a voltage v, a current i, and the relation $v = ri$, with i or v arbitrarily chosen.

The only possibly unfamiliar element in the list of circuit elements is the

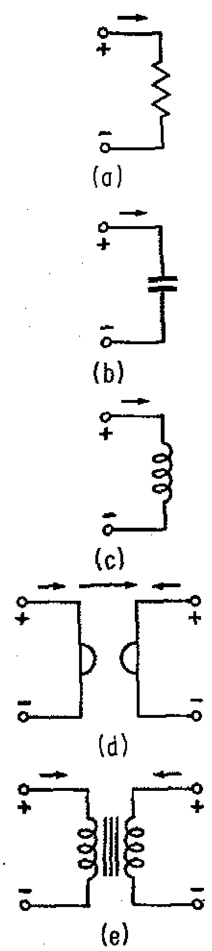


FIGURE 2.2.2. Representations and Definitions of Basic Circuit Elements.

gyrator. At audio frequencies the gyrator is very much an idealized sort of component, since it may only be constructed from resistor, capacitor, and transistor or other active components, in the sense that a two-port network using these types of elements may be built having a performance very close to that of the ideal component of Fig. 2.2.2d. In contrast, at microwave frequencies gyrators can be constructed that do not involve the use of active elements (and thus of an external power supply) for their operation. (They do however require a permanent magnetic field.)

Passive Circuit Elements

In the sequel we shall be concerned almost exclusively with circuit elements with constant element values, and, in the case of resistor, inductor, and capacitor components, with nonnegative element values. The class of all such elements (including transformer and gyrator elements) will be termed *linear, lumped, time invariant*, and *passive*. The term linear arises from the fact that the port variables are constrained by a linear relation; the word lumped arises because the port variables are constrained either via a memoryless transformation or an ordinary differential equation (as opposed to a partial differential equation or an ordinary differential equation with delay); the term time invariant arises because the element values are constant. The reason for use of the term passive is not quite so transparent. Passivity of a component is defined as follows:

> Suppose that a component contains no stored energy at some arbitrary time t_0. Then the total energy delivered to the component from any generating source connected to the component, computed over any time interval $[t_0, T]$, is always nonnegative.

Since the instantaneous power flow into a terminal pair at time t is $v(t)i(t)$, sign conventions being as in Fig. 2.2.1, passivity requires for the one-port components shown in Fig. 2.2.2 that

$$\mathcal{E}(T) = \int_{t_0}^{T} v(t)i(t)\,dt \geq 0 \tag{2.2.1}$$

for all initial times t_0, all $T \geq t_0$, and all possible voltage–current pairs satisfying the constraints imposed by the component. It is easy to check (and checking is requested in the problems) that with real constant r, l, and c, (2.2.1) is fulfilled *if and only if* r, l, and c are nonnegative.

For a multiport circuit element, i.e., the two-port transformer and gyrator already defined or the multiport transformer yet to be defined, (2.2.1) is replaced by

$$\mathcal{E}(T) = \int_{t_0}^{T} v'(t)i(t)\,dt \geq 0 \tag{2.2.2}$$

Notice that $v'(t)i(t)$, being $\sum v_j(t)i_j(t)$, represents the instantaneous power flow into the element, being computed by summing the power flows at each port.

For both the transformer and gyrator, the relation $v'i = v_1 i_1 + v_2 i_2 = 0$ holds for all t and all possible excitations. This means that it is never possible for there to be a net flow of power into a transformer and gyrator, and, therefore, that it is never possible for there to be a nonzero value of stored energy. Hence (2.2.2) is satisfied with $\mathcal{E}(T)$ identically zero.

Lossless Circuit Elements

A concept related to that of passivity is losslessness. Roughly speaking, a circuit element is lossless if it is passive and if, when a finite amount of energy is put into the element, all the energy can be extracted again. More precisely, losslessness requires passivity and, assuming zero excitation at time t_0,

$$\mathcal{E}(\infty) = \int_{t_0}^{\infty} v(t)i(t)\,dt = 0 \tag{2.2.3a}$$

or, for a multiport circuit element,

$$\mathcal{E}(\infty) = \int_{t_0}^{\infty} v'(t)i(t)\,dt = 0 \tag{2.2.3b}$$

for all compatible pairs $v(\cdot)$ and $i(\cdot)$, *which are also square integrable*; i.e.,

$$\int_{t_0}^{\infty} v'(t)v(t)\,dt < \infty \qquad \int_{t_0}^{\infty} i'(t)i(t)\,dt < \infty \tag{2.2.4}$$

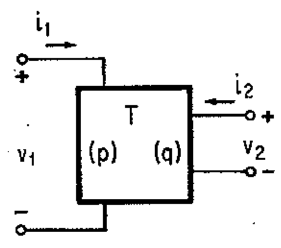
The gyrator and transformer are lossless for the reason that $\mathcal{E}(T) = 0$ for all T, as noted above, independently of the excitation. The capacitor and inductor are also lossless—a proof is called for in the problems.

In constructing our m-port networks, we shall restrict the number of elements to being finite. *An m-port network comprised of a finite number of linear, lumped, time-invariant, and passive components will be called a finite, linear, lumped, time-invariant, passive m port; unless otherwise stated, we shall simply call such a network an m port. An m port containing only lossless components will be called a lossless m port.*

The Multiport Transformer

A generalization is now presented of the ideal two-port transformer, viz., the *ideal multiport transformer*, introduced by Belevitch [4]. It will be seen later in our discussion of synthesis procedures that the multiport transformer indeed plays a major role in almost all synthesis methods that we shall discuss.

Consider the $(p+q)$-port transformer N_T shown in Fig. 2.2.3. Figure 2.2.3a shows a convenient symbolic representation that can frequently replace the more detailed arrangement of Fig. 2.2.3b. In this figure are depicted secondary currents and voltages $(i_2)_1, (i_2)_2, \ldots, (i_2)_q$ and $(v_2)_1, \ldots, (v_2)_q$ at the q secondary ports, primary currents $(i_1)_1, (i_1)_2, \ldots, (i_1)_p$ at the p primary ports, and one of the primary voltages $(v_1)_1$. (The definition of the other



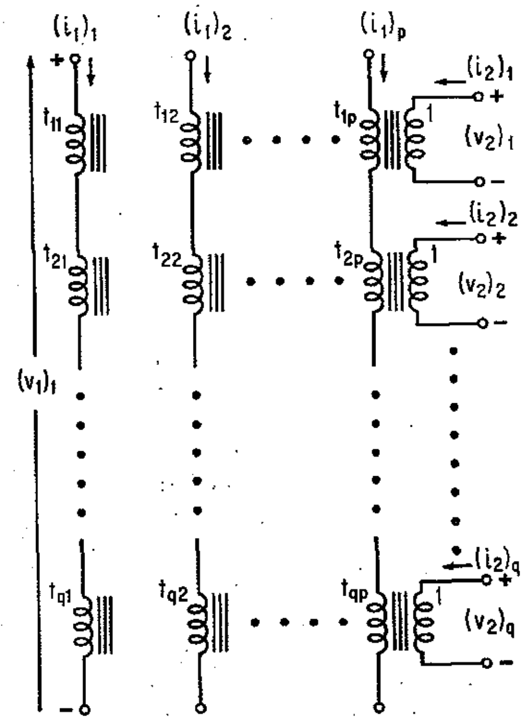


FIGURE 2.2.3. The Multiport Ideal Transformer.

primary voltages is clear from the figure.) The symbols t_{ij} denote turns ratios, and the figure is meant to depict the relations

$$(v_1)_i = \sum_{j=1}^{q} t_{ji}(v_2)_j \qquad i = 1, 2, \ldots, p$$

and

$$(i_2)_i = -\sum_{j=1}^{p} t_{ij}(i_1)_j \qquad i = 1, 2, \ldots, q$$

or, in matrix notation

$$v_1 = T'v_2 \qquad i_2 = -Ti_1 \tag{2.2.5}$$

where $T = [t_{ij}]$ is the $q \times p$ *turns-ratio matrix*, and v_1, v_2, i_1, and i_2 are vectors representing, respectively, the primary voltage, secondary voltage, primary current, and secondary current. The ideal multiport transformer is a lossless circuit element (see Problem 2.2.1), and, as is reasonable, is the same as the two-port transformer already defined if $p = q = 1$.

Generating Sources

Frequently, at each port of an m-port network a *voltage source* or a *current source* is connected. These are shown symbolically in Fig. 2.2.4.

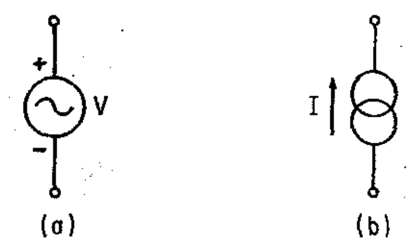


FIGURE 2.2.4. Independent Voltage and Current Sources.

The voltage source has the characteristic that the voltage across its terminals takes on a specified value or is a specified function of time independent of the current flowing through the source; similarly with a current source, the current entering and leaving the source is fixed as a specified value or specified function of time for all voltages across the source. In the sense that the terminal voltage for a voltage source and the current entering and leaving the terminal pair for a current source are invariant and are independent of the rest of the network, these sources are termed ***independent sources.***

If, for example, an independent voltage source V is connected to a terminal pair T_i and T_i' of an m port, then the voltage v_i across the ith port is constrained to be V or $-V$ depending on the polarity of the source. The two situations are depicted in Fig. 2.2.5.

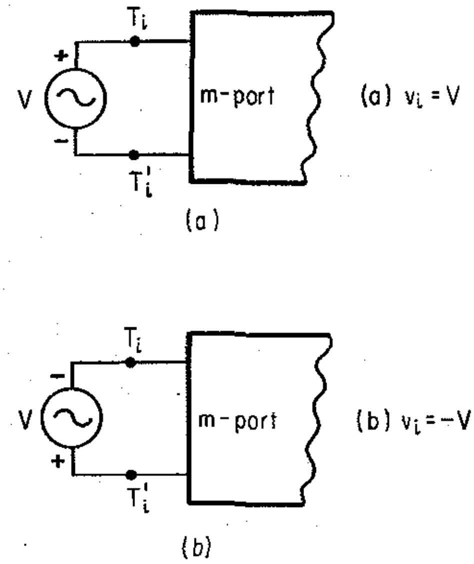


FIGURE 2.2.5. Connection of an Independent Voltage Source.

In contrast, there exists another class of sources, *dependent* or *controlled* voltage and current sources, *the values of voltage or current of which depend on one or more of the variables of the network under consideration.* Dependent sources are generally found in active networks—in fact, dependent sources are the basic elements of active device modeling. A commonly found example is the hybrid-pi model of a transistor in the common-emitter configuration, as shown in Fig. 2.2.6. The controlled current source is made to depend on $V_{b'e}$, the voltage across terminals B' and E. Basically, we may consider four types of controlled sources: a voltage-controlled voltage source,

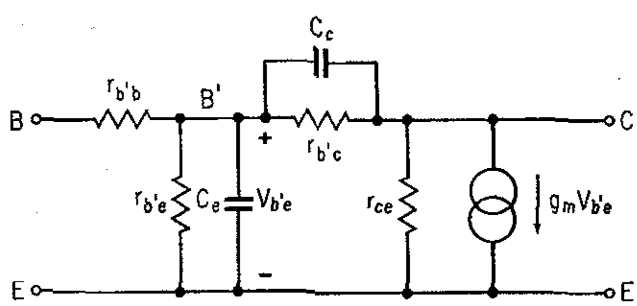


FIGURE 2.2.6. The Common Emitter Hybrid—pi Model of a Transistor.

a current-controlled voltage source, a voltage-controlled current source, and a current-controlled current source.

In the usual problems we consider, we hope that if a source is connected to a network, there will result voltages and currents in the network and at its ports that are well-defined functions of time. This may not always be possible. For example, if a one-port network consists of simply a short circuit, connection of an arbitrary voltage source at the network's port will not result in a well-defined current. Again, if a one-port network consists of simply a 1-henry (H) inductor, connection of a current source with the current a nondifferentiable function of time will not lead to a well-behaved voltage. Yet another example is provided by connecting at time $t_0 = -\infty$ a constant current source to a 1-farad (F) capacitor; at any finite time the voltage will be infinite. The first sort of difficulty is so basic that one cannot avoid it unless application of one class of sources is simply disallowed. The second difficulty can be avoided by assuming that *all excitations are suitably smooth*, while the third sort of difficulty can be avoided by assuming that *all excitations are first applied at some finite time* t_0; i.e., prior to t_0 all excitations are zero. We shall make no further explicit mention of these last two assumptions unless required by special circumstances; they will be assumed to apply at all times unless statements to the contrary are made.

Problem 2.2.1 Show that the linear time-invariant resistor, capacitor, and inductor are passive if the element values are nonnegative, and that the capacitor, inductor, and multiport transformer are lossless.

Problem 2.2.2 A time-variable capacitor $c(\cdot)$ constrains its voltage and current by $i(t) = (d/dt)[c(t)v(t)]$. What are necessary and sufficient conditions on $c(\cdot)$ for the capacitor to be (1) passive, and (2) lossless?

Problem 2.2.3 Establish the equivalents given in Fig. 2.2.7.

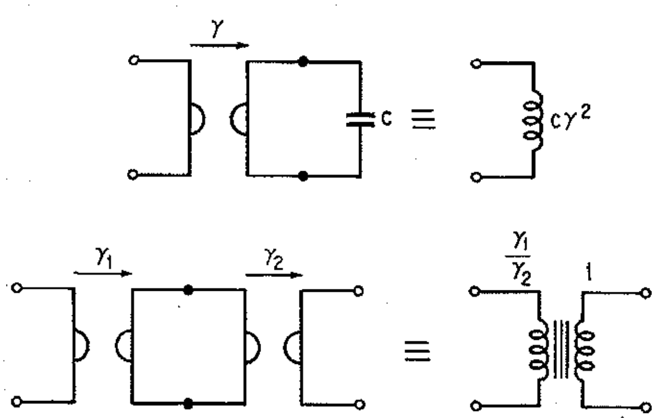


FIGURE 2.2.7. Some Equivalent Networks

Problem 2.2.4 Devise a simple network employing a controlled source that simulates a negative resistance (one whose element value is negative).

Problem 2.2.5 Figure 2.2.8a shows a one-port device called the nullator, while Fig. 2.2.8b depicts the norator. Show that the nullator has simultaneously zero current and zero voltage at its terminals, and that the norator can have arbitrary and independent current and voltage at its terminals.

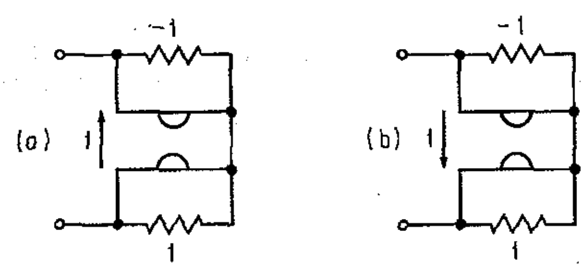


FIGURE 2.2.8. (a) The Nullator; and (b) the Norator.

2.3 TELLEGEN'S THEOREM—PASSIVE AND LOSSLESS NETWORKS

Passivity and Losslessness of m ports

The definitions of passivity and losslessness given in the last section were applied to circuit elements only; in this section we shall note the application of these definitions to m-port networks. We shall also introduce an important theorem that will permit us to connect circuit element properties, including passivity, with network properties.

The definition of passivity for an m-port network is straightforward.

An m-port network, assumed to be storing no energy* at time t_0, is said to be passive if

$$\mathcal{E}(T) = \int_{t_0}^{T} v'(t)i(t)\,dt \geq 0 \tag{2.3.1}$$

for all t_0, T, and all port voltage vectors $v(\cdot)$ and current vectors $i(\cdot)$ satisfying constraints imposed by the network.

An m port is said to be active if it is not passive, while the losslessness property is defined as follows.

An m-port network, assumed to be storing no energy at time t_0, is said to be lossless if it is passive and if

*A network is said to be storing no energy at time t_0 if none of its elements is storing energy at time t_0.

$$\mathcal{E}(\infty) = \int_{t_0}^{\infty} v'(t)i(t)\,dt = 0 \tag{2.3.2}$$

for all t_0 and all port voltage vectors $v(\cdot)$ and current vectors $i(\cdot)$ satisfying constraints imposed by the network together with

$$\int_{t_0}^{\infty} v'(t)v(t)\,dt < \infty \qquad \int_{t_0}^{\infty} i'(t)i(t)\,dt < \infty$$

Intuitively, we would feel that a network all of whose circuit elements are passive should itself be passive; likewise, a network all of whose circuit elements are lossless should also be lossless. These intuitive feelings are certainly correct, and as our tool to verify them we shall make use of *Tellegen's theorem.*

Tellegen's Theorem

The theorem is one of the most general results of all circuit theory, as it applies to virtually any kind of lumped network. Linearity, time invariance, and passivity are not required of the elements; there is, though, a requirement that the number of elements be finite. Roughly speaking, the reason the theorem is valid for such a large class of networks is that it depends on the validity of two laws applicable to all such networks—viz., Kirchhoff's current law and Kirchhoff's voltage law. We shall first state the theorem, then make brief comments concerning the theorem statement, and finally we shall prove the theorem. The statement of the theorem requires an understanding of the concepts of a *graph* of a network, and the *branches, nodes,* and *loops* of a graph. Kirchhoff's current law requires that the sum of the currents in *all* branches incident on *any one* node, with positive direction of current entering the node, be zero. Kirchhoff's voltage law requires that the sum of the voltages in all branches forming a *loop*, with consistent definition of signs, be zero.

Tellegen's Theorem. Suppose that N is a lumped finite network with b branches and n nodes. For the kth branch of the graph, suppose that v_k is the branch voltage under one set of operating conditions* at any one instant of time, and i_k the branch current under any other set of operating conditions at any other instant of time, with the sign convention for v_k and i_k being as shown

*By the term *set of operating conditions* we mean the set of voltages and currents in the various circuit elements of the network, arising from certain source voltages and currents and certain initial stored energies in the energy storage elements. Different operating conditions will result from different choices of these quantities. In the main theorem statement it is not assumed that element values can be varied, though a problem does extend the theorem to this case.

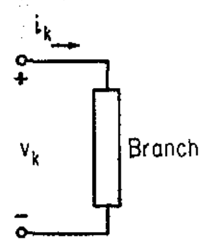


FIGURE 2.3.1. Reference Directions for Tellegen's Theorem.

in Fig. 2.3.1. Assuming the validity of the two Kirchhoff laws, then

$$\sum_{k=1}^{b} v_k i_k = 0 \tag{2.3.3}$$

We ask the reader to note carefully the following points.

1. The theorem assumes that the network N is representable by a graph, with branches and nodes. Certainly, if N contains only inductor, resistor, and capacitor elements, this is so. But it remains so if N contains transformers or gyrators; Fig. 2.3.2 shows, for example, how a transformer can be drawn as a network containing "simple" branches.

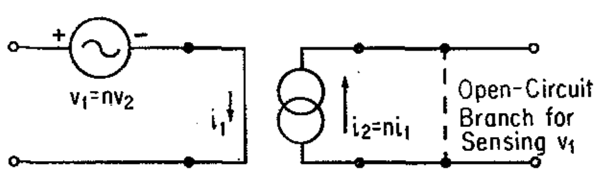


FIGURE 2.3.2. Redrawing of a Transformer.

2. N can contain sources, controlled or independent. These are treated just like any other element.
3. Most importantly, v_k and i_k are *not necessarily* determined under the same set of operating conditions—though they may be.

Proof. We can suppose, without loss of generality, that between every pair of nodes of the graph of N there is one and only one branch. For if there is no branch, we can introduce one on the understanding that under all operating conditions there will be zero current through it, and if there is more than one branch, we can replace this by a single branch with the current under any operating condition equal to the sum of the currents through the separate branches.

Denote the node voltage of the αth node by V_α, and the current from node α to node β by $I_{\alpha\beta}$. If the kth branch connects node α to node β, then

$$v_k = V_\alpha - V_\beta$$

relates the branch voltage with the node voltages under one set of operating conditions, and

$$i_k = I_{\alpha\beta}$$

relates the branch current to the current flowing from node α to node β under some other operating conditions (note that these equations are consistent with the sign convention of Fig. 2.3.1).

It follows that

$$\begin{aligned} v_k i_k &= (V_\alpha - V_\beta)I_{\alpha\beta} \\ &= (V_\beta - V_\alpha)I_{\beta\alpha} \\ &= \tfrac{1}{2}[(V_\alpha - V_\beta)I_{\alpha\beta} + (V_\beta - V_\alpha)I_{\beta\alpha}] \end{aligned}$$

Note that for every node pair α, β there will be one and only one branch k such that the above equation holds and, conversely, for every branch k there will be one and only one node pair α, β such that the equation holds. This means that if we sum the left side over all branches, we should sum the right side over all possible node pairs; i.e.,

$$\begin{aligned} \sum_{k=1}^{b} v_k i_k &= \sum_{\substack{\text{all pairs} \\ \text{of nodes}}} \tfrac{1}{2}[(V_\alpha - V_\beta)I_{\alpha\beta} + (V_\beta - V_\alpha)I_{\beta\alpha}] \\ &= \tfrac{1}{2}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}[(V_\alpha - V_\beta)I_{\alpha\beta}] \end{aligned}$$

(Why is the right-hand side not twice the quantity shown?) Now we have

$$\sum_{k=1}^{b} v_k i_k = \tfrac{1}{2}\sum_{\alpha=1}^{n} V_\alpha\left(\sum_{\beta=1}^{n} I_{\alpha\beta}\right) - \tfrac{1}{2}\sum_{\beta=1}^{n} V_\beta\left(\sum_{\alpha=1}^{n} I_{\alpha\beta}\right)$$

For fixed α, $\sum_{\beta=1}^{n} I_{\alpha\beta}$ is the sum of all currents leaving node α. It is therefore zero. Likewise, $\sum_{\alpha=1}^{n} I_{\alpha\beta} = 0$. Thus

$$\sum_{k=1}^{b} v_k i_k = 0 \qquad \nabla\nabla\nabla^*$$

*The symbol $\nabla\nabla\nabla$ will denote the end of the proof of a theorem or lemma.

The reader may wonder where the Kirchhoff voltage law was used in the above proof; actually, there was a use, albeit a very minor one. In setting $v_k = V_\alpha - V_\beta$, where branch k connects nodes α and β, we were making use of a very elementary form of the law.

Example 2.3.1 Just to convince the reader that the theorem really does work, we shall consider a very simple example, illustrated in Fig. 2.3.3. A circuit is

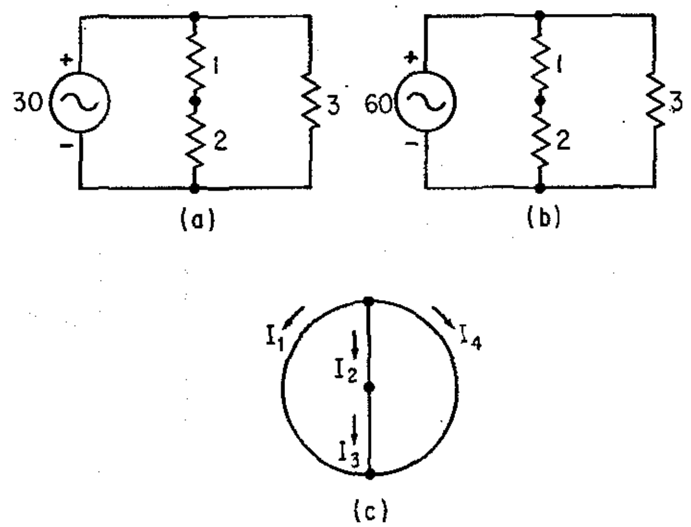


FIGURE 2.3.3. Example of Tellegen's Theorem.

shown under two conditions of excitation in Fig. 2.3.3a and b, with its branches and reference directions for current in Fig. 2.3.3c. The reference directions for voltage are automatically determined.

For the arrangement of Fig. 2.3.3a and b

$V_1 = 30$	$I_1 = -40$	$V_1I_1 = -1200$
$V_2 = 10$	$I_2 = 20$	$V_2I_2 = 200$
$V_3 = 20$	$I_3 = 20$	$V_3I_3 = 400$
$V_4 = 30$	$I_4 = 20$	$V_4I_4 = 600$
		$\sum V_kI_k = 0$

Passive Networks and Passive Circuit Elements

Tellegen's theorem yields a very simple proof of the fact that a finite number of *passive elements* when interconnected yields a *passive network*. (Recall that the term *passive* has been used hitherto in two *independent* contexts; we can now justify this double use.) Consider a network N_2 comprising an m port N_1 with m sources at each port (see Fig. 2.3.4).

Whether the sources are current or voltage sources is irrelevant. We shall argue that if the elements of N_1 are passive, then N_1 itself is passive. Let us number the branches of N_2 from 1 to k, with the sources being numbered as branches 1 through m. Voltage and current reference directions are chosen for the sources as shown in Fig. 2.3.4, and for the remaining branch elements

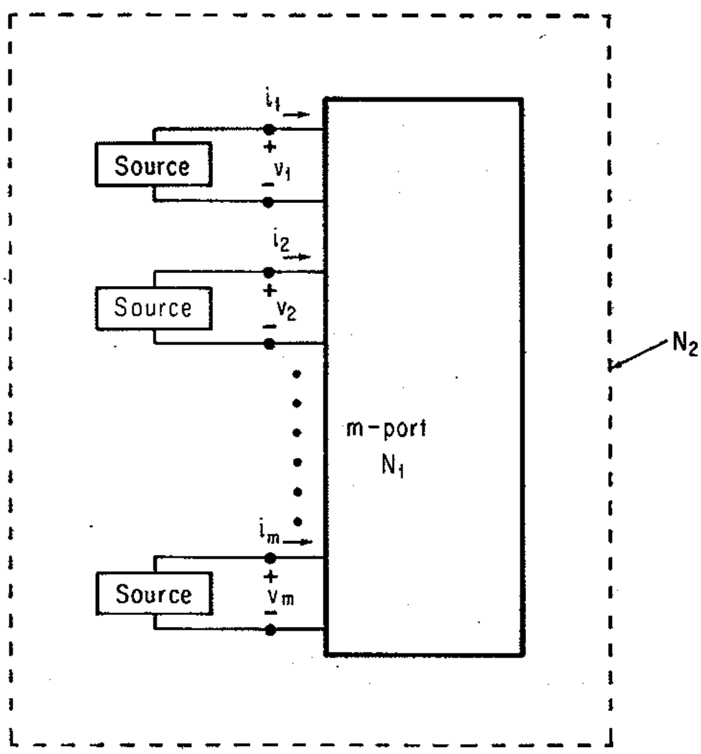


FIGURE 2.3.4. Passivity of N_1 is Related to Passivity of Elements of N_1.

of N_2, i.e., for the branch elements of N_1, in accordance with the usual convention required for application of Tellegen's theorem. (Note that the sign convention for the source currents and voltages differs from that required for application of Tellegen's theorem, but agrees with that adopted for port voltages and currents.)

Tellegen's theorem now yields that

$$-\sum_{j=1}^{m} v_j(t)i_j(t) + \sum_{j=m+1}^{k} v_j(t)i_j(t) = 0$$

or

$$\sum_{j=1}^{m} v_j(t)i_j(t) = \sum_{j=m+1}^{k} v_j(t)i_j(t)$$

Taking into account the sign conventions, we see that the left side represents the instantaneous power flow into N_1 at time t through its ports, while $v_j(t)i_j(t)$ for each branch of N_1 represents the instantaneous power flow into that branch. Thus with v the port voltage vector and i the port current vector

$$v'i = \sum_{j=m+1}^{k} v_j(t)i_j(t)$$

and

$$\int_{t_0}^{T} v'i\,dt \geq 0 \tag{2.3.4}$$

by the passivity of the m-port circuit elements, provided these elements are all storing no energy at time t_0. Equation (2.3.4) of course holds for all such t_0, all T, and all possible excitations of the m port; it is the equation that defines the passivity of the m port.

In a similar fashion to the above argument, one can argue that an m port consisting entirely of lossless circuit elements is lossless. (*Note:* The word lossless has hitherto been applied to two different entities, circuit elements and m ports.) There is a minor technical difficulty in the argument however; this is to show that if the port voltage v and current i are square integrable, the same is true of each branch voltage and current.* We shall not pursue this matter further at this point, but shall accept that the two uses of the word lossless can be satisfactorily tied together.

Problem 2.3.1 Consider an m-port network driven by m independent sinusoidal sources. Describe each branch voltage and current by a phasor, a complex number with amplitude equal to $1/\sqrt{2}$ times the amplitude of the sine wave variation of the quantity, and with phase equal to the phase, relative to a reference phase, of the sine wave. Then for branch k, $V_k I_k^*$ represents the complex power flowing into the branch. Show that the sum of the complex powers delivered by the sources to the m port is equal to the sum of the complex powers received by all branches of the network.

Problem 2.3.2 Let N_1 and N_2 be two networks with the same graph. In each network choose reference directions for the voltage and current of each branch in the same way. Let v_{ki} and i_{ki} be the voltage and current in branch k of network i for $i = 1, 2$, and assume that the two Kirchhoff laws are valid. Show that

$$\sum_k v_{k1}i_{k2} = \sum_k v_{k2}i_{k1} = 0$$

*Actually, the time invariance of the circuit elements is needed to prove this point, and counterexamples can be found if the circuit elements are permitted to be time varying. This means that interconnections of lossless time-varying elements need not be lossless [14].

2.4 IMPEDANCE, ADMITTANCE, HYBRID, AND SCATTERING DESCRIPTIONS OF A NETWORK

Since the m-port networks under consideration are time invariant, they may be described in the frequency domain—a standard technique used in all classical network theory. Thus instead of working in terms of the vector functions of time $v(\cdot)$ and $i(\cdot)$, the Laplace transforms of these quantities, $V(s)$ and $I(s)$, may be used, the elements of $V(s)$ and $I(s)$ being functions of the complex variable $s = \sigma + j\omega$.

Impedance and Admittance Descriptions

Now suppose that for a certain m port it is possible to connect arbitrary current sources at each of the ports and obtain a well-defined voltage response at each port. (This will not always be the case of course—for example, a one-port consisting simply of an open circuit does not have the property that an arbitrary current source can be applied to it.) In case connection of arbitrary current sources is possible, we can conceive of the port current vector $I(s)$ as being an independent variable, and the port voltage vector $V(s)$ as a dependent variable. Then network analysis provides procedures for determining an $m \times m$ matrix $Z(s) = [z_{ij}(s)]$ that maps $I(s)$ into $V(s)$ through

$$V(s) = Z(s)I(s) \tag{2.4.1}$$

[Subsequently, we shall investigate procedures using *state-space equations* for the determination of $Z(s)$.] We call $Z(\cdot)$ the *impedance matrix* of N, and say that N possesses an impedance description. Conversely, if it is possible to take $V(s)$ as the independent variable, i.e., if there exists a well-defined set of port currents for an arbitrary selection of port voltages, N possesses an admittance description, and there exists an admittance matrix $Y(s)$ relating $I(s)$ and $V(s)$ by

$$I(s) = Y(s)V(s) \tag{2.4.2}$$

As elementary network analysis shows, and as we shall see subsequently in discussing the analysis of networks via state-space ideas, the impedance and admittance matrices of m ports (if they exist) are $m \times m$ matrices of real rational functions of s. If both $Y(s)$ and $Z(s)$ exist, then $Y(s)Z(s) = I$, as is clear from (2.4.1) and (2.4.2). The term *immittance matrix* is often used to denote either an impedance or an admittance matrix. Physically, the (i, j) element $z_{ij}(s)$ of $Z(s)$ represents the Laplace transform of the voltage appearing at the ith port when a unit impulse of current is applied at port j, with all other ports open circuited to make $I_k(s) = 0$, $k \neq j$. A dual interpretation can be made for $y_{ij}(s)$.

Hybrid Descriptions

There are situations when one or both of $Z(s)$ and $Y(s)$ do not exist. This means that one cannot, under these circumstances, choose the excitations to be all independent voltages or all independent currents. For example, an open circuit possesses no impedance matrix, but does possess an admittance matrix—a scalar $Y(s) = 0$. Conversely, the short circuit possesses no admittance matrix, but does possess an impedance matrix. As shown in Example 2.4.1, a simple two-port transformer possesses neither an impedance matrix nor an admittance matrix. *But for any m port, it can be shown* [5, 6] *that there always exists at least one set of independent excitations, described by* $U(s)$, *where* $u_i(s)$ *may be the* ith *port voltage or the* ith *port current, such that the corresponding responses are well defined.* Of course, it is understood that the network consists of only a finite number of passive, linear, time-invariant resistors, inductors, capacitors, transformers, and gyrators. With $r_i(s)$ denoting the ith port current when $u_i(s)$ denotes the ith port voltage, or conversely, there exists a matrix $H(s) = [h_{ij}(s)]$ such that

$$R(s) = H(s)U(s) \tag{2.4.3}$$

Such a matrix $H(s)$ is called a *hybrid matrix*, and its elements are again rational fuctions in s with real coefficients. It is intuitively clear that a network in general may possess several alternative hybrid matrices; however, any one describes completely the externally observable behavior of the network.

Example 2.4.1 Consider the two-port transformer of Fig. 2.2.2e. By inspection of its defining equations

$$v_1 = Tv_2 \tag{2.4.4a}$$

$$i_2 = -Ti_1 \tag{2.4.4b}$$

we can see immediately that the port currents i_1 and i_2 cannot be chosen independently. Here, according to (2.4.4b), the admissible currents are those for which $i_2 = -Ti_1$. Consequently, the transformer does not possess an impedance-matrix description. Similarly, by looking at (2.4.4a), we see that it does not possess an admittance-matrix description. These conclusions may alternatively be deduced on rewriting (2.4.4) in the frequency domain as

$$\begin{bmatrix} 1 & -T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ T & 1 \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} \tag{2.4.5}$$

from which it is clear that an equation of the form of (2.4.1) or (2.4.2) cannot be obtained.

However, the two-port transformer has a hybrid-matrix description, as may be seen by rearranging (2.4.5) in the form

$$\begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} 0 & T \\ -T & 0 \end{bmatrix} \begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix} \tag{2.4.6}$$

Scattering Descriptions

Another port description of networks commonly used in network theory is the *scattering matrix* $S(s)$. In the field of microwave systems, scattering matrices actually provide the most natural and convenient way of describing networks, which usually consist of *distributed* as well as lumped elements.*

To define $S(s)$ for an m port N, consider the *augmented m-port network* of Fig. 2.4.1, formed from the originally given m port N by adding unit

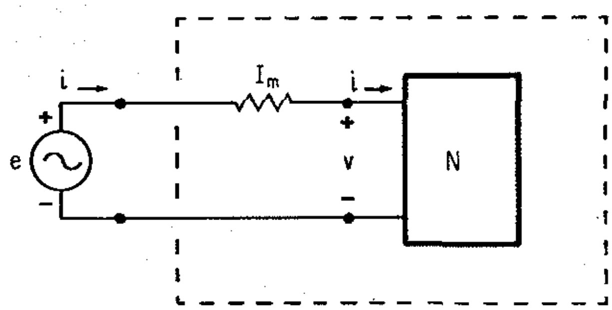


FIGURE 2.4.1. The Augmented m-Port Network.

resistors in series with each of its m ports.† We may excite the augmented network with voltage sources designated by e. It is straightforward to deduce from Fig. 2.4.1 that these sources are related to the original port voltage and current of N via

$$e(t) = v(t) + i(t)$$

We shall suppose for the moment that $e(\cdot)$ can be arbitrary, and any response of interest is well defined. Now the most obvious physical response to the voltage sources $[e_j]$ is the currents flowing through these sources, which are

*A well-known example of a distributed element is a transmission line.

†In this figure the symbol I_m denotes the $m \times m$ unit matrix. Whether I denotes a current in the Laplace-transform domain or the unit matrix should be quite clear from the context.

the entries of the port current vector i. But since

$$e - (v - i) = 2i$$

from the mathematical point of view we may as well use $v - i$ instead as the response vector. So for N we can think mathematically of $v + i$ as the excitation vector and $v - i$ as the response vector, with the understanding that physically this excitation may be applied by identifying e in Fig. 2.4.1 with $v + i$ and taking $e - 2i$ as the response. Actually, excitation and response vectors of $\frac{1}{2}(v + i)$ and $\frac{1}{2}(v - i)$ are more commonly adopted, these two quantities often being termed the ***incident voltage*** v^i and ***reflected voltage*** v^r. Then, with the following definition in the frequency domain,

$$2V^i = V + I = E \qquad 2V^r = V - I \tag{2.4.7}$$

we define the scattering matrix $S(s) = [s_{ij}(s)]$, an $m \times m$ matrix of real rational functions of s, such that

$$V^r(s) = S(s)V^i(s) \tag{2.4.8}$$

The matrix $S(s)$ so defined is sometimes called the ***normalized*** scattering matrix, since the augmenting resistances are of unit value.

Example 2.4.2 Consider once again the two-port transformer in Fig. 2.2.2e; it is simple to calculate its scattering matrix using (2.4.7) and (2.4.8); the end result is

$$S = \begin{bmatrix} \dfrac{T^2 - 1}{1 + T^2} & \dfrac{2T}{1 + T^2} \\ \dfrac{2T}{1 + T^2} & \dfrac{1 - T^2}{1 + T^2} \end{bmatrix} \tag{2.4.9}$$

The above example also illustrates an important result in network theory that ***although an immittance-matrix description may not necessarily exist for an m-port network, a scattering-matrix description does always exist*** [6]. (A sketch of the proof will be given shortly.) For this reason, scattering matrices are often preferred in theoretical work.

Since it is sometimes necessary to convert from S to Y or Z, or vice versa, interrelations between these matrices are of interest and are summarized in Table 2.4.1. Of course, Y or Z may not exist, but if they do, the table gives the correct formula. Problem 2.4.1 asks for the derivation of this table.

Table 2.4.1
SUMMARY OF MATRIX INTERRELATIONS

$S =$	S	$(I - Y)(I + Y)^{-1}$	$(Z - I)(Z + I)^{-1}$
$Y =$	$(I - S)(I + S)^{-1}$	Y	Z^{-1}
$Z =$	$(I + S)(I - S)^{-1}$	Y^{-1}	Z

Problem 2.4.1 Establish, in accordance with the definitions of the impedance, admittance, and scattering matrices, the formulas of Table 2.4.1.

Problem 2.4.2 In defining the normalized scattering matrix S, the augmented network, Fig. 2.4.1, is used when the reference terminations of N are 1-ohm (Ω) resistors connected in series with N. Consider the general situation of Fig. 2.4.2, in which a resistive network of symmetric impedance matrix

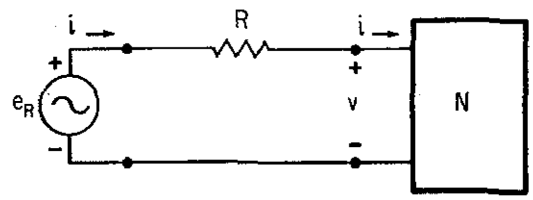


FIGURE 2.4.2. An m-port Network with Resistive Network Termination of Symmetric Impedance Matrix R.

R is connected in series with N. By analogy with (2.4.7) and (2.4.8), we define incident and reflected voltages *with reference to* R by

$$2V_R^i = V + RI = E_R$$
$$2V_R^r = V - RI$$

and *the scattering matrix with the reference* R, S_R, by

$$V_R^r = S_R V_R^i$$

Express S_R in terms of S. Hence show that if $R = I$, then $S_R = S$.

Problem 2.4.3 Find the scattering matrix (actually a scalar) of the resistor, inductor, and capacitor.

Problem 2.4.4 Evaluate the scattering matrix of the multiport transformer and check that it is symmetric. Specialize the result to the case in which the turns-ratio matrix T is square and orthogonal—defining the *orthogonal transformer*.

Problem 2.4.5 Find the impedance, admittance, and scattering matrices of the gyrator. Observe that they are not symmetric.

Problem 2.4.6 This problem relates the scattering matrix to the augmented admittance matrix of a network. Consider the m-port network defined by the dotted lines on Fig. 2.4.1. Show that it has an admittance matrix Y_a (the augmented admittance matrix of N) if and only if N possesses a scattering matrix S, and that

$$S = I - 2Y_a$$

2.5 NETWORK INTERCONNECTIONS

In this section we consider some simple interconnections of networks. The most frequently encountered forms of interconnection are the series, parallel, and cascade-load connections.

Series and Parallel Connections

Consider the arrangement of Fig. 2.5.1 in which two m ports N_1 and N_2 are connected in series; i.e., the terminal $(T'_j)_1$ of the jth port of N_1

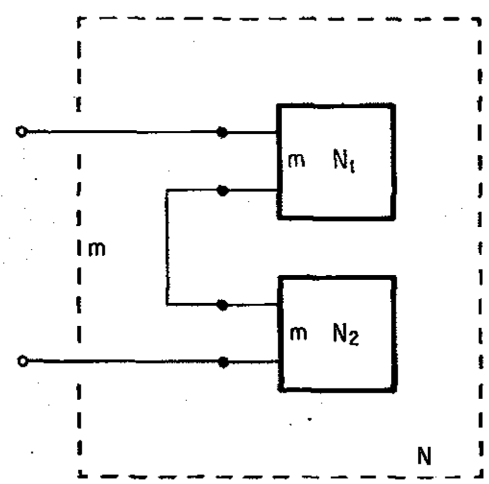


FIGURE 2.5.1. Series Connection.

is connected in series with the terminal $(T_j)_2$ of the jth port of N_2, and the remaining terminals $(T_j)_1$ and $(T'_j)_2$ are paired as new port terminals. Since the port currents of N_1 and N_2 are constrained to be the same and the port voltages to add, the series connection has an impedance matrix

$$Z = Z_1 + Z_2 \tag{2.5.1}$$

where Z_1 and Z_2 are the respective impedance matrices of N_1 and N_2.

It is important in drawing conclusions about the series connection (and other connections) of two networks that, *after connection*, it must remain true that the same current enters terminal T_j of N_1 or N_2 as leaves the associated terminal T'_j of the network. It is possible for this arrangement to be disturbed. Consider, for example, the arrangement of Fig. 2.5.2, where two two-port networks are series connected, the connections being shown

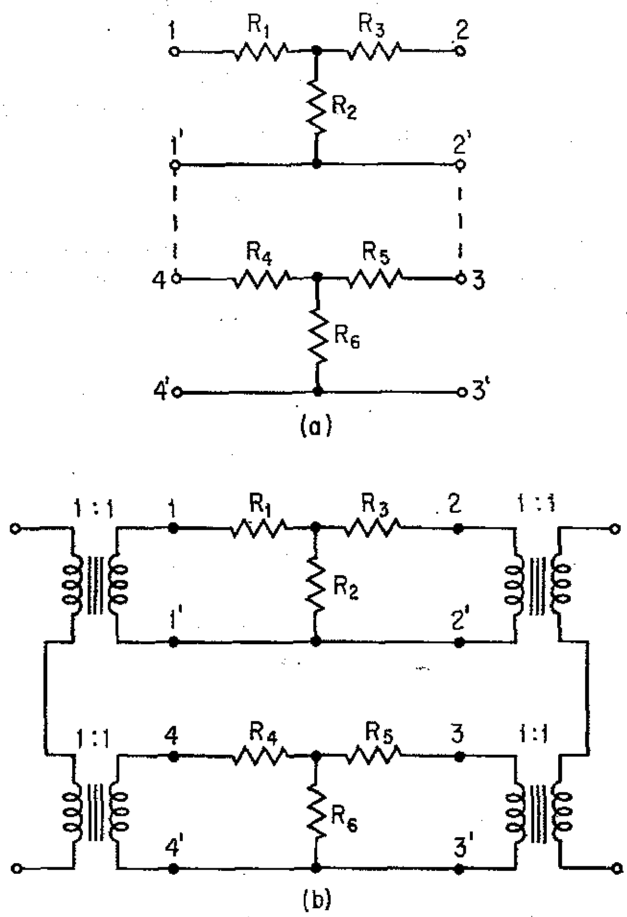


FIGURE 2.5.2. Series Connections Illustrating Nonadditivity and Additivity of Impedance Matrices.

with dotted lines. After connection a two port is obtained with terminal pairs 1–4′ and 2–3′. Suppose that the second pair is left open circuit, but a voltage is applied at the pair 1–4′. It is easy to see that there will be no current entering at terminal 2, but there will be current leaving the top network at terminal 2′. Therefore, the impedance matrix of the interconnected network will not be the sum of the separate impedance matrices. However, introduction of one-to-one turns-ratio transformers, as in Fig. 2.5.2b, will always guarantee that impedance matrices can be added. In our subsequent work we shall assume, without further comment on the fact, that the impedance matrix of any series connection can be derived according to (2.5.1), with, if necessary, the validity of the equation being guaranteed by the use of transformers.

The parallel connection of N_1 and N_2, shown in Fig. 2.5.3, has the corre-

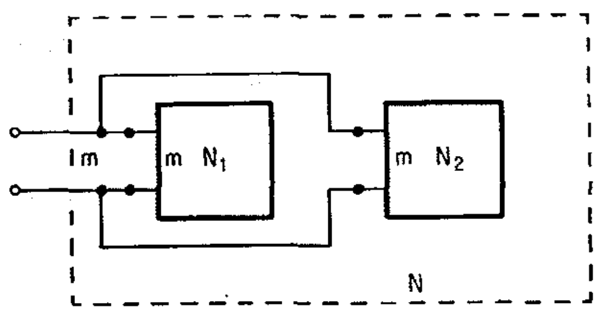


FIGURE 2.5.3. Parallel Connection.

sponding terminal pairs of N_1 and N_2 joined together to form a set of common terminal pairs, which constitute the ports of the resulting m port. Here, since the currents of N_1 and N_2 add and the voltages are the same, the parallel connection has an admittance matrix

$$Y = Y_1 + Y_2 \tag{2.5.2}$$

where Y_1 and Y_2 are the respective admittance matrices of N_1 and N_2. Similar cautions apply in writing (2.5.2) as apply in writing (2.5.1).

Cascade-Load Connection

Now consider the arrangement of Fig. 2.5.4, where an n port N_l *cascade loads* an $(m + n)$ port N_c to produce an m-port network N. If N_l

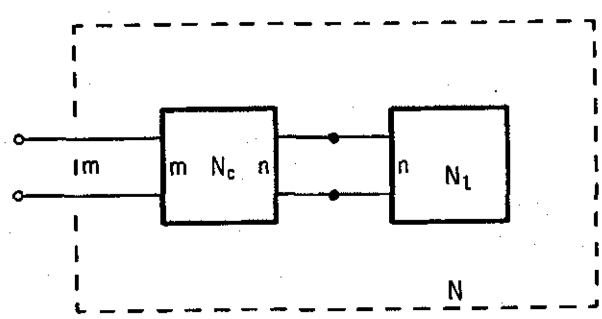


FIGURE 2.5.4. Cascade-Load Connection.

and N_c possess impedance matrices Z_l and Z_c, the latter being partitioned like the ports of N_c as

$$Z_c = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

where Z_{11} is $m \times m$ and Z_{22} is $n \times n$, then simple algebraic manipulation shows that N possesses an impedance matrix Z given by

$$Z = Z_{11} - Z_{12}(Z_{22} + Z_l)^{-1}Z_{21} \tag{2.5.3}$$

assuming the inverse exists. (Problem 2.5.1 requests verification of this result.)

Likewise, if N_l and N_c are described by admittance matrices Y_l and Y_c, and an analogous partition of Y_c defines submatrices Y_{ij}, it can be checked that N possesses an admittance matrix

$$Y = Y_{11} - Y_{12}(Y_{22} + Y_l)^{-1}Y_{21} \tag{2.5.4}$$

Again, we must assume the inverse exists.

For scattering-matrix descriptions of N, the situation is a little more tricky. Suppose that the scattering matrices for N_l and N_c are, respectively, S_l and S_c, with the latter partitioned as

$$S_c = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

where S_{11} is $m \times m$ and S_{22} is $n \times n$. Then the scattering matrix S of the cascade-loading interconnection N is given by (see Problem 2.5.1)

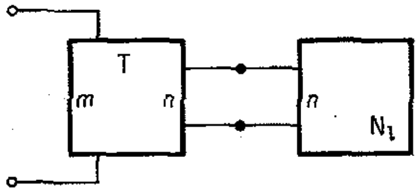
$$S = S_{11} + S_{12}S_l(I - S_{22}S_l)^{-1}S_{21} \tag{2.5.5a}$$

or equivalently

$$S = S_{11} + S_{12}(I - S_lS_{22})^{-1}S_lS_{21} \tag{2.5.5b}$$

In (2.5.3), (2.5.4), and (2.5.5) it is possible for the inverses not to exist, in which case the formulas are not valid. However, it is shown in [6] that if N_l and N_c are passive, the resultant cascade-connected network N *always* possesses a scattering matrix S. The scattering matrix is given by (2.5.5), with the inverse replaced by a pseudo inverse if necessary.

An important situation in which (2.5.3) and (2.5.4) break down arises when N_c consists of a multiport transformer. Here N_c possesses neither an impedance nor an admittance matrix. Consider the arrangement of Fig. 2.5.5a, and suppose that N_l possesses an impedance matrix Z_l. Let v_l and i_l denote the port voltage and current of N_l, and v and i those of N. Application of the



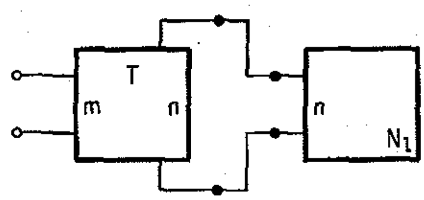


FIGURE 2.5.5. Cascade-Loaded Transformer Arrangements.

multiport-transformer definition with due regard to sign conventions yields

$$V(s) = T'V_l(s) \qquad I_l(s) = TI(s)$$

Since $V_l(s) = Z_l(s)I_l(s)$, it follows that $V(s) = T'Z_l(s)TI(s)$ or

$$Z = T'Z_lT \tag{2.5.6}$$

The dual case for admittances has the transformer turned around (see Fig. 2.5.5b), and the equation relating Y and Y_l is

$$Y = TY_lT' \tag{2.5.7}$$

We note also another important result of the arrangement of Fig. 2.5.5a, the development of which is called for in the problems. If T is orthogonal, i.e., $TT' = T'T = I$, and if N_l possesses a scattering matrix S_l, then N possess a scattering matrix S given by

$$S = T'S_lT \tag{2.5.8}$$

Finally, we consider one important special case of (2.5.5) in which N_l consists of uncoupled 1-Ω resistors, as in Fig. 2.5.6. In this case, one computes $S_l = 0$, and then Eq. (2.5.5) simplifies to

$$S = S_{11} \tag{2.5.9}$$

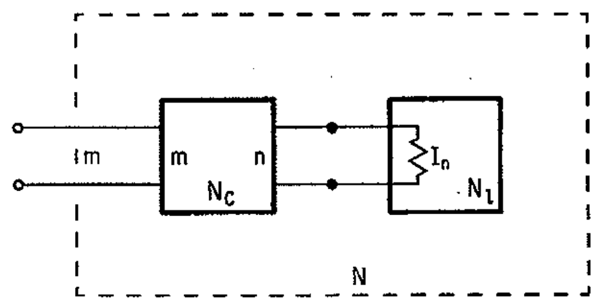


FIGURE 2.5.6. Special Cascade Load Connection.

Thus the termination of ports in unit resistors blocks out corresponding rows and columns of the scattering matrix.

The three types of interconnections—series, parallel, and cascade-load connections—are equally important in synthesis, since, as we shall see, *one of the basic principles in synthesis is to build up a complicated structure or network from simple structures.* It is perhaps interesting to note that the cascade-load connection is the most general form of all, in that the series and parallel connection may be regarded as special cases of the cascade-load connection. This result is illustrated in Fig. 2.5.7.

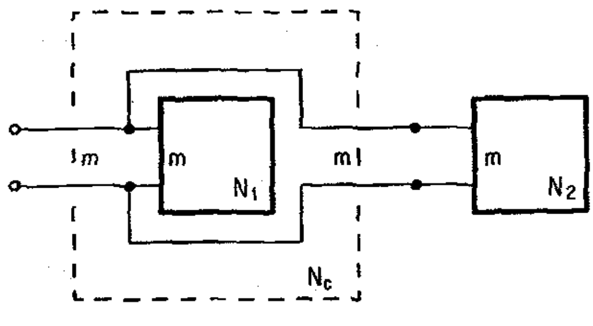
Scattering-Matrix Existence

More generally, it is clear that any finite linear time-invariant m-port network N composed of resistors, capacitors, inductors, transformers, and gyrators can be regarded as the cascade connection of a network N_c composed only of connecting wires, by themselves constituting opens and shorts, and terminated in a network N_l consisting of the circuit elements of N uncoupled from one another. The situation is shown in Fig. 2.5.8.

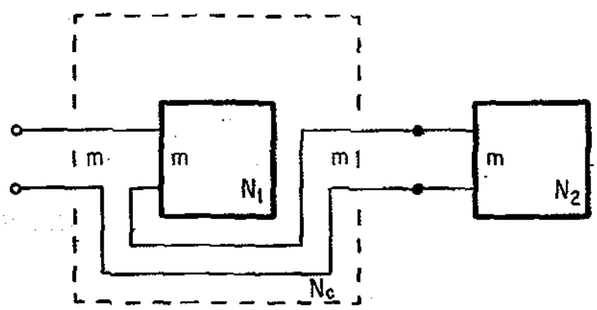
Now we have already quoted the result of [6], which guarantees that if N_c and N_l are passive and individually possess scattering matrices, then N possesses a scattering matrix, computable in accordance with (2.5.5) or a minor modification thereof. Accordingly, if we can show that N_c and N_l possess scattering matrices, it will follow, as claimed earlier, that every m port (with the usual restrictions, such as linearity) possesses a scattering matrix. Let us therefore now note why N_c and N_l possess scattering matrices.

First, it is straightforward to see that each of the network elements—the resistor, inductor, capacitor, gyrator, and two-port transformer—possesses a scattering matrix. In the case of a multiport transformer with a turns-ratio matrix T, its scattering matrix also exists and is given by

$$S = \begin{bmatrix} (I + T'T)^{-1}(T'T - I) & 2(I + T'T)^{-1}T' \\ 2T(I + T'T)^{-1} & (I + TT')^{-1}(I - TT') \end{bmatrix} \tag{2.5.10}$$



(a)



(b)

FIGURE 2.5.7. Cascade-Load Equivalents for:
(a) Parallel; (b) Series Connection.

(Note that S in (2.5.10) reduces to that of (2.4.9) for the two-port transformer when T is a 1×1 matrix.) The scattering matrix of the network N_t is now seen to exist. It is simply the *direct sum** in the matrix sense of a number of scattering matrices for resistors, inductors, capacitors, gyrators, and transformers.

The network N_c composed entirely of opens and shorts is evidently linear, time invariant, passive, and, in fact, lossless. We now outline in rather technical terms an argument for the existence of S_c. The argument can well be omitted by the reader unfamiliar with network topology, who should merely note the main result, viz., that any m-port possesses a scattering matrix.

Consider the augmented network corresponding to N_c as shown in Fig. 2.5.9. The only network elements other than the voltage sources are the $m + n$ unit resistors. With the aid of algorithms detailed in standard textbooks on network topology, e.g., [7], one can form a tree for this augmented

*The direct sum of A and B, denoted by $[A \dotplus B]$, is $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

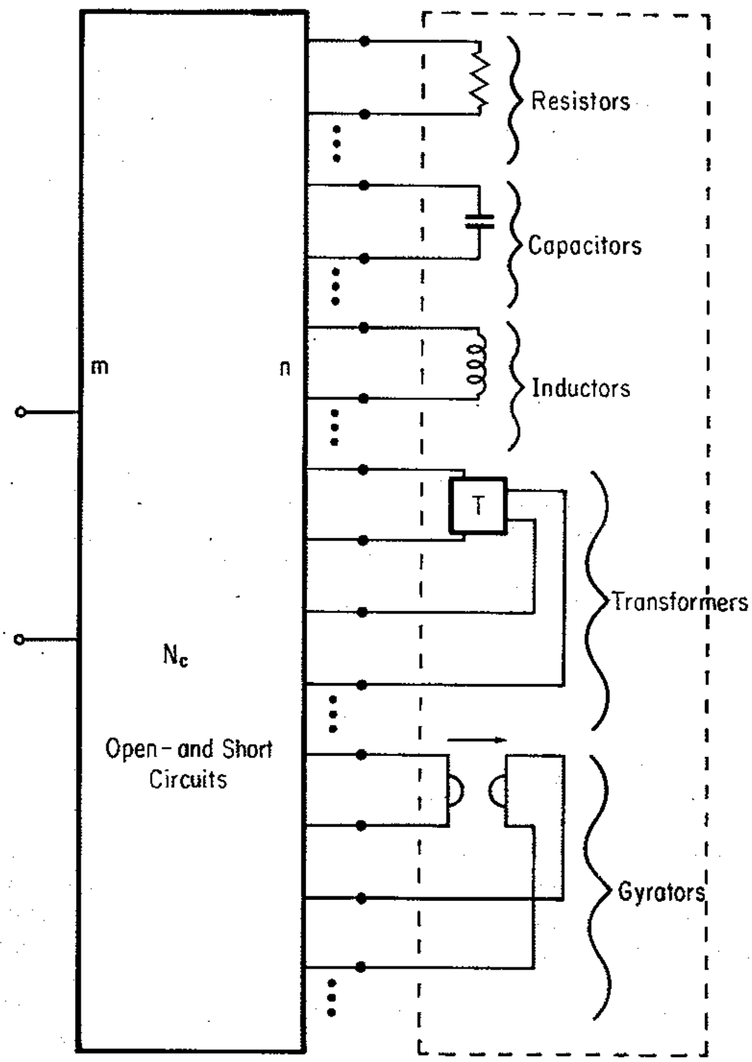


FIGURE 2.5.8. Cascade-Loading Representation for N.

network and select as independent variables a certain set of corresponding link currents, arranged in vector form as I_1. The remaining tree branch currents, in vector form I_2, are then related to I_1 via a relation of the form $I_2 = BI_1$. Notice that I_1 and I_2 between them include all the port currents, but no other currents since there are no branches in the circuit of Fig. 2.5.9 other than those associated with the unit resistors. One is free to choose reference directions for the branch-current entries of I_1 and I_2, and this is

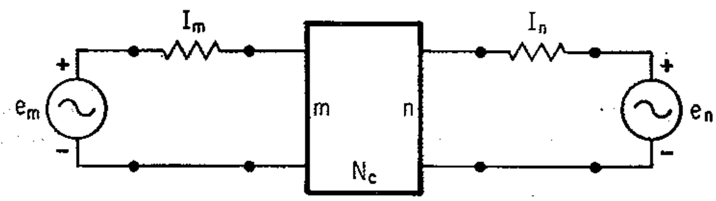


FIGURE 2.5.9. Augmented Network for N_c of Figure 2.5.8.

done so that they coincide with reference directions for the port currents of N_c. Denoting the corresponding port voltages by V_1 and V_2, the losslessness of N_c implies that

$$V_1'I_1 + V_2'I_2 = 0$$

or, using $I_2 = BI_1$,

$$(V_1' + V_2'B)I_1 = 0$$

Since I_1 is arbitrary, it must be true therefore that

$$V_1 = -B'V_2$$

Comparing the results $V_1 = -B'V_2$ and $I_2 = BI_1$ with (2.2.5), we see that constraining relations on the port variables of N_c are identical to those for a multiport transformer with a turns-ratio matrix $T = -B$. Therefore, N_c possesses a scattering matrix. [See Eq. (2.5.10).]

Problem 2.5.1 Consider the arrangement of Fig. 2.5.4.

(a) Let the impedance matrices for N, N_l, and N_c be, respectively, Z, Z_l, and Z_c with Z_c partitioned as the ports

$$Z_c = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

(Here Z_{11} is $m \times m$, etc.) Show that Z is given by

$$Z = Z_{11} - Z_{12}(Z_{22} + Z_l)^{-1}Z_{21}$$

(b) Establish the corresponding result for scattering matrices.

Problem 2.5.2 Suppose that a multiport transformer with turns-ratio matrix T is terminated at its secondary ports in a network of scattering matrix S_l. Suppose also that T is orthogonal. Show that the scattering matrix of the overall network is $T'S_lT$.

Problem 2.5.3 Show, by working from the definitions, that if an $(m + n)$ port with scattering matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

(S_{11} being $m \times m$, etc.) is terminated at its last n ports in 1-Ω resistors, the scattering matrix of the resulting m-port network is simply S_{11}.

2.6 THE BOUNDED REAL PROPERTY

We recall that the m-port networks we are considering are finite, linear, lumped, time invariant, and passive. Sometimes they are lossless also. The sort of mathematical properties these physical properties of the network impose on the scattering, impedance, admittance, and hybrid matrices will now be discussed.

In this section we shall define the *bounded real* property; then we shall show that the scattering matrix of an m port N is necessarily bounded real. Next we shall consider modifications of the bounded real property that are possible when a bounded real matrix is rational. Finally, we shall study the lossless bounded real property for rational matrices.

In the next section we shall define the *positive real* property, following which we show that any immittance or hybrid matrix of an m port is necessarily positive real. Though immittance matrices will probably be more familiar to the reader, it turns out that it is easier to prove results ab initio for scattering matrices, and then to deduce from them the results for immittance matrices, than to go the other way round, i.e., than to prove results first for immittance matrices, and deduce from them results for scattering matrices.

Bounded real and positive real matrices occur in a number of areas of system science other than passive network theory. For example, scattering matrices arise in characterizing sensitivity reduction in control systems and in inverse linear optimal control problems [8]. Positive real matrices arise in problems of control-system stability, particularly those tackled via the circle or Popov criteria [9], and they arise also in analyzing covariance properties of stationary random processes [10]. A discussion now of applications would take us too far afield, but the existence of these applications does provide an extra motivation for the study of bounded real and positive real matrices.

The Bounded Real Property

We assume that there is given an $m \times m$ matrix $A(\cdot)$ of functions of a complex variable s. With no assumption at this stage on the rationality or otherwise of the entries of $A(\cdot)$, the matrix $A(\cdot)$ is termed *bounded real* if the following conditions are satisfied [1]:

1. All elements of $A(\cdot)$ are analytic in Re $[s] > 0$.
2. $A(s)$ is real for real positive s.
3. For Re $[s] > 0$, $I - A'^*(s)A(s)$ is nonnegative definite Hermitian, that is, $x'^*[I - A'^*(s)A(s)]x \geq 0$ for all complex m vectors x. We shall write this simply as $I - A'^*(s)A(s) \geq 0$.

Example 2.6.1 Suppose that $A_1(s)$ and $A_2(s)$ are two $m \times m$ bounded real matrices. We show that $A_1(s)A_2(s)$ is bounded real. Properties 1 and 2 are clearly true for the product if they are true for each $A_i(s)$. Property 3 is a little more tricky. Since for Re $[s] > 0$,

$$I - A_1'^*(s)A_1(s) \geq 0$$

we have

$$A_2'^*(s)A_2(s) - A_2'^*(s)A_1'^*(s)A_1(s)A_2(s) \geq 0$$

Also for Re $[s] > 0$,

$$I - A_2'^*(s)A_2(s) \geq 0$$

Adding the second and third inequalities,

$$I - A_2'^*(s)A_1'^*(s)A_1(s)A_2(s) \geq 0$$

for Re $[s] > 0$, which establishes property 3 for $A_1(s)A_2(s)$.

We now wish to show the connection between the bounded real property and the scattering matrix of an m port. Before doing so, we note the following easy lemma.

Lemma 2.6.1. Let v and i be the port voltage and current vectors of an m port, and v^i and v^r the incident and reflected voltage vectors. Then

1. $v'i = v^{i\prime}v^i - v^{r\prime}v^r$. (2.6.1)
2. The passivity of the m port is equivalent to

$$\mathcal{E}(T) = \int_{t_0}^{T} (v^{i\prime}v^i - v^{r\prime}v^r)\,dt \geq 0 \tag{2.6.2}$$

for all times t_0 at which the network is unexcited, and all incident voltages v^i, with v^i, v^r fulfilling the constraints imposed by the network.

3. The losslessness of the m port is equivalent to passivity in the sense just noted, and

$$\int_{t_0}^{\infty} (v^{i\prime}v^i - v^{r\prime}v^r)\,dt = 0 \tag{2.6.3}$$

for all square integrable v^i; i.e.,

$$\int_{t_0}^{\infty} v^{i'} v^i \, dt < \infty \tag{2.6.4}$$

The formal proof will be omitted. Part 1 is a trivial consequence of the definitions of v^i and v^r, part 2 a consequence of the earlier passivity definition in terms of v and i, and part 3 a consequence of the earlier losslessness definition.

Now we can state one of the major results of network theory. The reader may well omit the proof if he desires; the result itself is one of network analysis rather than network synthesis, and its statement is far more important than the details of its proof, which are unpleasantly technical. The reader should also note that the result is a statement for a broader class of m ports than considered under our usual convention; very shortly, we shall revert to our usual convention and indicate the appropriate modifications to the result.

> **Theorem 2.6.1.** Let N be an m port that is linear, time invariant, and passive—but *not necessarily lumped or finite.* Suppose that N possesses a scattering matrix $S(s)$. Then $S(\cdot)$ is bounded real.

Before presenting a proof of the theorem (which, we warn the reader, will draw on some technical results of Laplace-transform theory), we make several remarks.

1. The theorem statement assumes the existence of $S(s)$; it does not give conditions for existence. As we know though, if N is finite and lumped, as well as satisfying the conditions of the theorem, $S(s)$ exists.
2. When we say "$S(s)$ exists," we should really qualify this statement by saying for what values of s it exists. The bounded real conditions only require certain properties to be true in $\text{Re}\,[s] > 0$, and so, actually, we could just require $S(s)$ to exist in $\text{Re}\,[s] > 0$ rather than for all s.
3. It is the linearity of N that more or less causes there to be a linear operator mapping v^i into v^r, while it is the time invariance of N that permits this operator to be described in the s domain. It is essentially the passivity of N that causes properties 1 and 3 of the bounded real definition to hold; property 2 is a consequence of the fact that real v^i lead to real v^r.
4. In a moment we shall indicate a modification of the theorem applying when N is also lumped and finite. This will give *necessary* conditions on a matrix function for it to be the scattering matrix of an m port with the conventional properties. *One major synthesis task will be to show that these conditions are also sufficient,* by starting with a matrix $S(s)$ satisfying the conditions and exhibiting a network with $S(s)$ as its scattering matrix.

Proof. Consider the passivity property (2.6.2), and let T approach infinity. Then if v^i is square integrable, we have

$$\infty > \int_{t_0}^{\infty} v^{i\prime} v^i \, dt \geq \int_{t_0}^{\infty} v^{r\prime} v^r \, dt$$

so that v^r is square integrable. Therefore, $S(s)$ is the Laplace transform of a convolution operator that maps square integrable functions into square integrable functions. This means (see [1, 11]) that $S(s)$ must have every element analytic in $\text{Re}\,[s] > 0$. This establishes property 1.

Let σ_0 be an arbitrary real positive constant, and let v^i be $xe^{\sigma_0 t}1(t - t_0)$, where x is a real constant m vector and $1(t)$ is the unit step function, which is zero for negative t, one for positive t. Notice that v^i will be smaller the more negative is t.

As $t_0 \to -\infty$ and $t \to \infty$, $v^r(t)$ will approach $S(\sigma_0)xe^{\sigma_0 t}1(t-t_0)$. Because x is arbitrary and $v^r(t)$ is real, $S(\sigma_0)$ must be real. This proves property 2.

Now let $s_0 = \sigma_0 + j\omega_0$ be an arbitrary point in $\text{Re}\,[s] > 0$. Let x be an arbitrary constant complex m vector, and let v^i be $\text{Re}\,[xe^{s_0 t}1(t - t_0)]$. As $t_0 \to -\infty$, v^r approaches

$$\text{Re}\,[S(s_0)xe^{s_0 t}\, 1(t - t_0)]$$

and the instantaneous power flow $v^{i\prime}v^i - v^{r\prime}v^r$ at time t becomes

$$\begin{aligned} p(t) &= \sum_{j=1}^{m} |x_j|^2 e^{2\sigma_0 t} \cos^2 (\omega_0 t + \theta_j) \\ &\quad - \sum_{j=1}^{m} |(S(s_0)x)_j|^2 e^{2\sigma_0 t} \cos^2 (\omega_0 t + \phi_j) \end{aligned}$$

where $\theta_j = \arg x_j$ and $\phi_j = \arg\,(S(s_0)x)_j$. Using the relation $\cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$, we have

$$\begin{aligned} p(t) &= \tfrac{1}{2}x'^*[I - S'^*(s_0)S(s_0)]xe^{2\sigma_0 t} \\ &\quad + \tfrac{1}{2} \sum_j |x_j|^2 e^{2\sigma_0 t} \cos (2\omega_0 t + 2\theta_j) \\ &\quad - \tfrac{1}{2} \sum_j |(S(s_0)x)_j|^2 e^{2\sigma_0 t} \cos (2\omega_0 t + 2\phi_j) \end{aligned}$$

Integration leads to $\mathcal{E}(T)$:

$$\begin{aligned} \mathcal{E}(T) &= \int_{-\infty}^{T} p(t)\, dt \\ &= \frac{1}{4\sigma_0} x'^*[I - S'^*(s_0)S(s_0)]xe^{2\sigma_0 T} \\ &\quad + \frac{1}{4}\,\text{Re}\left\{\frac{1}{s_0} x'[I - S'(s_0)S(s_0)]xe^{2 s_0 T}\right\} \end{aligned}$$

Now if ω_0 is nonzero, the angle of $(1/s_0)x'[I - S'(s_0)S(s_0)]xe^{2s_0T}$ takes on all values between 0 and 2π as T varies, since $e^{2s_0T} = e^{2\sigma_0T}e^{2j\omega_0T}$, and the angle of $e^{2j\omega_0T}$ takes on all values between 0 and 2π as T varies. Therefore, for certain values of T,

$$\text{Re}\left\{\frac{1}{s_0}x'[I - S'(s_0)S(s_0)]xe^{2s_0T}\right\} = 0$$

The nonnegativity of $\mathcal{E}(T)$ then forces

$$x'^*[I - S'^*(s_0)S(s_0)]x = 0$$

If ω_0 is zero, let x be an arbitrary real constant. Then, recalling that $S(s_0)$ will be real, we have

$$\mathcal{E}(T) = \frac{1}{4\sigma_0}x'[I - S'(s_0)S(s_0)]xe^{2\sigma_0T} \geq 0$$

Therefore, for s_0 real or complex in Re $[s] > 0$, we have

$$I - S'^*(s_0)S(s_0) \geq 0$$

as required. This establishes property 3. $\nabla\nabla\nabla$

Once again, we stress that the m ports under consideration in the theorem need not be lumped or finite; equivalently, there is no requirement that $S(s)$ be rational. However, by adding in a constraint that $S(s)$ is rational, we can do somewhat better.

The Rational Bounded Real Property

We define an $m \times m$ matrix $A(\cdot)$ of functions of a complex variable s to be ***rational bounded real***, abbreviated simply BR, if

1. It is a matrix of rational functions of s.
2. It is bounded real.

Let us now observe the variations on the bounded real conditions that become possible. First, instead of saying simply that every element of $A(\cdot)$ is analytic in Re $[s] > 0$, we can clearly say that no element of $A(\cdot)$ possesses a pole in Re $[s] > 0$. Second, if $A(s)$ is real for real positive s, this means that each entry of $A(\cdot)$ must be a *real* rational function, i.e., a ratio of two polynomials with real coefficients.

The implications of the third property are the most interesting. Any function in the vicinity of a pole takes unbounded values. Now the inequality $I - A'^*(s)A(s) \geq 0$ implies that the $(i - i)$ term of $I - A'^*(s)A(s)$ is nonnegative, i.e.,

$$1 - \sum_j |a_{ji}(s)|^2 \geq 0$$

Therefore, $|a_{ij}(s)|$ must be bounded by 1 at any point in $\text{Re}[s] > 0$. Consequently, it is impossible for $a_{ij}(s)$ to have a pole on the imaginary axis $\text{Re}[s] = 0$, for if it did have a pole there, $|a_{ij}(s)|$ would take on arbitrarily large values in the half-plane $\text{Re}[s] > 0$ in the vicinity of the pole. It follows that

$$A(j\omega) = \lim_{\substack{\sigma \to 0 \\ \sigma > 0}} A(\sigma + j\omega)$$

exists for all real ω and that

$$I - A'^*(j\omega)A(j\omega) \geq 0 \tag{2.6.5}$$

for all real ω. What is perhaps most interesting is that, knowing only that: (1) every element of $A(\cdot)$ is analytic in $\text{Re}[s] \geq 0$; and (2) Eq. (2.6.5) holds for all real ω, we can deduce, by an extension of the maximum modulus theorem of complex variable theory, that $I - A'^*(s)A(s) \geq 0$ for all s in $\text{Re}[s] > 0$. (See [1] and the problems at the end of this section.) In other words, if $A(\cdot)$ is rational and known to have analytic elements in $\text{Re}[s] \geq 0$, Eq. (2.6.5) carries as much information as

$$I - A'^*(s)A(s) \geq 0 \tag{2.6.6}$$

for all s in $\text{Re}[s] > 0$.

We can sum up these facts by saying that an $m \times m$ matrix $A(\cdot)$ of *real rational* functions of a complex variable s is bounded real if and only if: (1) no element of $A(\cdot)$ possesses a pole in $\text{Re}[s] \geq 0$; and (2) $I - A'^*(j\omega)A(j\omega) \geq 0$ for all real ω. If we accept the fact that the scattering matrix of an m port that is finite and lumped must be rational, then Theorem 2.6.1 and the above statement yield the following:

Theorem 2.6.2. Let N be an m port that is linear, time invariant, lumped, finite, and passive. Let $S(s)$ be the scattering matrix of N. Then no element of $S(s)$ possesses a pole in $\text{Re}[s] \geq 0$, and $I - S'^*(j\omega)S(j\omega) \geq 0$ for all real ω.

In our later synthesis work we shall start with a real rational $S(s)$ satisfying the conditions of the theorem, and provide procedures for generating a network with scattering matrix $S(s)$.

Example 2.6.2 We shall establish the bounded real nature of

$$S(s) = \begin{bmatrix} 0 & 1 \\ \dfrac{s-1}{s+1} & 0 \end{bmatrix}$$

Obviously, $S(s)$ is real rational, and no element possesses a pole in Re $[s] \geq 0$. It remains to examine $I - S'^*(j\omega)S(j\omega)$. We have

$$I - S'^*(j\omega)S(j\omega) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \dfrac{-j\omega - 1}{-j\omega + 1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \dfrac{j\omega - 1}{j\omega + 1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \dfrac{1 + \omega^2}{1 + \omega^2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 0$$

The Lossless Bounded Real Property

We now want to look at the constraints imposed on the scattering matrix of a network if that network is known to be lossless. Because it is far more convenient to do so, we shall confine discussions to real rational scattering matrices.

In anticipation of the main result, we shall define an $m \times m$ real rational matrix $A(\cdot)$ to be lossless bounded real (abbreviated LBR) if

1. $A(\cdot)$ is bounded real.
2. $I - A'^*(j\omega)A(j\omega) = 0$ for all real ω. (2.6.7)

Comparison with the BR conditions studied earlier yields the equivalent conditions that (1) every element of $A(\cdot)$ is analytic in Re $[s] \geq 0$, and (2) Eq. (2.6.7) holds.

The main result, hardly surprising in view of the definition, is as follows:

Theorem 2.6.3. Let N be an m port (usual conventions apply) with $S(s)$ the scattering matrix. Then N is lossless if and only if

$$I - S'^*(j\omega)S(j\omega) = 0 \qquad \text{for all real } \omega \tag{2.6.8}$$

Proof. Suppose that the network is unexcited at time t_0. Then losslessness implies that

$$\int_{t_0}^{\infty} (v^{i\prime}v^i - v^{r\prime}v^r)\, dt = 0$$

for all square integrable v^i. Since v^r is then also square integrable, the Fourier transforms $V^i(j\omega)$ and $V^r(j\omega)$ exist and the equation becomes, by Parseval's theorem,

$$\int_{-\infty}^{\infty} V^{i\prime *}(j\omega)[I - S'^*(j\omega)S(j\omega)]V^i(j\omega)\, d\omega = 0$$

Since v^i is an arbitrary square integrable function, $V^i(j\omega)$ is sufficiently arbitrary to conclude that

$$I - S'^*(j\omega)S(j\omega) = 0 \qquad \text{for all real } \omega$$

The converse follows by retracing the above steps. $\nabla\nabla\nabla$

Example 2.6.3 *If S_1 and S_2 are* LBR, then S_1S_2 is LBR. Clearly the only nontrivial matter to verify is the equality (2.6.8) with S replaced by S_1S_2.

$$\begin{aligned} I - S_2'^*(j\omega)S_1'^*(j\omega)S_1(j\omega)S_2(j\omega) & \\ = I - S_2'^*(j\omega)S_2(j\omega) & \qquad \text{since } S_1 \text{ is LBR} \\ = 0 & \qquad \text{since } S_2 \text{ is LBR} \end{aligned}$$

Example 2.6.4 The scattering matrix of Example 2.6.2,

$$S(s) = \begin{bmatrix} 0 & 1 \\ \dfrac{s-1}{s+1} & 0 \end{bmatrix}$$

is LBR, since, as verified in Example 2.6.2, Eq. (2.6.8) holds.

In view of the fact that $S(\cdot)$ is real rational, it follows that $S'^*(j\omega) = S'(-j\omega)$, and so the condition (2.6.8) becomes

$$I - S'(-s)S(s) = 0 \tag{2.6.9}$$

for $s = j\omega$, ω real. But any analytic function that is zero on a line must be zero everywhere, and so (2.6.9) holds for all s. We can call this the *extended LBR* property. It is easy to verify, for example, that the scattering matrix of Example 2.6.4 satisfies (2.6.9).

Testing for the BR and LBR Properties

Examination of the BR conditions shows that testing for the BR property requires two distinct efforts. First, the poles of elements of the candidate matrix $A(s)$ must be examined. To check that they are all in the left half-plane, a *Hurwitz test* may be used (see, e.g., [12, 13]). Second, the nonnegativity of the matrix $I - A'^*(j\omega)A(j\omega) = I - A'(-j\omega)A(j\omega)$ must be checked. As it turns out, this is equivalent to checking whether certain polynomials have any real zeros, and various tests, e.g., the Sturm test, are available for this [12, 13]. To detail these tests would take us too far afield; suffice it to say that both tests require only a finite number of steps, and do not demand that any polynomials be factored.

Testing for the LBR property is somewhat simpler, in that the zero nature, rather than the nonnegativity, of a matrix must be established.

Summary

For future reference, we summarize the following sets of conditions.

Bounded Real Property

1. All elements of $A(\cdot)$ are analytic in $\text{Re}\,[s] > 0$.
2. $A(s)$ is real for real positive s.
3. $I - A'^*(s)A(s) \geq 0$ for $\text{Re}\,[s] > 0$.

BR Property

1. $A(\cdot)$ is real rational.
2. All elements of $A(\cdot)$ are analytic in $\text{Re}\,[s] \geq 0$.
3. $I - A'^*(j\omega)A(j\omega) \geq 0$ for all real ω.

LBR Property

1. $A(\cdot)$ is real rational.
2. All elements of $A(\cdot)$ are analytic in $\text{Re}\,[s] \geq 0$.
3. $I - A'^*(j\omega)A(j\omega) = 0$ for all real ω.
4. $I - A'(-s)A(s) = 0$ for all s (extended LBR property).

We also note the following result, the proof of which is called for in Problem 2.6.3. Suppose $A(s)$ is BR, and that $[I \pm A(s)]x = 0$ for some s in $\text{Re}\,[s] > 0$, and some constant x. Then $[I \pm A(s)]x = 0$ for all s in $\text{Re}\,[s] \geq 0$.

Problem 2.6.1 Prove statement 3 of Lemma 2.6.1, viz., losslessness of an m port is equivalent to passivity together with the condition

$$\int_{t_0}^{\infty} (v^{i\prime}v^i - v^{r\prime}v^r)\,dt = 0$$

for all compatible incident and reflected voltage vectors, with the network assumed to be storing no energy at time t_0, and with

$$\int_{t_0}^{\infty} v^{i\prime}v^i\,dt < \infty$$

You may need to use the fact that the sum and difference of square integrable quantities are square integrable.

Problem 2.6.2 The maximum modulus theorem states the following. Let $f(s)$ be an analytic function within and on a closed contour C. Let M be the upper bound of $|f(s)|$ on C. Then $|f(s)| \leq M$ within C, and equality holds at one point if and only if $f(s)$ is a constant. Use this theorem to prove that if all elements of $A(s)$ are analytic in $\text{Re}\,[s] \geq 0$, then $I - A'^*(j\omega)A(j\omega) \geq 0$ for all real ω implies that $I - A'^*(s)A(s) \geq 0$ for all s in $\text{Re}\,[s] > 0$.

Problem 2.6.3 Suppose that $A(s)$ is rational bounded real. Using the ideas contained in Problem 2.6.2, show that if

$$[I + A(s)]x = 0$$

or

$$[I - A(s)]x = 0$$

for some s in $\text{Re}[s] > 0$, then equality holds for all $\text{Re}[s] \geq 0$.

Problem 2.6.4 (Alternative derivation of Theorems 2.6.2 and 2.6.3.) Verify by direct calculation that the scattering matrices of the passive resistor, inductor, capacitor, transformer, and gyrator are BR, with the last four being LBR. Represent an arbitrary m port as a cascade load, with the loaded network comprising merely opens and shorts, and, as shown in the last section, possessing a scattering matrix like that of an ideal transformer. Assume the validity of Eq. (2.5.5) to prove Theorems 2.6.2 and 2.6.3.

Problem 2.6.5 Establish whether the following function is BR:

$$S(s) = \frac{s^2 + 4s - 1}{2s^3 + 5s^2 + 6s + 3}$$

2.7 THE POSITIVE REAL PROPERTY

In this section our task is to define three properties: the positive real property, the rational positive real property, and the lossless positive real property. We shall relate these properties to properties of immittance or hybrid matrices of various classes of networks, much as we related the bounded real properties to properties of scattering matrices.

The Positive Real Property

We assume that there is given an $m \times m$ matrix $B(\cdot)$ of functions of a complex variable s. With no assumption at this stage on the rationality or otherwise of the entries of $B(\cdot)$, the matrix $B(\cdot)$ is termed *positive real* if the following conditions are satisfied [1]:

1. All elements of $B(\cdot)$ are analytic in $\text{Re}[s] > 0$.
2. $B(s)$ is real for real positive s.
3. $B'^*(s) + B(s) \geq 0$ for $\text{Re}[s] > 0$.

Example 2.7.1 Suppose that $A(s)$ is bounded real and that $[I - A(s)]^{-1}$ exists for all s in $\text{Re}[s] > 0$. Define

$$B(s) = [I + A(s)][I - A(s)]^{-1}$$

Then $B(s)$ is positive real.

To verify condition 1, observe that an element of $B(s)$ will have a right-half-plane singularity only if $A(s)$ has a right-half-plane singularity, or $[I - A(s)]$ is singular in $\text{Re}\,[s] > 0$. The first possibility is ruled out by the bounded real property, the second by assumption. Verification of condition 2 is trivial, while to verify condition 3, observe that

$$\begin{aligned}B'^*(s) + B(s) &= [I - A'^*(s)]^{-1}[I + A'^*(s)] + [I + A(s)][I - A(s)]^{-1} \\ &= [I - A'^*(s)]^{-1}\{[I + A'^*(s)][I - A(s)] \\ &\quad + [I - A'^*(s)][I + A(s)]\}[I - A(s)]^{-1} \\ &= 2[I - A'^*(s)]^{-1}[I - A'^*(s)A(s)][I - A(s)]^{-1}\end{aligned}$$

and the nonnegativity follows from the bounded real property. The converse result holds *too; i.e., if $B(s)$ is positive real, then $A(s)$ is* bounded real.

In passing we note the following theorem of [1, 8]. Of itself, it will be of no use to us, although a modified version (Theorem 2.7.3) will prove important. The theorem here parallels Theorem 2.6.1, and a partial proof is requested in the problems.

Theorem 2.7.1. Let N be an m port that is linear, time invariant and passive—but not necessarily lumped or finite. Suppose that N possesses an immittance or hybrid matrix $B(s)$. Then $B(\cdot)$ is positive real.

The Rational Positive Real Property

We define an $m \times m$ matrix $B(\cdot)$ of functions of a complex variable s to be rational positive real, abbreviated simply PR, if

1. $B(\cdot)$ is a matrix of rational functions of s.
2. $B(\cdot)$ is positive real.

Given that $B(\cdot)$ is a rational matrix, conditions 1 and 2 of the positive real definition are equivalent to requirements that no element of $B(\cdot)$ has a pole in $\text{Re}\,[s] > 0$ and that each element of $B(\cdot)$ be a real rational function. It is clear that condition 3 implies, by a limiting operation, that

$$B'^*(j\omega) + B(j\omega) \geq 0 \qquad (2.7.1)$$

for all real ω such that no element of $B(s)$ has a pole at $s = j\omega$. If it were true that no element of $B(\cdot)$ could have a pole for $s = j\omega$, then (2.7.1) would be true for all real ω and, we might imagine, would imply that

$$B'^*(s) + B(s) \geq 0 \qquad \text{Re}\,[s] > 0 \qquad (2.7.2)$$

However, there do exist positive real *matrices* for which an element possesses a pure imaginary pole, as the following example shows.

Example 2.7.2 Consider the function $B(s) = 1/s$, which is the impedance of a 1-F capacitor. Then $B(s)$ is analytic in $\text{Re}\,[s] > 0$, is real for real positive s, and

$$B'^*(s) + B(s) = \frac{1}{s^*} + \frac{1}{s} = \frac{2\,\text{Re}\,[s]}{|s|^2} \geq 0$$

in $\text{Re}\,[s] > 0$. Thus $B(s)$ is positive real, but possesses a pure imaginary pole.

One might then ask the following question. Suppose that $B(s)$ is real rational, that no element of $B(s)$ has a pole in $\text{Re}\,[s] > 0$, and that (2.7.1) *holds for all* ω for which $j\omega$ is not a pole of any element of $B(s)$; is it then true that $B(s)$ must be PR; i.e., does (2.7.2) always follow? The answer to this question is no, as shown in the following example.

Example 2.7.3 Consider $B(s) = -1/s$. This is real rational, analytic for $\text{Re}\,[s] > 0$, with $B'^*(j\omega) + B(j\omega) = 0$ for all $\omega \neq 0$. But it is not true that $B'^*(s) + B(s) \geq 0$ for all $\text{Re}\,[s] > 0$.

At this stage, the reader might feel like giving up a search for any $j\omega$-axis conditions equivalent to (2.7.2). However, there are some conditions, and they are useful. A statement is contained in the following theorem.

Theorem 2.7.2. Let $B(s)$ be a real rational matrix of functions of s. Then $B(\cdot)$ is PR if and only if

1. No element of $B(\cdot)$ has a pole in $\text{Re}\,[s] > 0$.
2. $B'^*(j\omega) + B(j\omega) \geq 0$ for all real ω, with $j\omega$ not a pole of any element of $B(\cdot)$.
3. If $j\omega_0$ is a pole of any element of $B(\cdot)$, it is at most a simple pole, and the residue matrix, $K_0 = \lim_{s \to j\omega_0} (s - j\omega_0)B(s)$ in case ω_0 is finite, and $K_\infty = \lim_{\omega \to \infty} B(j\omega)/j\omega$ in case ω_0 is infinite, is nonnegative definite Hermitian.

Proof. In the light of earlier remarks it is clear that what we have to do is establish the equivalence of statements 2 and 3 of the theorem with (2.7.2), under the assumption that all entries of $B(\cdot)$ are analytic in $\text{Re}\,[s] > 0$. We first show that conditions 2 and 3 of the theorem imply (2.7.2); the argument is based on application of the maximum modulus theorem, in a slightly more sophisticated form than in our discussion on BR matrices. Set $f(s) = x'^*B(s)x$ for an arbitrary n vector x, and consider a contour C extending from $-j\Omega$ to $j\Omega$ along the $j\omega$ axis, with small semicircular indentations into $\text{Re}[s] > 0$ around those points $j\omega_0$ which are poles of elements of $B(s)$; the contour C is closed with a semicircular arc in $\text{Re}[s] > 0$. On the portion of C coincident with the $j\omega$ axis, $\text{Re} f \geq 0$ by condition 2. On the semicircular indentations,

$$f \simeq \frac{x'^*K_0x}{s - j\omega_0}$$

and $\text{Re} f \geq 0$ by condition 3 and the fact that $\text{Re}[s] > 0$. If $B(\cdot)$ has no element with a pole at infinity, $\text{Re} f \geq 0$ on the right-half-plane contour, taking the constant value $\lim_{\omega \to \infty} x'^*[B'^*(j\omega) + B(j\omega)]x$ there. If $B(\cdot)$ has an element with a pole at infinity, then

$$f \simeq sx'^*K_\infty x$$

with $\text{Re} f \geq 0$ on the large semicircular arc.

We conclude that $\text{Re} f \geq 0$ on C, and by the maximum modulus theorem applied to $\exp[-f(s)]$ and the fact that

$$\exp[-\text{Re} f(s)] = |\exp[-f(s)]|$$

we see that $\text{Re} f \geq 0$ in $\text{Re}[s] > 0$. This proves (2.7.2).

We now prove that Eq. (2.7.2) implies condition 3 of the theorem. (Condition 2 is trivial.) Suppose that $j\omega_0$ is a pole of order m of an element of $B(s)$. Then the values taken by $x'^*B(s)x$ on a semicircular arc of radius ρ, center $j\omega_0$, are for small ρ

$$x'^*B(s)x \simeq x'^*K_0x\rho^{-m}e^{-jm\theta} \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Therefore

$$\rho^m \text{Re}[x'^*B(s)x] \simeq \text{Re}[x'^*K_0x] \cos m\theta + \text{Im}[x'^*K_0x] \sin m\theta$$

which must be nonnegative by (2.7.2). Clearly $m = 1$; i.e., only simple poles are possible else the expression could have either sign. Choosing θ near $-\pi/2$, 0, and $\pi/2$ shows that $\text{Im}[x'^*K_0x] = 0$ and $\text{Re}[x'^*K_0x] \geq 0$. In other words, K_0 is nonnegative definite Hermitian. ∇∇∇

Example 2.7.4 $B(s) = -1/s$ fulfills conditions 1 and 2 of Theorem 2.7.2, but not condition 3 and is not positive real. On the other hand, $B(s) = 1/s$ is positive real.

Example 2.7.5 Let $B(s) = sL$. Then $B(s)$ is positive real if and only if L is nonnegative definite symmetric. We show this as follows. By condition 3 of Theorem 2.7.2, we know that L is nonnegative definite Hermitian. The Hermitian nature of L means that $L = L_1 + jL_2$, where L_1 is symmetric, L_2 skew symmetric, and L_1 and L_2 are real. However, $L = \lim_{s\to\infty} B(s)/s$, and if $s \longrightarrow \infty$ along the positive real axis, $B(s)/s$ is always real, so that L is real.

Example 2.7.6 We shall verify that

$$z(s) = \frac{s^3 + 3s^2 + 5s + 1}{s^3 + 2s^2 + s + 2}$$

is PR. Obviously, $z(s)$ is real rational. Its poles are the zeros of $s^3 + 2s^2 + s + 2 = (s^2 + 1)(s + 2)$, so that $z(s)$ is analytic in $\text{Re}\,[s] > 0$. Next

$$z(s) = \frac{-1}{s+2} + \frac{1}{s+j1} + \frac{1}{s-j1} + 1$$

Therefore, the residues at the purely imaginary poles are both positive. Finally,

$$\begin{aligned} z(j\omega) + z^*(j\omega) &= \frac{-1}{j\omega+2} + \frac{-1}{-j\omega+2} + \frac{1}{j\omega+j1} + \frac{1}{-j\omega+j1} \\ &\quad + \frac{1}{j\omega-j1} + \frac{1}{-j\omega-j1} + 2 \\ &= \frac{-4}{4+\omega^2} + 2 \\ &= \frac{2\omega^2+4}{\omega^2+4} \\ &\geq 0 \end{aligned}$$

for all real $\omega \neq 1$.

In Theorem 2.7.3 we relate the PR property to a property of port descriptions of passive networks. If we accept the fact that any immittance or hybrid matrix associated with an m port that is finite and lumped must be rational, then we have the following important result.

Theorem 2.7.3. Let N be an m port that is linear, time invariant, lumped, finite, and passive. Let $B(s)$ be an immittance or hybrid matrix for N. Then $B(s)$ is PR.

Proof. We shall prove the result for the case when $B(s)$ is an impedance. The other cases follow by minor variation. The steps of the proof are, first, to note that N has a scattering matrix $S(s)$ related in a certain way to $B(s)$, and, second, to use the BR constraint on $S(s)$ to deduce the PR constraint on $B(s)$.

The scattering matrix $S(s)$ exists because of the properties assumed for N. The impedance matrix $B(s)$ is related to $S(s)$ via a formula we noted earlier:

$$B(s) = [I + S(s)][I - S(s)]^{-1}$$

As we know, $S(s)$ is bounded real, and so, by the result in Example 2.7.1, $B(s)$ is positive real. Since $B(s)$ is rational, $B(s)$ is then PR. ∇∇∇

The Lossless Positive Real Property

Our task now is to isolate what extra constraints are imposed on the immittance or hybrid matrix $B(s)$ of an m port N when the m port is lossless. The result is simple to obtain and is as follows:

Theorem 2.7.4. Let N be an m port (usual conventions apply) with $B(s)$ an immittance or hybrid matrix of N. Then N is lossless if and only if

$$B'^*(j\omega) + B(j\omega) = 0 \tag{2.7.3}$$

for all real ω such that $j\omega$ is not a pole of any element of $B(\cdot)$.

Proof. We shall only prove the result for the case when $B(s)$ is an impedance. Then there exists a scattering matrix $S(s)$ such that

$$B(s) = [I + S(s)][I - S(s)]^{-1}$$

It follows that

$$\begin{aligned}
B'^*(j\omega) + B(j\omega) &= [I - S'^*(j\omega)]^{-1}[I + S'^*(j\omega)] \\
&\quad + [I + S(j\omega)][I - S(j\omega)]^{-1} \\
&= [I - S'^*(j\omega)]^{-1}\{[I + S'^*(j\omega)][I - S(j\omega)] \\
&\quad + [I - S'^*(j\omega)][I + S(j\omega)]\}[I - S(j\omega)]^{-1} \\
&= 2[I - S'^*(j\omega)]^{-1}[I - S'^*(j\omega)\,S(j\omega)] \\
&\quad \times [I - S(j\omega)]^{-1} \\
&= 0
\end{aligned}$$

if and only if $S(\cdot)$ is LBR, i.e., if and only if N is lossless. Note that if $j\omega$ is a pole of an element of $B(s)$, then $[I - S(j\omega)]$ will be singular; so the above calculation is only valid when $j\omega$ is not a pole of any element of $B(\cdot)$. ∇∇∇

Having in mind the result of the above theorem, we shall say that a matrix $B(s)$ is lossless positive real (LPR) if (1) $B(s)$ is positive real, and (2) $B'^*(j\omega) + B(j\omega) = 0$ for all real ω, with $j\omega$ not a pole of any element of $B(s)$.

In view of the fact that $B(s)$ is rational, we have that $B'^*(j\omega) = B'(-j\omega)$, and so (2.7.3) implies that

$$B'(-s) + B(s) = 0 \tag{2.7.4}$$

for all $s = j\omega$, where $j\omega$ is not a pole of any element of $B(\cdot)$. This means, since $B'(-s) + B(s)$ is an analytic function of s, that (2.7.4) holds for all s that are not poles of any element of $B(\cdot)$.

To this point, the arguments have paralleled those given for scattering matrices. But for impedance matrices, we can go one step farther by isolating the pole positions of LPR matrices. Suppose that s_0 is a pole of $B(s)$. Then as $s \longrightarrow s_0$ but $s \neq s_0$, (2.7.4) remains valid. It follows that some element of $B'(-s)$ becomes infinite as $s \longrightarrow s_0$, so that $-s_0$ is a pole of an element of $B(s)$.

Now if whenever s_0 is a pole, $-s_0$ is too, and if there can be no poles in $\text{Re}\,[s] > 0$, *all poles of elements of $B(s)$ must be pure imaginary.* It follows that necessary and sufficient conditions for a real rational matrix $B(s)$ to be LBR are

1. All poles of elements of $B(s)$ have zero real part* and the residue matrix at any pole is nonnegative definite Hermitian.
2. $B'(-s) + B(s) = 0$ for all s such that s is not a pole of any element of $B(s)$.

Example 2.7.7 We shall verify that

$$z(s) = \frac{s(s^2 + 2)}{s^2 + 1}$$

is lossless positive real. By inspection, $z(-s) + z(s) = 0$, and the poles of $z(s)$, viz., $s = \infty$, $s = \pm j1$ have zero real part. The residue at $s = \infty$ is 1 and at $s = j1$ is $\lim_{s \to j1} s(s^2 + 2)/(s + j1) = \frac{1}{2}$. Therefore, $z(s)$ is lossless positive real.

Testing for the PR and LPR Properties

The situation as far as testing whether the PR and LPR conditions are satisfied is much as for testing whether the BR and LBR conditions are satisfied, with one complicating feature. It is possible for elements of the matrices being tested to have $j\omega$-axis poles, which requires some minor adjustments of the procedures (see [12]). We shall not outline any of the tests here, but the reader should be aware of the existence of such tests.

Problem 2.7.1 Suppose that $Z(s)$ is positive real and that $Z^{-1}(s)$ exists. Suppose also that $[I + Z(s)]^{-1}$ is known to exist. Show that $Z^{-1}(s)$ is positive real. [*Hint:* Show that $A(s) = [Z(s) + I]^{-1}[Z(s) - I]$ is bounded real, that $-A(s)$ is bounded real, and thence that $Z^{-1}(s)$ is positive real.]

Problem 2.7.2 *Suppose that an m port N, assumed to be linear and time invariant,* is known to have an impedance matrix $Z(s)$ that is positive real. Show

*For this purpose, a pole at infinity is deemed as having zero real part.

that $[I + Z(s)]^{-1}$ exists for all s in $\text{Re}\,[s] > 0$, and deduce that N has a bounded real scattering matrix.

Problem 2.7.3 Find whether the following matrices are positive real:

(a)
$$\begin{bmatrix} \dfrac{2s^5 + 11s^3 + 10s}{s^4 + 3s^2 + 2} & \dfrac{s^5}{s^4 + 3s^2 + 2} \\ \dfrac{s^5}{s^4 + 3s^2 + 2} & \dfrac{s^5 + 6s^3 + 7s}{s^4 + 3s^2 + 2} \end{bmatrix}$$

(b)
$$\begin{bmatrix} \dfrac{2s^3 + 5s}{s^4 + 5s^2 + 4} & \dfrac{3s}{s^4 + 5s^2 + 4} \\ \dfrac{3s}{s^4 + 5s^2 + 4} & \dfrac{5s^3 + 11s}{s^4 + 5s^2 + 4} \end{bmatrix}$$

Problem 2.7.4 Suppose that $Z_1(s)$ and $Z_2(s)$ are $m \times m$ PR matrices. Show that $Z_1(s) + Z_2(s)$ is PR. Establish an analogous result for LPR matrices.

2.8 RECIPROCAL *m* PORTS

An important class of networks are those m ports that contain resistor, inductor, capacitor, and transformer elements, but no gyrators. As already noted, gyrators may need to be constructed from active elements, so that their use usually demands power supplies. This may not always be acceptable.

The resistor, inductor, capacitor, and multiport transformer are all said to be *reciprocal* circuit elements, while the gyrator is *nonreciprocal*. An m port consisting entirely of reciprocal elements will be termed a *reciprocal m-port network*. We shall not explore the origin of this terminology here, which lies in thermodynamics. Rather, we shall indicate what constraints are imposed on the port descriptions of a network if the network is reciprocal.

Our basic tool will be Tellegen's Theorem, which we shall use to prove Lemma 2.8.1. From this lemma will follow the constraints imposed on immittance, hybrid, and scattering matrices of reciprocal networks.

Lemma 2.8.1. Let N be a reciprocal m-port network with all but two of its ports arbitrarily terminated in open circuits, short circuits, or two-terminal circuit elements. Without loss of generality, take these terminated ports as the last $m - 2$ ports. Suppose also that sources are connected at the first two ports so that with one choice of excitations, the port voltages and currents have Laplace transforms $V_1^{(1)}(s)$, $V_2^{(1)}(s)$, $I_1^{(1)}(s)$ and $I_2^{(1)}(s)$, while with another choice of excitations, the port voltages and currents have Laplace transforms $V_1^{(2)}(s)$, $V_2^{(2)}(s)$, $I_1^{(2)}(s)$, and $I_2^{(2)}(s)$. As depicted in Fig. 2.8.1, the subscript denotes the port number

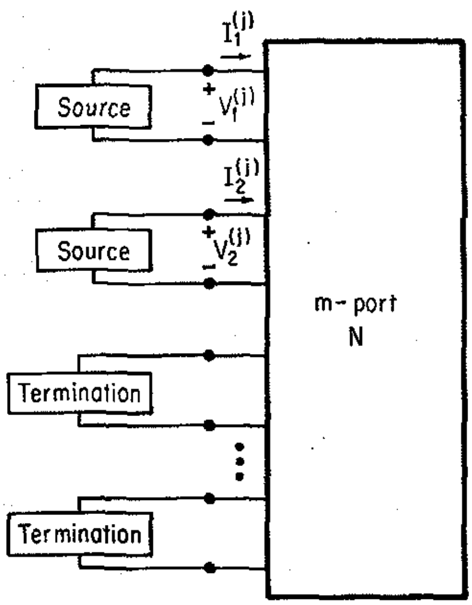


FIGURE 2.8.1. Arrangement to Illustrate Reciprocity; $j = 1, 2$ Gives Two Operating Conditions for Network.

and the superscript indexes the choice of excitations. Then

$$V_1^{(1)}(s)I_1^{(2)}(s) + V_2^{(1)}(s)I_2^{(2)}(s)$$
$$= V_1^{(2)}(s)I_1^{(1)}(s) + V_2^{(2)}(s)I_2^{(1)}(s) \qquad (2.8.1)$$

Before presenting the proof of this lemma, we ask the reader to note carefully the following points:

1. The terminations on ports 3 through m of the network N are unaltered when *different choices of excitations* are made. The network N is also unaltered. So one could regard N *with* the terminations as a two-port network N' with various excitations at its ports.
2. It is implicitly assumed that the excitations are Laplace transformable—it turns out that there is no point in considering the situation when they are not Laplace transformable.

Proof. Number the branches of the network N' (*not* N) from 3 to b, and consider a network N'' comprising N' together with the specified sources connected to its two ports. Assign voltage and current reference directions to the branches of N' in the usual way compatible with the application of Tellegen's theorem, with the voltage and current for the kth branch being denoted by $V_k^{(j)}(s)$ and $I_k^{(j)}(s)$, respectively. Here $j = 1$ or 2, depending on the choice of excitations. Notice that the pairs $V_1^{(j)}(s)$, $I_1^{(j)}(s)$ and $V_2^{(j)}(s)$, $I_2^{(j)}(s)$ have excitations opposite to those required by Tellegen's theorem.

Now $V_1^{(1)}(s), \ldots, V_b^{(1)}(s)$, being Laplace transforms of branch voltages, satisfy Kirchhoff's voltage law, and $I_1^{(2)}(s), \ldots, I_b^{(2)}(s)$, being Laplace transforms of branch currents, satisfy Kirchhoff's current law. Therefore, Tellegen's theorem implies that

$$V_1^{(1)}(s)I_1^{(2)}(s) + V_2^{(1)}(s)I_2^{(2)}(s) = \sum_{k=3}^{b} V_k^{(1)}(s)I_k^{(2)}(s)$$

Similarly,

$$V_1^{(2)}(s)I_1^{(1)}(s) + V_2^{(2)}(s)I_2^{(1)}(s) = \sum_{k=3}^{b} V_k^{(2)}(s)I_k^{(1)}(s)$$

Comparison with (2.8.1) shows that the lemma will be true if

$$\sum_{k=3}^{b} V_k^{(1)}(s)I_k^{(2)}(s) = \sum_{k=3}^{b} V_k^{(2)}(s)I_k^{(1)}(s) \tag{2.8.2}$$

Now if branch k comprises a two-terminal circuit element, i.e., a resistor, inductor, or capacitor, with impedance $Z_k(s)$, we have

$$\begin{aligned} V_k^{(1)}(s)I_k^{(2)}(s) &= I_k^{(1)}(s)Z_k(s)I_k^{(2)}(s) \\ &= I_k^{(1)}(s)V_k^{(2)}(s) \end{aligned} \tag{2.8.3}$$

If branches $k_1, \ldots, k_{p+q}$ are associated with a $(p + q)$-port transformer, it is easy to check that

$$\begin{aligned} &V_{k_1}^{(1)}(s)I_{k_1}^{(2)}(s) + \cdots + V_{k_{p+q}}^{(1)}(s)I_{k_{p+q}}^{(2)}(s) \\ &\qquad = V_{k_1}^{(2)}(s)I_{k_1}^{(1)}(s) + \cdots + V_{k_{p+q}}^{(2)}(s)I_{k_{p+q}}^{(1)} \end{aligned} \tag{2.8.4}$$

Summing (2.8.3) and (2.8.4) over all branches, we obtain (2.8.2), as required. $\nabla\nabla\nabla$

The lemma just proved can be used to prove the main results describing the constraints on the port descriptions of m ports. We only prove one of these results in the text, leaving to the problem of this section a request to the reader to prove the remaining results. All proofs are much the same, demanding application of the lemma with a cleverly selected set of exciting variables.

Theorem 2.8.1. Let N be a reciprocal m port. Then

1. If N possesses an impedance matrix $Z(s)$, it is symmetric.
2. If N possesses an admittance matrix $Y(s)$, it is symmetric.

3. If $H(s)$ is a hybrid matrix of N, then for $i \neq j$, one has

$$h_{ij}(s) = \pm h_{ji}(s)$$

with the $+$ sign holding if the exciting variables at ports i and j are both currents *or* both voltages, and the $-$ sign holding if one is a current and the other a voltage.

4. The scattering matrix $S(s)$ of N is symmetric.

Proof. We shall prove part 1 only. Let i, j be arbitrary different integers with $1 \leq i, j \leq m$. We shall show that $z_{ij}(s) = z_{ji}(s)$, where $Z(s) = [z_{ij}(s)]$ is the impedance matrix of N. Let us renumber temporarily the ports of N so that the new port 1 is the old port i and the new port 2 is the old port j. We shall show that with the new numbering, $z_{12}(s) = z_{21}(s)$.

First, suppose that all ports other than the first one are terminated in open circuits, and a current source with Laplace transform of the current equal to $I_1^{(1)}(s)$ is connected to port 1. Then the voltage present at port 2, viz., $V_2^{(1)}(s)$, is $z_{21}(s)I_1^{(1)}(s)$, while $I_2^{(1)}(s) = 0$, by the open-circuit constraint.

Second, suppose that all ports other than the second are terminated in open circuits, and a current source with Laplace transform of current equal to $I_2^{(2)}(s)$ is connected to port 2. Then the voltage present at port 1, viz., $V_1^{(2)}(s)$, is $z_{12}(s)I_2^{(2)}(s)$, while $I_1^{(2)}(s) = 0$.

Now apply Lemma 2.8.1. There results the equation

$$z_{21}(s)I_1^{(1)}(s)I_2^{(2)}(s) = z_{12}(s)I_1^{(1)}(s)I_2^{(2)}(s)$$

Since $I_1^{(1)}(s)$ and $I_2^{(2)}(s)$ can be arbitrarily chosen,

$$z_{21}(s) = z_{12}(s) \qquad \nabla\nabla\nabla$$

Complete proofs of the remaining points of the theorem are requested in the problem. Note that if $Z(s)$ and $Y(s)$ are both known to exist, then $Z(s) = Z'(s)$ implies that $Y(s) = Y'(s)$, since $Z(\cdot)$ and $Y(\cdot)$ are inverses. Also, if $Z(s)$ is known to exist, then symmetry of $S(s)$ follows from $S(s) = [Z(s) - I][Z(s) + I]^{-1}$ and the symmetry of $Z(s)$.

In the later work on synthesis we shall pay some attention to the reciprocal synthesis problem, which, roughly, is the problem of passing from a port description of a network fulfilling the conditions of Theorem 2.8.1 to a reciprocal network with port description coinciding with the prescribed description.

The reader should note exactly how the presence of a gyrator destroys reciprocity. First, the gyrator has a skew-symmetric impedance matrix—so that by Theorem 2.8.1, *it cannot* be a *reciprocal* network. Second, the arguments ending in Eq. (2.8.3) and (2.8.4) simply cannot be repeated for the gyrator, so the proof of Lemma 2.8.1 breaks down if gyrators are brought into the picture.

Problem 2.8.1 Complete the proof of Theorem 2.8.1. To prove part 4, it may be helpful to augment N by connecting series resistors at its ports, or to use the result of Problem 2.4.6.

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3

State-Space Equations and Transfer-Function Matrices

3.1 DESCRIPTIONS OF LUMPED SYSTEMS

In this chapter our aim is to explore (independently of any network associations) connections between the description of linear, lumped, time-invariant systems by transfer-function matrices and by state-space equations. We assume that the reader has been exposed to these concepts before, though perhaps not in sufficient depth to understand all the various connections between the two system descriptions. While this chapter does not purport to be a definitive treatise on linear systems, it does attempt to survey those aspects of linear-system theory, particularly the various connections between transfer-function matrices and state-space equations, that will be needed in the sequel. To do this in an appropriate length requires a terse treatment; we partially excuse this on the grounds of the reader having had prior exposure to most of the ideas.

In this section we do little more than summarize what is to come, and indicate how a transfer-function matrix may be computed from a set of state-space equations. In Section 2 we outline techniques for solving state-space equations, and in Sections 3, 4, and 5 we cover the topics of complete controllability and observability, minimal realizations, and the generation of state-space equations from a prescribed transfer-function matrix. In Sections 6 and 7 we examine two specific connections between transfer-function matrices and state-space equations—the relation between the degree of a transfer-function matrix and the dimension of the associated minimal realizations, and the stability properties of a system that can be inferred from transfer-function matrix and state-space descriptions. For more complete discussions of linear system theory, see, e.g., [1–3].

Two Linear System Descriptions

The most obvious way to describe a lumped physical system is via differential equations (or, sometimes, integro-differential equations). Though the most obvious procedure, this is not necessarily the most helpful. If there are well-identified *input*, or *excitation* or *control* variables, and well-identified *output* or *response* variables, and if there is little or no interest in the behavior of other quantities, a convenient description is provided by the *system impulse response* or its Laplace transform, the *system transfer-function matrix*.

Suppose that there are p scalar inputs and m scalar outputs. We arrange the inputs in vector form, to yield a p-vector function of time, $u(\cdot)$. Likewise the outputs may be arranged as an m-vector function of time $y(\cdot)$. The impulse response is an $m \times p$ matrix function* of time, $w(\cdot)$ say, which is zero for negative values of its argument and which relates $u(\cdot)$ to $y(\cdot)$ according to

$$y(t) = \int_0^t w(t - \tau)u(\tau)\, d\tau \tag{3.1.1}$$

where it is assumed that (1) $u(t)$ is zero for $t \leq 0$, and (2) there are zero initial conditions on the physical system at $t = 0$.

Of course, the time invariance of the system allows shifting of the time origin, so we can deal with an initial time $t_0 \neq 0$ if desired.

Introduce the notation

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{3.1.2}$$

[Here, $F(\cdot)$ is the Laplace transform of $f(\cdot)$.]

Then, with mild restrictions on $u(\cdot)$ and $y(\cdot)$, (3.1.1) may be written as

$$Y(s) = W(s)U(s) \tag{3.1.3}$$

The $m \times p$ matrix $W(\cdot)$, which is the Laplace transform of the impulse response $w(\cdot)$, is a matrix of real rational functions of its argument, provided the system with which it is associated is a lumped system.† Henceforth in

*More properly, a generalized function, since $w(\cdot)$ may contain delta functions and even their derivatives.

†Existence of $W(s)$ for all but a finite set of points in the complex plane is guaranteed when $W(s)$ is the transfer-function matrix of a lumped system.

this chapter we shall almost always omit the use of the word "real" in describing $W(s)$.

A second technique for describing lumped systems is by means of state-variable equations. These are differential equations of the form

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= H'x + Ju \end{aligned} \tag{3.1.4}$$

where x is a vector, say of dimension n, and F, G, H, and J are constant matrices of appropriate dimension. The vector x is termed the *state vector*, and (3.1.4) are known as *state-space equations*. Of course, $u(\cdot)$ and $y(\cdot)$ are as before.

As is immediately evident from (3.1.4), the state vector serves as an intermediate variable linking $u(\cdot)$ and $y(\cdot)$. Thus to compute $y(\cdot)$ from $u(\cdot)$, one would first compute $x(\cdot)$—by solving the first equation in (3.1.4)—and then compute $y(\cdot)$ from $x(\cdot)$ and $u(\cdot)$. But a state vector is indeed much more than this; we shall develop briefly here one of its most important interpretations, which is as follows. Suppose that an input $u_0(\cdot)$ is applied over the interval $[t_{-1}, t_0]$ and a second input $u_1(\cdot)$ is applied over $[t_0, t_1]$. To compute the resulting output over $[t_0, t_1]$, one could use

1. the value of $x(t_{-1})$ as an initial condition for the first equation in (3.1.4),
2. the control $u_0(\cdot)$ over $[t_{-1}, t_0]$, and
3. the control $u_1(\cdot)$ over $[t_0, t_1]$.

Alternatively, if we knew $x(t_0)$, it is evident from (3.1.4) that we could compute the output over $[t_0, t_1]$ using

1. the value of $x(t_0)$ as an initial condition for the first equation in (3.1.4), and
2. the control $u_1(\cdot)$ over $[t_0, t_1]$.

Let us think of t_0 as a present time, t_{-1} as a past time, and t_1 as a future time. What we have just said is that *in computing the future behavior of the system, the present state carries as much information as a past state and past control*, or we might say that *the present state sums up all that we need to know of the past in order to compute future behavior.*

State-space equations may be derived from the differential or integro-differential equations describing the lumped system; or they may be derived from knowledge of a transfer-function matrix $W(s)$ or impulse response $w(t)$; or, again, they may be derived from other state-space equations by mechanisms such as change of coordinate basis. The derivation of state-space equations directly from other differential equations describing the system cannot generally proceed by any systematic technique; we shall devote one chapter to such a derivation when the lumped system in question is a network. Derivation from a transfer-function matrix or other state-space equations will be discussed subsequently in this chapter.

The advantages and disadvantages of a state-space-equation description as opposed to a transfer-function-matrix description are many and often subtle. One obvious advantage of state-space equations is that the incorporation of nonzero initial conditions is possible.

In the remainder of this section we shall indicate *how to compute a transfer-function matrix from state-space equations.*

From (3.1.4) it follows by taking Laplace transforms that

$$sX(s) = FX(s) + GU(s)$$
$$Y(s) = H'X(s) + JU(s)$$

[Zero initial conditions have been assumed, since only then will the transfer function matrix relate $U(s)$ and $Y(s)$.] Straightforward manipulation yields

$$X(s) = (sI - F)^{-1}GU(s)$$

and

$$Y(s) = [J + H'(sI - F)^{-1}G]U(s) \tag{3.1.5}$$

Now compare (3.1.3) and (3.1.5). Assuming that (3.1.3) and (3.1.5) describe the same physical system, it is immediate that

$$W(s) = J + H'(sI - F)^{-1}G \tag{3.1.6}$$

Equation (3.1.6) shows how to pass from a set of state-space equations to the associated transfer-function matrix. Several points should be noted. First,

$$W(\infty) = \lim_{s\to\infty} W(s) = J \tag{3.1.7}$$

It follows that if $W(s)$ is prescribed, and if $W(\infty)$ is not finite, there cannot be a set of state-space equations of the form (3.1.4) describing the same physical system that $W(s)$ describes.* Second, we remark that although the computation of $(sI - F)^{-1}$ might seem a difficult task for large dimension F, an efficient algorithm is available (see, e.g., [4]). Third, it should be noted that the impulse response $w(t)$ can also be computed from the state-space equations, though this requires the ability to solve (3.1.4) when u is a delta function.

*Note that a rational $W(s)$ will become infinite for $s = \infty$ if and only if there is differentiation of the input to the system with transfer-function matrix $W(s)$ in producing the system output. This is an uncommon occurrence, but will arise in some of our subsequent work.

The end result is

$$w(t) = J\,\delta(t) + H'e^{Ft}G1(t) \tag{3.1.8}$$

where $\delta(\cdot)$ is the unit delta function, $1(t)$ the unit step function, and e^{Ft} is *defined as* the sum of the series

$$I + \frac{t}{1!}F + \frac{t^2}{2!}F^2 + \cdots$$

The matrix e^{Ft} is well defined for all F and all t, in the sense that the partial sums S_n of n terms of the above series are such that each entry of S_n approaches a unique finite limit as $n \longrightarrow \infty$ [4]. We shall explore additional properties of the matrix e^{Ft} in the next section.

Problem 3.1.1 Compute the transfer function associated with state-space equations

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$

$$y = [3 \quad 4]x + 2u$$

Problem 3.1.2 Show that if F is in block upper triangular form, i.e.,

$$F = \begin{bmatrix} F_1 & F_2 \\ 0 & F_3 \end{bmatrix}$$

then $(sI - F)^{-1}$ can be expressed in terms of $(sI - F_1)^{-1}$, $(sI - F_3)^{-1}$, and F_2.

Problem 3.1.3 Find state-space equations for the transfer functions

(a) $\dfrac{b}{s + a}$

(b) $\dfrac{c}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

Deduce a procedure for computing state-space equations from a scalar transfer function $W(s)$, assuming that $W(s)$ is expressible as a constant plus a sum of partial fraction terms with linear and quadratic denominators.

Problem 3.1.4 Suppose that $W(s) = J + H'(sI - F)^{-1}G$ and that $W(\infty) = J$ is nonsingular. Show that

$$W^{-1}(s) = J^{-1} - J^{-1}H'(sI - \overline{F - GJ^{-1}H'})^{-1}GJ^{-1}$$

This result shows how to write down a set of state-space equations of a system that is the *inverse* of a prescribed system whose state-space equations are known.

3.2 SOLVING THE STATE-SPACE EQUATIONS

In this section we outline briefly procedures for solving

$$\dot{x} = Fx + Gu$$
$$y = H'x + Ju \tag{3.2.1}$$

We shall suppose that $u(t)$ is prescribed for $t \geq t_0$, and that $x(t_0)$ is known. We shall show how to compute $y(t)$ for $t \geq t_0$.

In Section 3.1 we defined the matrix e^{Ft} as the sum of an infinite series of matrices. This matrix can be shown to satisfy the following properties:

1. $e^{Ft_1}e^{Ft_2} = e^{F(t_1+t_2)}$.
2. $e^{Ft}e^{-Ft} = I$.
3. $p(F)e^{Ft} = e^{Ft}p(F)$ for any polynomial $p(\cdot)$ in the matrix F.
4. $d/dt(e^{Ft}) = Fe^{Ft} = e^{Ft}F$.

One way to prove these properties is by manipulations involving the infinite-sum definition of e^{Ft}.

Now we turn to the solution of the first equation of (3.2.1).

Theorem 3.2.1. The solution of

$$\dot{x} = Fx + Gu \tag{3.2.2}$$

for prescribed $x(t_0) = x_0$ and $u(\tau)$, $\tau \geq t_0$, is unique and is given by

$$x(t) = e^{F(t-t_0)}x_0 + \int_{t_0}^{t} e^{F(t-\tau)}Gu(\tau)\,d\tau \tag{3.2.3}$$

In particular, the solution of the homogeneous equation

$$\dot{x} = Fx \tag{3.2.4}$$

is

$$x(t) = e^{F(t-t_0)}x_0 \tag{3.2.5}$$

Proof. We shall omit a proof of uniqueness, which is somewhat technical and not of great interest. Also, we shall simply verify that (3.2.3) is a solution of (3.2.2), although it is possible to deduce (3.2.3) knowing the solution (3.2.5) of the homogeneous equation (3.2.4) and using the classical device of variation of parameters.

To obtain $\dot{x}$ using (3.2.3), we need to differentiate $e^{F(t-t_0)}$ and $e^{F(t-\tau)}$. The derivative of $e^{F(t-t_0)}$ follows easily using properties 1 and 4:

$$\frac{d}{dt}e^{F(t-t_0)} = \frac{d}{dt}e^{Ft}e^{-Ft_0} = Fe^{Ft}e^{-Ft_0} = Fe^{F(t-t_0)}$$

Similarly, of course,

$$\frac{d}{dt}e^{F(t-\tau)} = Fe^{Ft}e^{-F\tau}$$

Differentiation of (3.2.3) therefore yields

$$\begin{aligned}\dot{x}(t) &= Fe^{F(t-t_0)}x_0 + F\int_{t_0}^{t} e^{F(t-\tau)}Gu(\tau)\,d\tau + Gu(t)\\ &= Fx(t) + Gu(t)\end{aligned}$$

on using (3.2.3). Thus (3.2.2) is recovered. Setting $t = t_0$ in (3.2.3) also yields $x(t_0) = x_0$, as required. This suffices to prove the theorem. ∇ ∇ ∇

Notice that $x(t)$ in (3.2.3) is the sum of two parts. The first, $\exp[F(t - t_0)]x_0$, is the *zero-input response* of (3.2.2), or the response that would be observed if $u(\cdot)$ were identically zero. The remaining part on the right side of (3.2.3) is the *zero-initial-state response*, or the response that would be observed if the initial state $x(t_0)$ were zero.

Example 3.2.1 Consider

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Suppose that $x(0) = [2 \quad 0]'$ and $u(t)$ is a unit step function, zero until $t = 0$. The series formula for e^{Ft} yields simply that

$$\exp\left\{\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} t\right\} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

Therefore

$$\begin{aligned}x(t) &= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & 0 \\ 0 & e^{-2(t-\tau)} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 2e^{-t} \\ 0 \end{bmatrix} + \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} e^{t} - 1 \\ \frac{1}{2}e^{2t} - \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} + 1 \\ \dfrac{-e^{-2t}}{2} + \dfrac{1}{2} \end{bmatrix}\end{aligned}$$

The complete solution of (3.2.1), rather than just the solution of the differential equation part of (3.2.1), is immediately obtainable from (3.2.3). The result is as follows.

Theorem 3.2.2. The solution of (3.2.1) for prescribed $x(t_0) = x_0$ and $u(\tau)$, $\tau \geq t_0$, is

$$y(t) = H'e^{F(t-t_0)}x_0 + \int_{t_0}^{t} H'e^{F(t-\tau)}Gu(\tau)\,d\tau + Ju(t) \tag{3.2.6}$$

Notice that by setting $x_0 = 0$, $t_0 = 0$, and

$$w(t) = J\,\delta(t) + H'e^{Ft}G1(t) \tag{3.2.7}$$

in the formula for $y(t)$, we obtain

$$y(t) = \int_0^t w(t-\tau)u(\tau)\,d\tau \tag{3.2.8}$$

In other words, $w(t)$ as defined by (3.2.7) is the impulse response associated with (3.2.1), as we claimed in the last section.

To apply (3.2.6) in practice, it is clearly necessary to be able to compute e^{Ft}. We shall now examine a procedure for calculating this matrix.

Calculation of e^{Ft}

We assume that F is known and that we need to find e^{Ft}. An exact technique is based on reduction of F to diagonal, or, more exactly, Jordan form.

If F is diagonalizable, we can find a matrix T (whose columns are actually eigenvectors of F) and a diagonal matrix Λ (whose diagonal entries are actually eigenvalues of F) such that

$$T^{-1}FT = \Lambda \tag{3.2.9}$$

(Procedures for finding such a T are outlined in, for example, [4] and [5]. The details of these procedures are inessential for the comprehension of this book.) Observe that

$$\Lambda^2 = T^{-1}FTT^{-1}FT = T^{-1}F^2T$$

More generally

$$\Lambda^n = T^{-1}F^nT$$

Therefore

$$e^{\Lambda t} = I + \Lambda t + \frac{\Lambda^2}{2!}t^2 + \cdots$$

$$= I + T^{-1}FTt + \frac{T^{-1}F^2T}{2!}t^2 + \cdots$$

$$= T^{-1}\left[I + Ft + \frac{F^2}{2!}t^2 + \cdots\right]T$$

$$= T^{-1}e^{Ft}T$$

Evidently

$$e^{Ft} = Te^{\Lambda t}T^{-1} \tag{3.2.10}$$

It remains to compute $e^{\Lambda t}$. This is easy however in view of the diagonal nature of Λ. Direct application of the infinite-sum definition shows that if

$$\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \tag{3.2.11}$$

then

$$e^{\Lambda t} = \operatorname{diag}\{e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}\} \tag{3.2.12}$$

Thus the problem of computing e^{Ft} is essentially a problem of computing eigenvalues and eigenvectors of F.

If F has complex eigenvalues, the matrix T is complex. If it is desired to work with real T, a T is determined such that

$$\begin{aligned} T^{-1}FT = [\lambda_1] \dotplus [\lambda_2] \dotplus \cdots \dotplus \begin{bmatrix} \lambda_{i+1} & -\mu_{i+1} \\ \mu_{i+1} & \lambda_{i+1} \end{bmatrix} \\ \dotplus \begin{bmatrix} \lambda_{i+2} & -\mu_{i+2} \\ \mu_{i+2} & \lambda_{i+2} \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} \lambda_{i+r} & -\mu_{i+r} \\ \mu_{i+r} & \lambda_{i+r} \end{bmatrix} \end{aligned} \tag{3.2.13}$$

where $\dotplus$ denotes the direct sum operation; $\lambda_1, \lambda_2, \ldots, \lambda_i$ are real eigenvalues of F; and $(\lambda_{i+1} \pm j\mu_{i+1}), (\lambda_{i+2} \pm j\mu_{i+2}), \ldots, (\lambda_{i+r} \pm j\mu_{i+r})$ are complex eigenvalues of F.

It may easily be verified that

$$\exp\left\{\begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix} t\right\} = \begin{bmatrix} e^{\lambda t}\cos\mu t & -e^{\lambda t}\sin\mu t \\ e^{\lambda t}\sin\mu t & e^{\lambda t}\cos\mu t \end{bmatrix} \tag{3.2.14}$$

The simplest way to see this is to observe that for arbitrary x_0,

$$\begin{aligned} &\frac{d}{dt}\left\{\begin{bmatrix} e^{\lambda t}\cos\mu t & -e^{\lambda t}\sin\mu t \\ e^{\lambda t}\sin\mu t & e^{\lambda t}\cos\mu t \end{bmatrix} x_0\right\} \\ &= \begin{bmatrix} \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t & -\lambda e^{\lambda t}\sin\mu t - \mu e^{\lambda t}\cos\mu t \\ \lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t & \lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t \end{bmatrix} x_0 \\ &= \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix}\left\{\begin{bmatrix} e^{\lambda t}\cos\mu t & -e^{\lambda t}\sin\mu t \\ e^{\lambda t}\sin\mu t & e^{\lambda t}\cos\mu t \end{bmatrix} x_0\right\} \end{aligned}$$

Observe also that with $t = 0$,

$$\begin{bmatrix} e^{\lambda t}\cos \mu t & -e^{\lambda t}\sin \mu t \\ e^{\lambda t}\sin \mu t & e^{\lambda t}\cos \mu t \end{bmatrix} x_0 = x_0$$

Therefore

$$x(t) = \begin{bmatrix} e^{\lambda t}\cos \mu t & -e^{\lambda t}\sin \mu t \\ e^{\lambda t}\sin \mu t & e^{\lambda t}\cos \mu t \end{bmatrix} x_0$$

is the solution of

$$\dot{x} = \begin{bmatrix} \lambda & -\mu \\ \mu & \lambda \end{bmatrix} x$$

which is equal to x_0 at $t = 0$. Equation (3.2.14) is now an immediate consequence of Theorem 3.2.1.

From (3.2.13) and (3.2.14), we have

$$\exp\{[T^{-1}FT]t\} = [e^{\lambda_1 t}] + [e^{\lambda_2 t}] + \cdots + [e^{\lambda_i t}]$$
$$+ \begin{bmatrix} e^{\lambda_{i+1} t}\cos \mu_{i+1} t & -e^{\lambda_{i+1} t}\sin \mu_{i+1} t \\ e^{\lambda_{i+1} t}\sin \mu_{i+1} t & e^{\lambda_{i+1} t}\cos \mu_{i+1} t \end{bmatrix} + \cdots$$
$$+ \begin{bmatrix} e^{\lambda_{i+r} t}\cos \mu_{i+r} t & -e^{\lambda_{i+r} t}\sin \mu_{i+r} t \\ e^{\lambda_{i+r} t}\sin \mu_{i+r} t & e^{\lambda_{i+r} t}\cos \mu_{i+r} t \end{bmatrix}$$

The left side is $T^{-1}e^{Ft}T$, and so, again, evaluation of e^{Ft} is immediate.

Example 3.2.2 Suppose that

$$F = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -10 \\ 0 & 1 & -5 \end{bmatrix}$$

With

$$T = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

it follows that

$$T^{-1}FT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

Hence

$$e^{Ft} = T\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t}\cos t & -e^{-2t}\sin t \\ 0 & e^{-2t}\sin t & e^{-2t}\cos t \end{bmatrix} T^{-1}$$

$$
= \begin{bmatrix} e^{-t} & -2e^{-t} + 2e^{-2t}\cos t + e^{-2t}\sin t & 5e^{-t} - 5e^{-2t}\cos t - 5e^{-2t}\sin t \\ 0 & e^{-2t}\cos t + 3e^{-2t}\sin t & -10e^{-2t}\sin t \\ 0 & e^{-2t}\sin t & e^{-2t}\cos t - 3e^{-2t}\sin t \end{bmatrix}
$$

If F is not diagonalizable, a similar approach can still be used based on evaluating e^{Jt}, where J is the Jordan form associated with F. Suppose that we find a T such that

$$
T^{-1}FT = J \tag{3.2.15}
$$

with J a direct sum of blocks of the form

$$
J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_i & 1 & & & & \cdot \\ \cdot & & \lambda_i & & & & \cdot \\ & & & \cdot & & & \\ \cdot & & & & \cdot & & \cdot \\ & & & & & \cdot & \\ \cdot & & & & & \lambda_i & 1 \\ 0 & \cdot & \cdot & \cdot & & 0 & \lambda_i \end{bmatrix} \tag{3.2.16}
$$

(Procedures for finding such a T are outlined in [4] and [5].) Clearly

$$
e^{Jt} = e^{J_1t} \dotplus e^{J_2t} \dotplus \cdots \tag{3.2.17}
$$

and

$$
e^{Ft} = Te^{Jt}T^{-1} \tag{3.2.18}
$$

The main problem is therefore to evaluate e^{J_it}. If J_i is an $r \times r$ matrix, one can show (and a derivation is called for in the problems) that

$$
e^{J_it} = \begin{bmatrix} e^{\lambda_it} & te^{\lambda_it} & \frac{1}{2!}t^2e^{\lambda_it} & \cdot & \cdot & \cdot & \frac{1}{(r-1)!}t^{r-1}e^{\lambda_it} \\ 0 & e^{\lambda_it} & te^{\lambda_it} & & & & \cdot \\ \cdot & & e^{\lambda_it} & & & & \cdot \\ & & & \cdot & & & \\ \cdot & & & & \cdot & & \cdot \\ & & & & & \cdot & \\ \cdot & & & & & e^{\lambda_it} & te^{\lambda_it} \\ 0 & \cdot & \cdot & \cdot & & 0 & e^{\lambda_it} \end{bmatrix} \tag{3.2.19}
$$

From (3.2.10) through (3.2.12) notice that for a diagonalizable F, each entry of e^{Ft} will consist of sums of terms $a_ie^{\lambda_it}$, where λ_i is an eigenvalue of F. Each eigenvalue λ_i of F shows up in the form $a_ie^{\lambda_it}$ in at least one entry of

e^{Ft}, though not necessarily in all. As a consequence, if all eigenvalues of F have negative real parts, then all entries of e^{Ft} decay with time, while if one or more eigenvalues of F have positive real part, some entries of e^{Ft} increase exponentially with time, and so on. If F has a nondiagonal Jordan form we see from (3.2.15) through (3.2.19) that some entries of e^{Ft} contain terms of the form $a_i t^\alpha e^{\lambda_i t}$ for positive integer α. If the λ_i have negative real parts, such terms also decay for large t.

Approximate Solutions of the State-Space Equations

The procedures hitherto outlined for solution of the state-space equations are applicable provided the matrix e^{Ft} is computable. Here we wish to discuss procedures that are applicable when e^{Ft} is not known. Also, we wish to note procedures for evaluating

$$\int_{t_0}^{t} e^{F(t-\tau)} Gu(\tau)\, d\tau$$

when either e^{Ft} is not known, or e^{Ft} is known but the integral is not analytically computable because of the form of $u(\cdot)$.

The problem of solving

$$\dot{x} = Fx + Gu \tag{3.2.2}$$

is a special case of the problem of solving the general nonlinear equation

$$\dot{x} = f(x, u, t) \tag{3.2.20}$$

Accordingly, techniques for the numerical solution of (3.2.20)—and there are many—may be applied to any specialized version of (3.2.20), including (3.2.2). We can reasonably expect that the special form of (3.2.2), as compared with (3.2.20), may result in those techniques applicable to (3.2.20) possessing extra properties or even a simple form when applied to (3.2.2). Also, again because of the special form of (3.2.2), we can expect that there may be numerical procedures for solving (3.2.2) that are not derived from procedures for solving (3.2.20), but stand independently of these procedures. We shall first discuss procedures for solving (3.2.20), indicating for some of these procedures how they specialize when applied to (3.2.2). Then we shall discuss special procedures for solving (3.2.2).

Our discussion will be brief. Material on the solution of (3.2.2) and (3.2.20) with particular reference to network analysis may be found in [6–8]. A basic reference dealing with the numerical solution of differential equations is [9].

The solution of (3.2.20) commences in the selection of a step size h. This is the interval separating successive instants of time at which $x(t)$ is computed; i.e., we compute $x(h)$, $x(2h)$, $x(3h)$, . . . given $x(0)$. We shall have more to say about the selection of h subsequently. Let us denote $x(nh)$ by x_n, $u(nh)$ by u_n, and $f(x(nh), u(nh), nh)$ by f_n. Numerical solution of (3.2.20) proceeds by specifying an algorithm for obtaining x_n in terms of $x_{n-1}, \ldots, x_{n-k}, f_n, \ldots, f_{n-k}$. The general form of the algorithm will normally be a recursive relation of the type

$$x_n = \sum_{i=1}^{k} a_i x_{n-i} + h \sum_{j=0}^{k} b_j f_{n-j} \tag{3.2.21}$$

where the a_i and b_j are certain numerical constants. We shall discuss some procedures for the selection of the a_i and b_j in the course of making several comments concerning (3.2.21).

1. *Predictor and predictor-corrector formulas.* If $b_0 = 0$, then f_n is not required in (3.2.21), and x_n depends only on $x_{n-1}, x_{n-2}, \ldots, x_{n-k}$ and $u_{n-1}, u_{n-2}, \ldots, u_{n-k}$ both linearly and through $f(\cdot, \cdot, \cdot)$. The formula (3.2.21) is then called a *predictor* formula. But if $b_0 \neq 0$, then the *right-hand* side of (3.2.21) depends on x_n; the equation cannot then necessarily be explicitly solved for x_n. But one may proceed as follows. First, replace $f_n = f(x_n, u_n, nh)$ by $f(x_{n-1}, u_n, nh)$ and with this replacement evaluate the right-hand side of (3.2.21). Call the result $\hat{x}_n$, since, in general, x_n will not result. Second, replace $f_n = f(x_n, u_n, nh)$ by $f(\hat{x}_n, u_n, nh)$ and with this replacement, evaluate the right side of (3.2.21). One may now take this quantity as x_n and think of it as a *corrected and predicted* value of x_n. [Alternatively, one may call the new right side $\hat{\hat{x}}_n$, replace f_n by $f(\hat{\hat{x}}_n, u_n, nh)$, and reevaluate, etc.] The formula (3.2.21) with $b_0 \neq 0$ is termed a *predictor-corrector* formula.

Example 3.2.3 A Taylor series expansion of (3.2.20) yields

$$x_n = x_{n-1} + hf(x_{n-1}, u_{n-1}, (n-1)h) \tag{3.2.22}$$

which is known as the Euler formula. It is a predictor formula. An approximation of

$$\int_{(n-1)h}^{nh} f(x, u, t)\, dt$$

by a trapezoidal integration formula yields

$$x_n = x_{n-1} + \frac{h}{2}[f(x_{n-1}, u_{n-1}, (n-1)h) + f(x_n, u_n, nh)] \tag{3.2.23}$$

which is a predictor-corrector formula. In practice, one could implement the slightly simpler formula

$$x_n = x_{n-1} + \frac{h}{2}[f_{n-1} + f(x_{n-1} + hf_{n-1}, u_n, nh)] \tag{3.2.24}$$

which is sometimes known as the *Heun algorithm.*

When specialized to the linear equation (3.2.2), Eq. (3.2.22) becomes

$$\begin{aligned} x_n &= x_{n-1} + hFx_{n-1} + hGu_{n-1} \\ &= (I + hF)x_{n-1} + hGu_{n-1} \end{aligned} \tag{3.2.25}$$

Equation (3.2.24) becomes

$$x_n = \left(I + hF + \frac{h^2F^2}{2}\right)x_{n-1} + \frac{hG}{2}(u_{n-1} + u_n) + \frac{h^2}{2}FGu_{n-1} \tag{3.2.26}$$

2. *Single-step and multistep formulas.* If in the basic algorithm equation (3.2.21) we have $k = 1$, then x_n depends on the values of x_{n-1}, u_{n-1}, and u_n, but not—at least directly—on the earlier values of $x_{n-2}, u_{n-2}, x_{n-3}, \ldots$. In this instance the algorithm is termed a *single-step* algorithm. If x_n depends on values of x_i and u_i for $i < n - 1$, the algorithm is called a *multistep algorithm.* Single-step algorithms are self-starting, in the sense that if $x(0) = x_0$ is known, $x_1, x_2, \ldots$, can be computed straightforwardly; in contrast, multistep algorithms are not self-starting. Knowing only x_0, there is no way of obtaining x_1 from the algorithm. Use of a multistep algorithm has generally to be preceded by use of a single-step algorithm to generate sufficient initial data for the multistep algorithm. Against this disadvantage, it should be clear that multistep algorithms are inherently more accurate—because at each step they use more data—than their single-step counterparts.

Example 3.2.4 (Adam's predictor-corrector algorithm) *Note:* This algorithm uses a different equation for obtaining the corrected value of x_n than for obtaining the predicted value; it is therefore a generalization of the predictor-corrector algorithms described. The algorithm is also a multistep algorithm; the predicted $\hat{x}_n$ is given by

$$\begin{aligned} \hat{x}_n = x_{n-1} + \frac{h}{24}[&-9f(x_{n-4}, u_{n-4}, (n-4)h) + 37f(x_{n-3}, u_{n-3}, (n-3)h) \\ &- 59f(x_{n-2}, u_{n-2}, (n-2)h) + 55f(x_{n-1}, u_{n-1}, (n-1)h)] \end{aligned}$$

The corrected x_n is given by

$$\begin{aligned} x_n = x_{n-1} + \frac{h}{24}[&9f(x_{n-3}, u_{n-3}, (n-3)h) - 5f(x_{n-2}, u_{n-2}, (n-2)h) \\ &+ 19f(x_{n-1}, u_{n-1}, (n-1)h) + 9f(\hat{x}_n, u_n, nh)] \end{aligned} \tag{3.2.27}$$

Additional correction by repeated use of (3.2.27) may be employed. Of course, the algorithm must be started by a single-step algorithm. Derivation of the appropriate formulas for the linear equation (3.2.2) is requested in the problems.

3. *Order of an algorithm.* Suppose that in a single-step algorithm x_{n-1} agrees exactly with the solution of the differential equation (3.2.20) for $t = (n-1)h$. In general x_n will not agree with the solution at time nh, but the difference, call it ϵ_n, will depend on the algorithm and on the step size h. *If* $\|\epsilon_n\| = 0(h^{p+1})$, *the algorithm is said to be of order** p. The definition extends to multi-step algorithms. High-order algorithms are therefore more accurate in going from step to step than are low-order ones.

Example 3.2.6 The Euler algorithm is of order 1, as is immediately checked. The Heun algorithm is of order 2—this is not difficult to check. An example of an algorithm of order p is obtained from a Taylor series expansion in (3.2.20):

$$x_n = x_{n-1} + hf_{n-1} + \frac{h^2}{2!}\frac{d}{dt}f_{n-1} + \cdots + \frac{h^p}{p!}\frac{d^{p-1}}{dt^{p-1}}f_{n-1} \tag{3.2.28}$$

assuming that the derivatives exist. A popular fourth-order algorithm is the *Runge-Kutta algorithm:*

$$x_n = x_{n-1} + hR_{n-1} \tag{3.2.29}$$

with

$$R_{n-1} = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{3.2.30}$$

and

$$\begin{aligned}
k_1 &= f_{n-1} \\
k_2 &= f\left(x_{n-1} + \frac{h}{2}k_1, u\left(\overline{n-1}h + \frac{h}{2}\right), \overline{n-1}h + \frac{h}{2}\right) \\
k_3 &= f\left(x_{n-1} + \frac{h}{2}k_2, u\left(\overline{n-1}h + \frac{h}{2}\right), \overline{n-1}h + \frac{h}{2}\right) \\
k_4 &= f(x_{n-1} + hk_3, u_n, nh)
\end{aligned}$$

This algorithm may be interpreted in geometric terms. First, the value k_1 of $f(x, u, t)$ at $t = (n-1)h$ is computed. Using this to estimate x at a time $h/2$ later, we compute $k_2 = f(x, u, t)$ at $t = (n-1)h + h/2$. Using this new value as an estimate of $\dot{x}$, we reestimate x at time $(n-1)h + (h/2)$ and compute $k_3 = f(x, u, t)$ at this time. Finally, using this value as an estimate of $\dot{x}$, we compute an estimate of $x(nh)$ and thus of $k_4 = f(x, u, t)$ at $t = nh$. The various values of $f(x, u, t)$ are then averaged, and this averaged value used in (3.2.29). It is interesting to note that this algorithm reduces to Simpson's rule if $f(x, u, t)$ is independent of x.

4. *Stability of an algorithm.* There are two kinds of errors that arise in applying the numerical algorithms: *truncation error,* which is due to the fact

*We write $f(h) = 0(h^p)$ when, as $h \longrightarrow 0$, $f(h) \longrightarrow 0$ at least as fast as h^p.

that any algorithm is an approximation, and *roundoff error*, due to rounding off of numbers arising in the calculations. An algorithm is said to be numerically stable if the errors do not build up as iterations proceed. Stability depends on the form of $f(x, u, t)$ and the step size; many algorithms are stable only for small h.

Example 3.2.6 In [8] the stability of the Euler algorithm applied to the linear equation (3.2.2) is discussed. The algorithm as noted earlier is

$$x_n = (I + hF)x_{n-1} + hGu_{n-1} \tag{3.2.31}$$

It is stable if the eigenvalues of F all possess negative real parts *and* if

$$h < \min_i \left[-2\frac{Re\,\lambda_i}{|\lambda_i|^2} \right] \tag{3.2.32}$$

where λ_i is an eigenvalue of F.

It would seem that of those methods for solving (3.2.2) which are specializations of methods for solving (3.2.20), the Runge-Kutta algorithm has proved to be one of the most attractive. It is not clear from the existing literature how well techniques above compare with special techniques for (3.2.2), two of which we now discuss.

The simple approximation of e^{Ft} by a finite number of terms in its series expansion has been suggested as one approach for avoiding the difficulties in computing e^{Ft}. Since the approximation by a fixed number of terms will be better the smaller the interval $[0, T]$ over which it is used, one possibility is to approximate e^{Ft} over $[0, h]$ and use an integration step size of h. Thus, following [8],

$$x_n = e^{Fh}x_{n-1} + e^{Fh}\int_0^h e^{-F\sigma}Gu(\overline{n-1}h + \sigma)\,d\sigma$$

If Simpson's rule is applied to the integral, one obtains

$$x_n = e^{Fh}\left(x_{n-1} + \frac{h}{6}Gu_{n-1}\right) + \frac{2h}{3}e^{1/2Fh}Gu\left(nh - \frac{h}{2}\right) + \frac{h}{6}Gu_n \tag{3.2.33}$$

The matrices e^{Fh} and $e^{1/2Fh}$ are approximated by polynomials in F obtained by power series truncation and the standard form of algorithm is obtained.

A second suggestion, discussed in [6], makes use of the fact that the exponential of a diagonal matrix is easy to compute. Suppose that $F = D + A$, where D is diagonal. We may rewrite (3.2.2) as

$$\dot{x} = Dx + [Ax + Gu] \tag{3.2.34}$$

from which it follows that

$$x(t) = e^{Dt}x(0) + \int_0^t e^{D(t-\tau)}[Ax(\tau) + Gu(\tau)]\, d\tau$$

and

$$x_n = e^{Dh}x_{n-1} + \int_0^h e^{D(h-\sigma)}[Ax(\overline{n-1}h + \sigma) + Gu(\overline{n-1}h + \sigma)]\, d\sigma \quad (3.2.35)$$

Again, Simpson's rule (or some other such rule) may be applied to evaluate the integral. Of course, there is now no need to approximate e^{Dh}. A condition for stability is reported in [6] as

$$h < \min_i \frac{c}{|\lambda_i(A)|} \quad (3.2.36)$$

where c is a small constant and $\lambda_i(A)$ is an eigenvalue of A.

The homogeneous equation (3.2.4) has, as we know, the solution $x(t) = e^{Ft}x_0$, when $x(0) = x_0$. Suggestions for approximate evaluation of e^{Ft} have been many (see, e.g., [10–17]). The three most important ideas used in these various techniques are as follows (note that any one technique may use only one idea):

1. Approximation of e^{Ft} by a truncated power series.
2. Evaluation of $e^{Fk\Delta}$ by evaluating $e^{F\Delta}$ for small Δ (with consequent early truncation of the power series), followed by formation of $(e^{F\Delta})^k$.
3. Making use of the fact that, by the Cayley–Hamilton theorem, e^{Ft} may be written as

$$e^{Ft} = \sum_{i=0}^{n-1} \alpha_i(t)F^i \quad (3.2.37)$$

where F is $n \times n$, and the $\alpha_i(\cdot)$ are analytic functions for which power series are available (see Problem 3.2.7). In actual computation, each $\alpha_i(\cdot)$ is approximated by truncating its power series.

Problem 3.2.1 Let F be a prescribed $n \times n$ matrix and X an unknown $n \times n$ matrix. Show that a solution (actually the unique solution) of

$$\dot{X} = FX$$

with $X(t_0) = X_0$ is

$$X(t) = e^{F(t-t_0)}X_0$$

If matrices A and B, respectively $n \times n$ and $m \times m$, are prescribed, and X is an unknown $n \times m$ matrix, show that a solution of

$$\dot{X} = AX + XB$$

is

$$X(t) = e^{A(t-t_0)}X_0 e^{B(t-t_0)}$$

when $X(t_0) = X_0$ is the prescribed boundary condition.
What is the solution of

$$\dot{X} = AX + XB + C(t)$$

where $C(t)$ is a prescribed $n \times m$ matrix?

Problem 3.2.2 Suppose that

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & \cdot \\ \cdot & & \lambda & & \cdot \\ & & & \ddots & \\ \cdot & & & & \cdot \\ \cdot & & & \lambda & 1 \\ 0 & \cdot & \cdot & \cdot & \lambda \end{bmatrix}$$

is an $r \times r$ matrix. Show that

$$e^{Jt} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2!}t^2 e^{\lambda t} & \cdots & \frac{1}{(r-1)!}t^{r-1}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} & & \cdot \\ \cdot & & e^{\lambda t} & & \cdot \\ \cdot & & & \ddots & \cdot \\ \cdot & & & e^{\lambda t} & te^{\lambda t} \\ 0 & \cdot & \cdot & \cdot & e^{\lambda t} \end{bmatrix}$$

Problem 3.2.3 (Sinusoidal response) Consider the state-space equations

$$\dot{x} = Fx + Gu$$
$$y = H'x + Ju$$

and suppose that for $t \geq 0$, $u(t) = u_0 \cos(\omega t + \phi)$ for some constant u_0, ϕ, and ω, with $j\omega$ not an eigenvalue of F. Show from the differential equation that

$$x(t) = x_c \cos \omega t + x_s \sin \omega t$$

is a solution if $x(0)$ is appropriately chosen. Evaluate x_c and x_s and conclude that

$$y(t) = \mathrm{Re}\{[J + H'(j\omega I - F)^{-1}G]e^{j\phi}\}u_0 \cos \omega t$$
$$+ \mathrm{Im}\{[J + H'(j\omega I - F)^{-1}G]e^{j\phi}\}u_0 \sin \omega t$$

Show also that if all eigenvalues of F have negative real parts, so that the zero-input component of $y(t)$ approaches zero as $t \longrightarrow \infty$, then the above formula for $y(t)$ is asymptotically correct as $t \longrightarrow \infty$, irrespective of $x(0)$.

Problem 3.2.4 Derive an Adams predictor-corrector formula for the integration of $\dot{x} = Fx + Gu$.

Problem 3.2.5 Derive Runge-Kutta formulas for the integration of $\dot{x} = Fx + Gu$.

Problem 3.2.6 Suppose that e^{AT} is approximated by the first $K + 1$ terms of its Taylor series. Let R denote the remainder matrix; i.e.,

$$R = \sum_{K+1}^{\infty} \frac{A^j T^j}{j!}$$

Let us introduce the constant ϵ, defined by

$$\epsilon = \frac{\|A\| T}{K + 2}$$

Show that [10]

$$\|R\| \leq \frac{(\|A\| T)^{K+1}}{(K + 1)!} \frac{1}{1 - \epsilon}$$

This error formula can be used in determining the point at which a power series should be truncated to achieve satisfactory accuracy.

Problem 3.2.7 The Cayley–Hamilton theorem says that if an $n \times n$ matrix A has characteristic polynomial $s^n + a_n s^{n-1} + \cdots + a_1$, then $A^n + a_n A^{n-1} + \cdots + a_1 I = 0$. Use this to show that there exist functions $\alpha_i(t)$, $i = 0, 1, \ldots, n - 1$ such that $\exp[At] = \sum_{i=0}^{n-1} \alpha_i(t) A^i$.

3.3 PROPERTIES OF STATE-SPACE REALIZATIONS

In this section our aim is to describe the properties of complete controllability and complete observability. This will enable statement of one of the fundamental results of linear system theory concerning *minimal realizations* of a transfer-function matrix, and enable solution of the problem of passing from a prescribed transfer-function matrix to a set of associated state-space equations.

Complete Controllability

The term complete controllability refers to a condition on the matrices F and G of a state-space equation

$$\dot{x} = Fx + Gu \tag{3.3.1}$$

that guarantees the ability to get from one state to another with an appropriate control.

Complete Controllability Definition. The pair $[F, G]$ is completely controllable if, given any $x(t_0)$ and t_0, there exists $t_1 > t_0$ and a control $u(\cdot)$ defined over $[t_0, t_1]$ such that if this control is applied to (3.3.1), assumed in state $x(t_0)$ at t_0, then at time t_1 there results $x(t_1) = 0$.

We shall now give a number of equivalent formulations of the complete controllability statement.

Theorem 3.3.1. The pair $[F, G]$ is completely controllable if and only if any of the following conditions hold:

1. There does not exist a nonzero constant vector w such that $w'e^{Ft}G = 0$ for all t.
2. If F is $n \times n$, rank $[G\ FG\ F^2G \cdots F^{n-1}G] = n$.
3. $\int_{t_0}^{t_1} e^{-Ft}GG'e^{-F't}\,dt$ is nonsingular for all t_0 and t_1, with $t_1 > t_0$.
4. Given t_0 and t_1, $t_0 < t_1$, and two states $x(t_0)$ and $x(t_1)$, there exists a control $u(\cdot)$ defined over $[t_0, t_1]$ taking $x(t_0)$ at time t_0 to $x(t_1)$ at time t_1.

Proof. We shall prove that the definition implies 1, that 1 and 2 imply each other, that 1 implies 3, that 3 implies 4, and that 4 implies the definition.

Definition implies 1. Suppose that there exists a nonzero constant vector w such that $w'e^{Ft}G = 0$ for all t. We shall show that this contradicts the definition. Take $x(t_0) = e^{-F(t_1-t_0)}w$. By definition, there exists $u(\cdot)$ such that

$$0 = e^{F(t_1-t_0)}[e^{-F(t_1-t_0)}w] + \int_{t_0}^{t_1} e^{F(t_1-\tau)}Gu(\tau)\,d\tau$$

or

$$0 = w + \int_{t_0}^{t_1} e^{F(t_1-\tau)}Gu(\tau)\,d\tau$$

Multiply on the left by w'. Then, since $w'e^{F't}G = 0$ for all t, we obtain

$$0 = w'w$$

This is a contradiction.

1 and 2 imply each other. Suppose that $w'e^{Ft}G = 0$ for all t and some nonzero constant vector w. Then

$$\frac{d^i}{dt^i}(w'e^{Ft}G) = w'F^i e^{Ft}G = 0$$

for all i. Set $t = 0$ in these relations for $i = 1, 2, \ldots, n - 1$ to conclude that

$$w'[G\ FG \cdots F^{n-1}G] = 0 \tag{3.3.2}$$

Noting that the matrix $[G\ FG \cdots F^{n-1}G]$ has n rows, it follows that if this matrix has rank n, there cannot exist w satisfying (3.3.2) and therefore there cannot exist a nonzero w such that $w'e^{Ft}G = 0$ for all t; i.e., *2 implies 1.*

To prove the converse, suppose that 2 fails. We shall show that 1 fails, from which it follows that 1 implies 2. Accordingly, suppose that there exists a nonzero w such that (3.3.2) holds. By the Cayley–Hamilton theorem, F^n, F^{n+1}, and higher powers of F are linear combinations of $I, F, \ldots, F^{n-1}$, and so

$$w'F^iG = 0 \qquad \text{for all } i$$

Therefore, using the series-expansion definition of e^{Ft},

$$w'e^{Ft}G = 0 \qquad \text{for all } t$$

This says that 1 fails; so we have proved that failure of 2 implies failure of 1, or that *1 implies 2.*

1 implies 3. We shall show that if 3 fails, then 1 fails. This is equivalent to showing that 1 implies 3. Accordingly, suppose that

$$\int_{t_0}^{t_1} e^{-Ft}GG'e^{-F't}\,dt\,w = 0$$

for some t_0, t_1, and nonzero constant w, with $t_1 > t_0$. Then

$$\int_{t_0}^{t_1} (w'e^{-Ft}G)(G'e^{-F't}w)\,dt = 0$$

The integrand is nonnegative; so

$$(w'e^{-Ft}G)(G'e^{-F't}w) = 0$$

for all t in $[t_0, t_1]$, whence

$$w'e^{-Ft}G = 0$$

for all t in $[t_0, t_1]$. By the analyticity of $w'e^{Ft}G$ as a function of t, it follows that $w'e^{Ft}G = 0$ for all t; i.e., 1 fails.

3 implies 4. What we have to show is that for given t_0, t_1, $x(t_0)$, and $x(t_1)$, with $t_1 > t_0$, there exists a control $u(\cdot)$ such that

$$x(t_1) = e^{F(t_1-t_0)}x(t_0) + \int_{t_0}^{t_1} e^{F(t_1-\tau)}Gu(\tau)\,d\tau$$

Observe that if we take

$$u(t) = G'e^{-F't}\left[\int_{t_0}^{t_1} e^{-F\sigma}GG'e^{-F'\sigma}\,d\sigma\right]^{-1}[e^{-Ft_1}x(t_1) - e^{-Ft_0}x(t_0)]$$

then

$$\begin{aligned}
&e^{F(t_1-t_0)}x(t_0) + \int_{t_0}^{t_1} e^{F(t_1-\tau)}Gu(\tau)\,d\tau \\
&\quad = e^{F(t_1-t_0)}x(t_0) + e^{Ft_1}\int_{t_0}^{t_1} e^{-F\tau}GG'e^{-F'\tau}\,d\tau\left[\int_{t_0}^{t_1} e^{-F\sigma}GG'e^{-F'\sigma}\,d\sigma\right]^{-1} \\
&\qquad \times [e^{-Ft_1}x(t_1) - e^{-Ft_0}x(t_0)] \\
&\quad = e^{F(t_1-t_0)}x(t_0) + x(t_1) - e^{F(t_1-t_0)}x(t_0) \\
&\quad = x(t_1)
\end{aligned}$$

as required.

4 implies the definition. This is trivial, since the definition is a special case of 4, corresponding to $x(t_1) = 0$. $\nabla\nabla\nabla$

Property 2 of Theorem 3.3.1 offers what is probably the most efficient procedure for *checking* complete controllability.

Example 3.3.1 Suppose that

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \qquad g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$[g \quad Fg \quad F^2g] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 6 \end{bmatrix}$$

This matrix has rank 3, and so $[F, g]$ is completely controllable.

Two other important properties of complete controllability will now be presented. Proofs are requested in the examples.

Theorem 3.3.2. Suppose that $[F, G]$ is completely controllable, with F $n \times n$ and G $n \times p$. Then for any $n \times p$ matrix K, the pair $[F + GK', G]$ is completely controllable.

In control-systems terminology, this says that complete controllability is invariant under state-variable feedback. The system

$$\dot{x} = (F + GK')x + Gu$$

may be derived from (3.3.1) by replacing u by $(u + K'x)$, i.e., by allowing the input to be composed of an external input and a quantity derived from the state vector.

Theorem 3.3.3. Suppose that $[F, G]$ is completely controllable, and T is an $n \times n$ nonsingular matrix. Then $[TFT^{-1}, TG]$ is completely controllable. Conversely, if $[F, G]$ is not completely controllable, $[TFT^{-1}, TG]$ is not completely controllable.

To place this theorem in perspective, consider the effect of the transformation

$$\hat{x} = Tx \tag{3.3.3}$$

on (3.3.1). Note that the transformation is invertible if T is nonsingular. It follows that

$$\dot{\hat{x}} = T\dot{x} = TFx + TGu = TFT^{-1}\hat{x} + TGu \tag{3.3.4}$$

Because the transformation (3.3.3) is invertible, it follows that x and $\hat{x}$ *abstractly* represent the same quantity, and that (3.3.1) and (3.3.4) are *abstractly* the same equation. The theorem guarantees that two versions of the same abstract equation inherit the same controllability properties.

The definition and property 4 of Theorem 3.3.1 suggest that if $[F, G]$ is not completely controllable, some part of the state vector, or more accurately, one or more linear functionals of it, are unaffected by the input. The next theorem pins this property down precisely. We state the theorem, then offer several remarks, and then we prove it.

Theorem 3.3.4. Suppose that in Eq. (3.3.1) the pair $[F, G]$ is not completely controllable. Then there exists a nonsingular constant matrix T such that

$$TFT^{-1} = \begin{bmatrix} \hat{F}_{11} & \hat{F}_{12} \\ 0 & \hat{F}_{22} \end{bmatrix} \qquad TG = \begin{bmatrix} \hat{G}_1 \\ 0 \end{bmatrix} \tag{3.3.5}$$

with $[\hat{F}_{11}, \hat{G}_1]$ completely controllable.

As we have remarked, under the transformation (3.3.3), Eq. (3.3.1) passes to

$$\dot{\hat{x}} = TFT^{-1}\hat{x} + TGu \tag{3.3.6}$$

Let us partition $\hat{x}$ conformably with the partitioning of TFT^{-1}. We derive, in obvious notation,

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{F}_{11}\hat{x}_1 + \hat{F}_{12}\hat{x}_2 + \hat{G}_1 u \\ \dot{\hat{x}}_2 &= \hat{F}_{22}\hat{x}_2 \end{aligned} \tag{3.3.7}$$

It is immediately clear from this equation that $\hat{x}_2$ will be totally unaffected by the input or $\hat{x}_2(t_0)$; i.e., the $\hat{x}_2$ part of $\hat{x}$ is uncontrollable, in terms of both the definition and common sense. Aside from the term $\hat{F}_{12}\hat{x}_2$ in the differential equation for $\hat{x}_1$, we see that $\hat{x}_1$ satisfies a differential equation with the complete controllability property; i.e., the $\hat{x}_1$ part of $\hat{x}$ is completely controllable. The purpose of the theorem is thus to explicitly show what is controllable and what is not.

Proof. Suppose that F is $n \times n$, and that $[F, G]$ is not completely controllable. Let

$$\operatorname{rank}\,[G\; FG\; \cdots\; F^{n-1}G] = r < n \tag{3.3.8}$$

Notice that the space spanned by $\{G, FG, \ldots, F^{n-1}G\}$ is an invariant F space; i.e., if any vector in the space is multiplied by F, then the resulting vector is in the space. This follows by noting that

$$\begin{aligned} F[G\; FG\; \cdots\; F^{n-1}G] &= [FG\; F^2G\; \cdots\; F^nG] \\ &= \left[FG\; F^2G\; \cdots\; F^{n-1}G\; \sum_{i=0}^{n-1} a_i F^i G\right] \end{aligned}$$

for some set of constants a_i, the existence of which is guaranteed by the Cayley–Hamilton theorem.

Define the matrix T as follows. The first r columns of T^{-1} are a basis for the space spanned by $\{G, FG, \dots, F^{n-1}G\}$, while the remaining $n - r$ columns are chosen so as to guarantee that T^{-1} is nonsingular. The existence of these column vectors is guaranteed by (3.3.8). Since G is in the space spanned by $\{G, FG, \dots, F^{n-1}G\}$ and since this space is spanned by the first r columns of T^{-1}, it follows that

$$G = T^{-1}\begin{bmatrix}\hat{G}_1 \\ 0\end{bmatrix} \tag{3.3.9}$$

for some $\hat{G}_1$ with r rows. The same argument shows that

$$[G\ FG \cdots F^{n-1}G] = T^{-1}\begin{bmatrix}\hat{S}_1 \\ 0\end{bmatrix} \tag{3.3.10}$$

where $\hat{S}_1$ has r rows. By (3.3.8), it follows that $\hat{S}_1$ has rank r. Also, we must have

$$F[G\ FG \cdots F^{n-1}G] = T^{-1}\begin{bmatrix}\hat{S}_2 \\ 0\end{bmatrix}$$

for some $\hat{S}_2$ with r rows. This implies

$$FT^{-1}\begin{bmatrix}\hat{S}_1 \\ 0\end{bmatrix} = T^{-1}\begin{bmatrix}\hat{S}_2 \\ 0\end{bmatrix}$$

or

$$TFT^{-1}\begin{bmatrix}\hat{S}_1 \\ 0\end{bmatrix} = \begin{bmatrix}\hat{S}_2 \\ 0\end{bmatrix}$$

Since $\hat{S}_1$ has rank r, this implies that

$$TFT^{-1} = \begin{bmatrix}\hat{F}_{11} & \hat{F}_{12} \\ 0 & \hat{F}_{22}\end{bmatrix} \tag{3.3.11}$$

where $\hat{F}_{11}$ is $r \times r$. Equations (3.3.9) and (3.3.11) jointly yield (3.3.5). It remains to be shown that $[\hat{F}_{11}, \hat{G}_1]$ is completely controllable. To see this, notice that (3.3.9) and (3.3.11) imply that

$$T[G\ FG \cdots F^{n-1}G] = \begin{bmatrix}\hat{G}_1 & \hat{F}_{11}\hat{G}_1 & \cdots & \hat{F}_{11}^{n-1}\hat{G}_1 \\ 0 & 0 & & 0\end{bmatrix}$$

From (3.3.10) and the fact that $\hat{S}_1$ has rank r, it follows that

$$\text{rank}\,[\hat{G}_1\ \hat{F}_{11}\hat{G}_1\ \cdots\ (\hat{F}_{11})^{n-1}\hat{G}_1] = \text{rank}\,\hat{S}_1 = r$$

Now $\hat{F}_{11}$ is $r \times r$, and therefore $\hat{F}_{11}^r, \hat{F}_{11}^{r+1}, \dots, \hat{F}_{11}^{n-1}$ depend linearly on $I, \hat{F}_{11}, \hat{F}_{11}^2, \dots, \hat{F}_{11}^{r-1}$, by the Cayley–Hamilton theorem. It follows that

$$\text{rank}\,[\hat{G}_1\ \ \hat{F}_{11}\hat{G}_1\ \cdots\ \hat{F}_{11}^{r-1}\hat{G}_1] = \text{rank}\,[\hat{G}_1\ \ \hat{F}_{11}\hat{G}_1\ \cdots\ \hat{F}_{11}^{n-1}\hat{G}_1] = r$$

This establishes the complete controllability result. ∇ ∇ ∇

Example 3.3.2 Suppose that

$$F = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} \qquad g = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Observe that

$$[g\ \ Fg\ \ F^2g] = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

which has rank 2, so that $[F, g]$ is not completely controllable. We now seek an invertible transformation of the state vector that will highlight the uncontrollability in the sense of the preceding theorem.

We take for T^{-1} the matrix

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The first two columns are a basis for the subspace spanned by g, Fg, and F^2g, while the third column is linearly independent of the first two. It follows then that

$$T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \qquad TFT^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -2 \\ 0 & 0 & -1 \end{bmatrix} \qquad Tg = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Observe that

$$\hat{F}_{11} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \qquad \hat{G}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad [\hat{G}_1\ \ \hat{F}_{11}\hat{G}_1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which exhibits the complete controllability of $[\hat{F}_{11}, \hat{G}_1]$. Also

$$(T^{-1})'h = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

We can write new state-space equations as

$$\dot{\hat{x}}_1 = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\hat{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u + \begin{bmatrix} -1 \\ -2 \end{bmatrix}\hat{x}_2$$
$$\dot{\hat{x}}_2 = -\hat{x}_2$$
$$y = [0 \quad 1]\hat{x}_1 + \hat{x}_2$$

If, for the above example, we compute the transfer function from the original F, g, and h, we obtain $1/(s + 1)^2$, which is also the transfer function associated with state equations from which the uncontrollable part has been removed:

$$\dot{\hat{x}}_1 = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\hat{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$
$$y = [0 \quad 1]\hat{x}_1$$

This is no accident, but a direct consequence of Theorem 3.3.4. The following theorem provides the formal result.

Theorem 3.3.5. Consider the state-space equations

$$\dot{x} = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}x + \begin{bmatrix} G_1 \\ 0 \end{bmatrix}u \qquad (3.3.12)$$
$$y = [H'_1 \quad H'_2]x + Ju$$

Then the transfer-function matrix relating $U(s)$ to $Y(s)$ is $J + H'_1(sI - F_{11})^{-1}G_1$.

The proof of this theorem is requested in the problems. In essence, the theorem says that the "uncontrollable part" of the state-space equations can be thrown away in determining the transfer-function matrix. In view of the latter being a description of input-output properties, this is hardly surprising, since the uncontrollable part of x is that part not connected to the input u.

Complete Observability

The term complete observability refers to a relationship existing between the state and the output of a set of state-space equations:

Under (3.3.3), we obtain

$$\begin{aligned} \dot{\hat{x}} &= TFT^{-1}\hat{x} + TGu \\ y &= [(T^{-1})'H]'\hat{x} = H'T^{-1}\hat{x} \end{aligned} \tag{3.3.14}$$

Thus transformations of the form (3.3.3) transform both state-space equations, according to the arrangement

$$F \longrightarrow TFT^{-1} \qquad G \longrightarrow TG \qquad H \longrightarrow (T^{-1})'H$$

In other words, if these replacements are made in a set of state-space equations, and if the initial condition vector is transformed according to (3.3.3), the input $u(\cdot)$ and output $y(\cdot)$ will be related in the same way. The only difference is that the variable x, which can be thought of as an intermediate variable arising in the course of computing $y(\cdot)$ from $u(\cdot)$, is changed.

This being the case, it should follow that Eq. (3.3.14) should define the same transfer-function matrix as (3.3.13). Indeed, this is so, since

$$\begin{aligned} [(T^{-1})'H]'(sI - TFT^{-1})^{-1}TG &= H'T^{-1}\{T(sI - F)T^{-1}\}^{-1}TG \\ &= H'T^{-1}T(sI - F)^{-1}T^{-1}TG \\ &= H'(sI - F)^{-1}G \end{aligned} \tag{3.3.15}$$

Let us sum up these important facts in a theorem.

Theorem 3.3.9. Suppose that the triple $\{F, G, H\}$ defines state-space equations in accord with (3.3.13). Then if T is any nonsingular matrix, $\{TFT^{-1}, TG, (T^{-1})'H\}$ defines state-space equations in accord with (3.3.14), where the relation between $\hat{x}$ and x is $\hat{x} = Tx$. The pairs $[F, G]$ and $[TFT^{-1}, TG]$ are simultaneously either completely controllable or they are not, and $[F, H]$ and $[TFT^{-1}, (T^{-1})'H]$ are simultaneously either completely observable or they are not. The variables $u(\cdot)$ and $y(\cdot)$ are related in the same way by both equation sets; in particular, the transfer-function matrices associated with both sets of equations are the same [see (3.3.15)].

Notice that by Theorems 3.3.3 and 3.3.8, the controllability and observability properties of (3.3.13) and (3.3.14) are the same.

Notice also that if the second equation of (3.3.13) were

$$y = H'x + Ju$$

then the above arguments would still carry through, the sole change being that the second equation of (3.3.14) would be

$$y = H'T^{-1}\hat{x} + Ju$$

The dual theorem corresponding to Theorem 3.3.4 is as follows:

Theorem 3.3.10. Suppose that in Eq. (3.3.13) the pair $[F, H]$ is not completely observable. Then there exists a nonsingular matrix T such that

$$TFT^{-1} = \begin{bmatrix} \hat{F}_{11} & 0 \\ \hat{F}_{21} & \hat{F}_{22} \end{bmatrix}$$

$$(T^{-1})'H = \begin{bmatrix} \hat{H}_1 \\ 0 \end{bmatrix}$$

with $[\hat{F}_{11}, \hat{H}_1]$ completely observable. The matrix $\hat{F}_{11}$ is $r \times r$, where $r = \text{rank } [H \;\; F'H \cdots (F')^{n-1}H]$.

It is not hard to see the physical significance of this structure of TFT^{-1} and $(T^{-1})'H$. With $\hat{x}$ as state vector, with obvious partitioning of $\hat{x}$, and with

$$TG = \begin{bmatrix} \hat{G}_1 \\ \hat{G}_2 \end{bmatrix}$$

we have

$$\dot{\hat{x}}_1 = \hat{F}_{11}\hat{x}_1 + \hat{G}_1 u$$
$$y = \hat{H}'_1\hat{x}_1$$
$$\dot{\hat{x}}_2 = \hat{F}_{21}\hat{x}_1 + \hat{F}_{22}\hat{x}_2 + \hat{G}_2 u$$

Observe that y depends directly on $\hat{x}_1$ and not $\hat{x}_2$, while the differential equation for $\hat{x}_1$ is also independent of $\hat{x}_2$. Thus y is not even indirectly dependent on $\hat{x}_2$. In essence, the $\hat{x}_1$ part of $\hat{x}$ is observable and the $\hat{x}_2$ part unobservable.

The dual of Theorem 3.3.5 is to the effect that the state-space equations

$$\dot{x} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} x + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u$$
$$y = [H'_1 \quad 0]x$$

define the same transfer-function matrix as the equations

$$\dot{x} = F_{11}x + G_1 u$$
$$y = H'_1 x$$

In other words, we can eliminate the unobservable part of the state-space equations in computing a transfer-function matrix. Theorem 3.3.5 of course said that we could scrap the uncontrollable part. Now this elimination of the unobservable part and uncontrollable part requires the F, G, and H matrices to have a special structure; so we might ask whether any form of elimination is possible when F, G, and H do not have the special structure. Theorem 3.3.9 may be used to obtain a result for all triples $\{F, G, H\}$, rather than specially structured ones, the result being as follows:

Theorem 3.3.11. Consider the state-space equations (3.3.13), and suppose that complete controllability of $[F, G]$ and complete observability of $[F, H]$ do not simultaneously hold. Then there exists a set of state-space equations with the state vector of lower dimension than the state vector in (3.3.13), together with a constructive procedure for obtaining them, such that if these equations are

$$\begin{aligned} \dot{x}_M &= F_M x_M + G_M u \\ y &= H'_M x_M \end{aligned} \tag{3.3.16}$$

then $H'(sI - F)^{-1}G = H'_M(sI - F_M)^{-1}G_M$, with $[F_M, G_M]$ completely controllable and $[F_M, H_M]$ completely observable.

Proof. Suppose that $[F, G]$ is not completely controllable. Use Theorem 3.3.4 to find a coordinate-basis transformation yielding state-space equations of the form of (3.3.12). By Theorem 3.3.9, the transfer-function matrix of the second set of equations is still $H'(sI - F)^{-1}G$. By Theorem 3.3.5, we may drop the uncontrollable states without affecting the transfer-function matrix. A set of state-space equations results that now have the complete controllability property, the transfer-function matrix of which is known to be $H'(sI - F)^{-1}G$.

Theorem 3.3.10 and further application of Theorem 3.3.9 allow us to drop the unobservable states and derive observable state-space equations. Controllability is retained—a proof is requested in the problems—and the transfer-function matrix of each set of state-space equations arising is still $H'(sI - F)^{-1}G$.

If $[F, G]$ is initially completely controllable, then we simply drop the unobservable states via Theorem 3.3.10. In any case, we obtain a set of state-space equations (3.3.16) that guarantees simultaneously the complete controllability and observability properties, and with transfer-function matrix $H'(sI - F)^{-1}G$. ∇ ∇ ∇

Example 3.3.3 Consider the triple $\{F, g, h\}$ described by

$$F = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} \qquad g = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad h = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

As noted in Example 3.3.2, the pair $[F, g]$ is not completely controllable. Moreover, the pair $[F, h]$ is not completely observable, because

$$[h \quad F'h \quad (F')^2h] = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

which has rank 2. We shall derive a set of state-space equations that are completely controllable and observable and have as their transfer function $h'(sI - F)^{-1}g$, which may be computed to be $-1/(s + 1)$.

First, we aim to get the state equation in a form that allows us to drop the uncontrollable part. This we did in Example 3.3.2, where we showed that with

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

we had

$$TFT^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -2 & -2 \\ 0 & 0 & -1 \end{bmatrix} \qquad Tg = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Also, it is easily found that

$$(T^{-1})'h = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Completely controllable state equations with transfer function $h'(sI - F)^{-1}g$ are therefore provided by

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = [-1 \quad 1]x$$

These equations are not completely observable because the "observability matrix"

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

does not have rank 2. Accordingly, we now seek to eliminate the unobservable part. The procedure now is to find a matrix T such that the columns of T' consist of a basis for the subspace spanned by the columns of the observability matrix, together with linearly independent columns yielding a nonsingular T'. Here, therefore,

$$T' = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

will suffice. It follows that

$$T = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

and the new state-space equations are

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}x + \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}u \\ &= \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}x + \begin{bmatrix} -1 \\ 0 \end{bmatrix}u \end{aligned}$$

and

$$\begin{aligned} y &= [-1 \quad 1]\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}x \\ &= [1 \quad 0]x \end{aligned}$$

Eliminating the unobservable part, we obtain

$$\begin{aligned} \dot{x} &= [-1]x + [-1]u \\ y &= x \end{aligned}$$

which has transfer function $-1/(s + 1)$.

Any triple $\{F, G, H\}$ such that $H'(sI - F)^{-1}G = W(s)$ for a prescribed transfer-function matrix $W(s)$ is called a *realization* or *state-space realization* of $W(s)$. If $[F, G]$ and $[F, H]$ are, respectively, completely controllable and observable, the realization is termed *minimal*, for reasons that will shortly become clear. The *dimension* of a realization is the dimension of the associated state vector.

Does an arbitrary real rational $W(s)$ always possess a realization? If $\lim_{s\to\infty} W(s) = W(\infty)$ is nonzero, then it is impossible for $W(s)$ to equal $H'(sI - F)^{-1}G$ for any triple $\{F, G, H\}$. However, if $W(\infty)$ is finite but $W(s)$ is otherwise arbitrary, there are always quadruples $\{F, G, H, J\}$, as we shall show in Section 3.5, such that $W(s)$ is precisely $J + H'(sI - F)^{-1}G$. We call such quadruples realizations and extend the use of the word minimal to such realizations. As we have noted earlier, the matrix J is uniquely determined—in contrast to F, G, and H—as $W(\infty)$.

Theorem 3.3.11 applies to realizations $\{F, G, H, J\}$ or state-space equations

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= H'x + Ju \end{aligned}$$

whether or not $J = 0$. In other words, Theorem 3.3.11 says that any nonminimal realization, i.e., any realization where $[F, G]$ is not completely controllable or $[F, H]$ is not completely observable, can be replaced by a realization $\{F_M, G_M, H_M, J\}$ that is minimal, the replacement preserving the associated transfer-function matrix; i.e.,

$$J + H'_M(sI - F_M)^{-1}G_M = J + H'(sI - F)^{-1}G$$

The appearance of $J \neq 0$ is irrelevant in all the operations that generate F_M, G_M, and H_M from F, G, and H.

Problem 3.3.1 Prove Theorem 3.3.2. (Use the rank condition or the original definition.)

Problem 3.3.2 Prove Theorem 3.3.3. (Use the rank condition.)

Problem 3.3.3 Prove Theorem 3.3.5.

Problem 3.3.4 Consider the state-space equations

$$\dot{x} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} x + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u$$
$$y = [H'_1 \quad 0]x$$

Suppose that $\begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}$ and $\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ form a completely controllable pair, and that $[F_{11}, H_1]$ is completely observable. Show that $[F_{11}, G_1]$ is completely controllable. This result shows that elimination of the unobservable part of state-space equations preserves complete controllability.

Problem 3.3.5 Find a set of state-space equations with a one-dimensional state vector and with the same transfer function relating input and output as

$$\dot{x} = \begin{bmatrix} -3 & +4 & -7 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} u$$
$$y = [2 \quad -3 \quad 4]x$$

3.4 MINIMAL REALIZATIONS

We recall that, given a rational transfer-function matrix $W(s)$ with $W(\infty)$ finite, a realization of $W(s)$ is any quadruple $\{F, G, H, J\}$ such that

$$W(s) = J + H'(sI - F)^{-1}G \tag{3.4.1}$$

and a minimal realization is one with $[F, G]$ completely controllable and $[F, H]$ completely observable. With $W(\infty) = 0$, we can think of triples $\{F, G, H\}$ rather than quadruples "realizing" $W(s)$.

We postpone until Section 3.5 a proof of the existence of realizations. Meanwhile, we wish to do two things. First, we shall establish additional meaning for the use of the word minimal by proving that there exist no realizations of a prescribed transfer function $W(s)$ with state-space dimension less than that of any minimal realization, while all minimal realizations have the same dimension. Second, we shall explain how different minimal realizations are related.

To set the stage, we introduce the *Markov matrices* A_i associated with a prescribed $W(s)$, assumed to be such that $W(\infty)$ is finite. These matrices are defined as the coefficients in a matrix power series expansion of $W(s)$ in powers of s^{-1}; thus

$$W(s) = A_{-1} + \frac{A_0}{s} + \frac{A_1}{s^2} + \frac{A_2}{s^3} + \cdots \tag{3.4.2}$$

Computation of the A_i from $W(s)$ is straightforward; one has $A_{-1} = \lim_{s\to\infty} W(s)$, $A_0 = \lim_{s\to\infty} s[W(s) - A_{-1}]$, etc. Notice that the A_i are defined independently of any realization of $W(s)$. Nonetheless, they may be related to the matrices of a realization, as the following lemma shows.

Lemma 3.4.1. Let $W(s)$ be a rational transfer-function matrix with $W(\infty) < \infty$. Let $\{A_i\}$ be the Markov matrices and let $\{F, G, H, J\}$ be a realization of $W(s)$. Then $A_{-1} = J$ and

$$A_i = H'F^iG \qquad i \geq 0 \tag{3.4.3}$$

Proof. Let $w(t)$ be the impulse response matrix associated with $W(s)$. From (3.4.2) it follows that for $t \geq 0$

$$w(t) = A_{-1}\,\delta(t) + A_0 + A_1 t + \frac{A_2 t^2}{2!} + \cdots$$

Also, as we know,

$$\begin{aligned} w(t) &= J\delta(t) + H'e^{Ft}G \\ &= J\delta(t) + H'G + H'FGt + \frac{H'F^2Gt^2}{2!} + \cdots \end{aligned}$$

The result follows by equating the coefficients of the two power series. ∇ ∇ ∇

We can now establish the first main result of this section. Let us define the *Hankel matrix* of order r by

$$\mathcal{H}_r = \begin{bmatrix} A_0 & A_1 & \cdots & A_{r-1} \\ A_1 & A_2 & \cdots & A_r \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{r-1} & A_r & \cdots & A_{2r-2} \end{bmatrix} \tag{3.4.4}$$

Theorem 3.4.1. All minimal realizations of $W(s)$ have the same dimension, n_M, where n_M is rank $\mathcal{H}_n$ for any $n \geq n_M$, or, to avoid the implicit nature of the definition, for suitably large n. Moreover, no realization of $W(s)$ has dimension less than n_M; i.e., minimal realizations are minimal dimension realizations.

Proof. For a prescribed realization $\{F, G, H, J\}$, define the matrices W_i and V_i for positive integer i by

$$W_i = [G \quad FG \cdots F^{i-1}G]$$
$$V_i = [H \quad F'H \cdots (F')^{i-1}H]$$

Let n be the dimension of the realization $\{F, G, H\}$; then

$$V_n'W_n = \begin{bmatrix} H' \\ H'F \\ \cdot \\ \cdot \\ \cdot \\ H'F^{n-1} \end{bmatrix} [G \quad FG \quad \cdots \quad F^{n-1}G]$$

$$= \begin{bmatrix} H'G & H'FG & \cdots & H'F^{n-1}G \\ H'FG & H'F^2G & & \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ H'F^{n-1}G & \cdot & \cdot \quad \cdot & H'F^{2n-2}G \end{bmatrix}$$

$$= \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ A_1 & A_2 & & A_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{n-1} & A_n & \cdots & A_{2n-2} \end{bmatrix}$$

If $\{F, G, H\}$ is minimal and of dimension n_M, then, by definition, $[F, G]$ is completely controllable. It therefore follows from an earlier theorem that W_{n_M} has rank n_M. A similar argument shows that V_{n_M} has rank n_M. But the dimensions of W_{n_M} and V_{n_M} are such that the product $V'_{n_M}W_{n_M}$ has the same rank; that is,

$$n_M = \text{rank}\, \mathcal{H}_{n_M} \tag{3.4.5}$$

But also, V_n and W_n for any $n > n_M$ will have rank n_M; to see this, consider for example $W_{n_M+1} = [W_{n_M} \;\; F^{n_M}G]$. By the Cayley–Hamilton theorem, F^{n_M} is a linear combination of $I, F, F^2, \ldots, F^{n_M-1}$, so $F^{n_M}G$ is a linear combination of $G, FG, \ldots, F^{n_M-1}G$. Hence rank W_{n_M+1} = rank W_{n_M}. The argument is obviously extendable to $n = n_M + 2, n_M + 3$, etc. Thus for all $n > n_M$, $n_M \geq$ rank (V'_nW_n) = rank $\mathcal{H}_n \geq$ rank $\mathcal{H}_{n_M} = n_M$ with the second inequality following, because $\mathcal{H}_{n_M}$ is a submatrix of $\mathcal{H}_n$. Evidently, then,

$$n_M = \text{rank}\, \mathcal{H}_{n_M} = \text{rank}\, \mathcal{H}_n \qquad \text{for all } n \geq n_M \tag{3.4.6}$$

Now suppose that n_M is not unique; i.e., there exist minimal realizations of dimensions n_{M1} and n_{M2}, with $n_{M1} \neq n_{M2}$. We easily find a contradiction. Suppose also without loss of generality that $n_{M1} > n_{M2}$. By the minimality of the realization of dimension n_{M1}, we have by (3.4.5)

$$\text{rank}\, \mathcal{H}_{n_{M1}} = n_{M1}$$

Likewise

$$\text{rank}\, \mathcal{H}_{n_{M2}} = n_{M2}$$

But (3.4.6) implies, because $n_{M1} > n_{M2}$, that

$$\text{rank}\, \mathcal{H}_{n_{M1}} = n_{M2}$$

The two expressions for rank $\mathcal{H}_{n_{M1}}$ are contradictory; hence minimal realizations have the same dimension.

To see that there do not exist realizations of lower dimension, suppose that $\{F, G, H\}$ is a realization of dimension n and is not minimal, i.e., is either not completely controllable or completely observable. Then, as we saw in the last section, there exists a realization of smaller dimension that is completely controllable and observable—in fact, we have a constructive procedure for obtaining such a realization. Such a realization, being minimal, has dimension n_M. Hence $n_M < n$, proving the theorem. $\nabla\nabla\nabla$

Example 3.4.1 Consider the scalar transfer function

$$W(s) = \frac{1}{(s+1)(s+2)}$$

Then

$$A_{-1} = \lim_{s\to\infty} W(s) = 0$$

$$A_0 = \lim_{s\to\infty} \{s[W(s) - A_{-1}]\} = 0$$

$$A_1 = \lim_{s\to\infty} \left\{s^2\left[W(s) - A_{-1} - \frac{A_0}{s}\right]\right\} = 1$$

$$A_2 = \lim_{s\to\infty} \left\{s^3\left[W(s) - A_{-1} - \frac{A_0}{s} - \frac{A_1}{s^2}\right]\right\} = -3$$

$$A_3 = \lim_{s\to\infty} \left\{s^4\left[W(s) - A_{-1} - \frac{A_0}{s} - \frac{A_1}{s^2} - \frac{A_2}{s^3}\right]\right\} = 7$$

$$A_4 = -15 \qquad A_5 = 31 \qquad A_6 = -63 \qquad \text{etc.}$$

It follows that

$$\mathcal{H}_1 = 0 \qquad \mathcal{H}_2 = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \qquad \mathcal{H}_3 = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

$$\mathcal{H}_4 = \begin{bmatrix} 0 & 1 & -3 & 7 \\ 1 & -3 & 7 & -15 \\ -3 & 7 & -15 & 31 \\ 7 & -15 & 31 & -63 \end{bmatrix} \qquad \text{etc.}$$

and one can verify that

$$\operatorname{rank} \mathcal{H}_2 = \operatorname{rank} \mathcal{H}_3 = \operatorname{rank} \mathcal{H}_4 = \cdots = 2$$

Therefore, all minimal realizations of $W(s)$ have dimension 2.

As the above example illustrates, the computation of the dimension of minimal realizations using Hankel matrices is not particularly efficient and suffers from the drastic disadvantage that, apparently, the ranks of an infinite number of matrices have to be examined. If $W(s)$ is rational, this is not actually the case, as we shall see in later sections. We shall also note later some other procedures for identifying the dimension of a minimal realization of a prescribed $W(s)$.

As background for the second main result of this section, we recall the following two facts, established in Theorem 3.3.9:

1. If $\{F, G, H, J\}$ is a realization for a prescribed $W(s)$, so is $\{TFT^{-1}, TG, (T^{-1})'H, J\}$ for any nonsingular T.
2. If $\{F, G, H, J\}$ is minimal, so is $\{TFT^{-1}, TG, (T^{-1})'H, J\}$, and conversely.

Evidently, we have a recipe for constructing an infinity of minimal realizations given one minimal realization. The natural question arises as to whether we can construct all minimal realizations by this technique. The answer is yes, but the result itself should not be thought of as trivial.

Theorem 3.4.2. Let $\{F_1, G_1, H_1, J_1\}$ and $\{F_2, G_2, H_2, J_2\}$ be two minimal realizations of a prescribed $W(s)$. Then there exists a nonsingular T such that

$$F_2 = TF_1T^{-1} \qquad G_2 = TG_1 \qquad H_2 = (T^{-1})'H_1 \quad (3.4.7)$$

Proof. Suppose that F_1 and F_2 are $n \times n$ matrices. Define W_j and V_j by

$$W_j = [G_j \;\; F_jG_j \cdots F_j^{n-1}G_j] \qquad V_j = [H_j \;\; F_j'H_j \cdots (F_j')^{n-1}H_j] \quad (3.4.8)$$

We established in the course of proving Theorem 3.4.1 that

$$V_j'W_j = \mathcal{H}_n$$

where $\mathcal{H}_n$ is the appropriately defined Hankel matrix. Hence

$$V_1'W_1 = V_2'W_2 \quad (3.4.9)$$

Now V_1, W_1, V_2, and W_2 all possess rank n and have n rows. This means that we can write

$$(V_2V_2')^{-1}(V_2V_1')W_1 = W_2 \quad (3.4.10)$$

and the inverse is guaranteed to exist. We claim that

$$T = (V_2V_2')^{-1}(V_2V_1') \quad (3.4.11)$$

carries one realization into the other. Note that T is invertible—otherwise W_1 and W_2 in (3.4.10) could not simultaneously possess rank n. Also, (3.4.10) and (3.4.11) imply that

$$T[G_1 \;\; F_1G_1 \cdots F_1^{n-1}G_1] = [G_2 \;\; F_2G_2 \cdots F_2^{n-1}G_2]$$

That $G_2 = TG_1$ is immediate. We shall now show that $F_2 = TF_1T^{-1}$. As well as (3.4.9), it is easy to show that

$$V_1'F_1W_1 = V_2'F_2W_2$$

whence

$$TF_1W_1 = F_2W_2$$

Because $TW_1 = W_2$, we have

$$TF_1T^{-1}W_2 = F_2W_2$$

and the fact that W_2 has rank n yields the desired result. To show that $H_2 = (T^{-1})'H_1$, we have from (3.4.9) and the equation $TW_1 = W_2$ that

$$V_1'T^{-1}W_2 = V_2'W_2$$

and thus, using the fact that W_2 has rank n,

$$V_1'T^{-1} = V_2'$$

The result follows by using the formula (3.4.8). $\nabla\nabla\nabla$

Notice that the proof of the above theorem contains a constructive procedure for obtaining T; Eqs. (3.4.8) and (3.4.11) define T in terms of F_j, G_j, and H_j for $j = 1, 2$.

Example 3.4.2 To illustrate the procedure for computing T, consider the following two minimal realizations of the scalar transfer function $1/(s+1)(s+2)$:

$$F_1 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad H_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad H_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We have

$$V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$V_2V_2' = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \qquad (V_2V_2')^{-1} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$$V_2V_1' = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \qquad T = (V_2V_2')^{-1}(V_2V_1') = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Observe that

$$TF_1 = \begin{bmatrix} -2 & -1 \\ -2 & -2 \end{bmatrix} \quad \text{and} \quad F_2T = \begin{bmatrix} -2 & -1 \\ -2 & -2 \end{bmatrix}$$

so that $TF_1T^{-1} = F_2$. Also

$$TG_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = G_2 \quad \text{and} \quad T'H_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = H_1$$

Problem 3.4.1 Suppose that $W(s)$ is a scalar transfer function, with minimal realization $\{F, G, H, J\}$. Because $W(s)$ is scalar, $W(s) = W'(s)$. Show that $\{F', H, G, J\}$ is a minimal realization of $W(s)$ and show that the matrix T such that

$$F' = TFT^{-1} \qquad H = TG \qquad G = (T^{-1})'H$$

is symmetric.

Problem 3.4.2 Suppose that $W(s)$ is a scalar transfer function with minimal realization $\{F, G, H, J\}$, and suppose that F is diagonal. The matrices G and H are vectors. Show that no entry of G or H is zero.

Problem 3.4.3 Suppose that $W(s)$ is an $m \times p$ transfer-function matrix with minimal realization $\{F, G, H, J\}$. Show that if F is $n \times n$, G has rank p, and H has rank m, then

$$\operatorname{rank}[G \quad FG \quad \cdots \quad F^{n-p}G] = n$$
$$\operatorname{rank}[H \quad F'H \quad \cdots \quad (F')^{n-m}H] = n$$

and that rank $\mathcal{H}_n = \operatorname{rank} \mathcal{H}_{n'}$, where

$$n' = \max(n - p + 1, n - m + 1)$$

Problem 3.4.4 In the text a formula for the matrix T relating two minimal realizations $\{F_j, G_j, H_j\}$ with $j = 1, 2$ of the same transfer-function matrix was given in terms of the matrices $[H_j \quad F_j'H_j \quad \cdots \quad (F_j')^{n-1}H_j]$, where F_j is $n \times n$. Obtain a formula in terms of the matrices $[G_j \quad F_jG_j \quad \cdots \quad F_j^{n-1}G_j]$.

3.5 CONSTRUCTION OF STATE-SPACE EQUATIONS FROM A TRANSFER-FUNCTION MATRIX

In this section we examine the problem of passing from a prescribed rational transfer-function matrix $W(s)$, with $W(\infty) < \infty$, to a quadruple of constant matrices $\{F, G, H, J\}$ such that

$$W(s) = J + H'(sI - F)^{-1}G \tag{3.5.1}$$

Of course, this is equivalent to the problem of finding state-space equations corresponding to a prescribed $W(s)$.

In general, we shall only be interested in *minimal* realizations of a prescribed $W(s)$, although the first results have to do with realizations that are completely controllable, but not necessarily completely observable, and vice versa. We follow these results by describing the *Silverman-Ho algorithm;* this is a technique for directly finding a minimal, or completely controllable and completely observable, realization of a transfer-function matrix $W(s)$.

We note first an almost obvious point—that in restricting ourselves to $W(s)$ with $W(\infty) < \infty$, we may as well restrict ourselves to the case when $W(\infty) = 0$. For suppose that $W(\infty)$ is nonzero; as we know, the matrix J of any realization of $W(s)$, minimal or otherwise, is precisely $W(\infty)$. Also,

$$W(s) = [W(s) - W(\infty)] + J$$

and

$$V(s) = W(s) - W(\infty)$$

has the property that $V(\infty) = 0$, with $V(s)$ rational. Evidently, if we can find a minimal realization $\{F, G, H\}$ for $V(s)$, it is immediate that $\{F, G, H, J\}$ is a minimal realization of $W(s)$.

Scalar $W(s)$.

It turns out that there are rapid ways of computing realizations for a scalar $W(s)$ that are always either completely controllable or completely observable. With a simple precaution, the realizations can be assured to be minimal also.

Suppose that $W(s)$ is given as

$$W(s) = \frac{b_n s^{n-1} + \cdots + b_2 s + b_1}{s^n + a_n s^{n-1} + \cdots + a_2 s + a_1} \tag{3.5.2}$$

Notice that we are restricting attention to the case $W(\infty) = 0$ without any real loss of generality, as we have noted. A completely controllable realization, often called the phase-variable realization, is described in the following theorem. The matrix F appearing in the theorem is termed in linear algebra a *companion matrix*, and the realization has sometimes been termed a companion-matrix realization.

Theorem 3.5.1. Let $W(s)$ be given as in Eq. (3.5.2). Then a completely controllable realization $\{F, G, H\}$ for $W(s)$ is as follows:

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{n-1} \\ b_n \end{bmatrix} \tag{3.5.3}$$

Moreover, this realization is also completely observable if and only if the polynomials $s^n + a_n s^{n-1} + \cdots + a_1$ and $b_n s^{n-1} + \cdots + b_1$ have no common factors.

Proof. The proof that $\{F, G, H\}$ is a realization of $W(s)$ divides into two main steps. First, we show that

$$|sI - F| = s^n + a_n s^{n-1} + \cdots + a_1 \tag{3.5.4}$$

Then we show that

$$(sI - F)^{-1}G = \frac{1}{|sI - F|}\begin{bmatrix} 1 \\ s \\ s^2 \\ \cdot \\ \cdot \\ \cdot \\ s^{n-1} \end{bmatrix} \tag{3.5.5}$$

That $\{F, G, H\}$ is a realization is then immediate from (3.5.5) and the definition of H.

Let us prove (3.5.4). Assume that the result holds for F of dimension $n = 1, 2, 3, \ldots, m - 1$. We shall prove it must hold for $n = m$. Direct calculation establishes that it holds for $n = 2$, so the inductive proof will be complete. We have, on expanding by the first row,

$$\det\begin{bmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & & 0 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & 0 & 0 & & -1 \\ a_1 & a_2 & a_3 & \cdots & s + a_n \end{bmatrix}$$

$$= s\det\begin{bmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & & 0 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & & & & -1 \\ a_2 & a_3 & a_4 & \cdots & s + a_n \end{bmatrix}$$

$$+ \det\begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & & 0 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & & & & -1 \\ a_1 & a_3 & a_4 & \cdots & s + a_n \end{bmatrix}$$

$$= s(s^{n-1} + a_n s^{n-2} + \cdots + a_3 s + a_2) + a_1$$

where we have used the inductive hypothesis. This equality is precisely (3.5.4).

Let us now prove (3.5.5). In view of the form of G, it follows that $(sI - F)^{-1}G$ is a vector that is the last column of $(sI - F)^{-1}$. The Jacobi formula for computing the inverse of a matrix in terms of the minors of its cofactors then establishes that, for example,

$$[(sI-F)^{-1}G]_3 = \frac{(-1)^{n+3}}{|sI-F|}\det\begin{bmatrix} s & -1 & 0 & 0 & \cdots & 0 \\ 0 & s & 0 & 0 & & 0 \\ 0 & 0 & -1 & 0 & & 0 \\ 0 & 0 & s & -1 & & 0 \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

$$= \frac{s^2}{|sI-F|}$$

with a similar calculation holding for the other entries of $(sI - F)^{-1}G$. Equation (3.5.5) follows.

To complete the theorem proof, we must argue that $[F, G]$ is completely controllable, and we must establish the complete observability condition. That $[F, G]$ is completely controllable follows from the fact that $[G \quad FG \cdots F^{n-1}G]$ is a matrix of the form

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & & 0 & 1 & x \\ 0 & & 1 & x & x \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ \cdot\ \cdot & & & & \cdot \\ 1 & \cdots & x & x & x \end{bmatrix}$$

where x denotes an arbitrary element. (This is easy to check.) Clearly, rank $[G \quad FG \cdots F^{n-1}G] = n$.

Finally, we examine the observability condition. If (3.5.2) is such that the numerator and denominator polynomial have common factors, then these factors may be canceled. The resulting denominator polynomial will have degree $n' < n$, and, by what we have already proved, there will exist a realization of dimension n'. Hence n is not the minimal dimension of a realization, and thus the pair $[F, H]$ in (3.5.3) cannot be completely observable. Thus, if $[F, H]$ is completely observable, the numerator and denominator of $W(s)$ as given in (3.5.2) have no common factors.

Now suppose that $[F, H]$ is not completely observable. Then, as we have seen, there exists a minimal realization $\{F_M, G_M, H_M\}$ constructible from $\{F, G, H\}$ that will, of course, have lower dimension, and for which

$$H'_M(sI - F_M)^{-1}G_M = H'(sI - F)^{-1}G$$

The matrix $(sI - F_M)^{-1}$ is expressible as a matrix of polynomials divided by $|sI - F_M|$, so $H'_M(sI - F_M)^{-1}G_M$ will consist of a polynomial divided by the polynomial $|sI - F_M|$. This means that $W(s)$ is a ratio of two polynomials, with the denominator polynomial of degree equal to the size of F_M, which is less than n. Consequently, the numerator and denominator of (3.5.2) must have common factors.

Since we have proved that lack of complete observability of $[F, H]$ implies existence of common factors in (3.5.2), it follows that if the numerator and denominator polynomials in (3.5.2) have no common factors, then $[F, H]$ is completely observable. The proof of the theorem is now complete. ∇ ∇ ∇

An important corollary follows immediately from the above theorem; this corollary relates the dimension of a minimal realization of a scalar $W(s)$ to properties of $W(s)$.

Corollary. Let $W(s)$ be given as in (3.5.2), with no common factors between the numerator and denominator. Then the dimension of a minimal realization of $W(s)$ is n.

Example 3.5.1 In an example in the last section we studied the transfer function

$$W(s) = \frac{1}{(s+1)(s+2)}$$

and went to some pains to show that the dimension of minimal realizations is 2. This is immediate from the corollary. By Theorem 3.5.1, also, a minimal realization is provided by

$$F = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

It is easy to check that $[F, H]$ is completely observable. On the other hand, if

$$W(s) = \frac{s+1}{(s+1)(s+2)}$$

we could have the same F and G but would have $H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$[H \quad F'H] = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

and lack of complete observability is evident.

In the problems of this section, proof of the following theorem is requested; the theorem is closely akin to, and can be proved rapidly by using, Theorem 3.5.1.

Theorem 3.5.2. Let $W(s)$ be given as in Eq. (3.5.2). Then a completely observable realization $\{F, G, H\}$ for $W(s)$ is as follows:

$$F = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & & 0 & -a_2 \\ 0 & 1 & & 0 & -a_3 \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_n \end{bmatrix} \qquad G = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix} \qquad H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} \tag{3.5.6}$$

Moreover, this realization is completely controllable if and only if the polynomials $s^n + a_n s^{n-1} + \cdots + a_1$ and $b_n s^{n-1} + \cdots + b_1$ have no common factors.

Matrix $W(s)$

Now we turn to transfer-function matrices. First, we shall give matrix generalizations of Theorems 3.5.1 and 3.5.2, following the treatment of [18]. Then we shall indicate without proof one of a number of other procedures, the Silverman-Ho algorithm, for direct computation of a minimal realization. (The matrix generalizations of Theorems 3.5.1 and 3.5.2 lead, in the first instance, to completely controllable or completely observable realizations, but not necessarily minimal realizations.) For a treatment of the Silverman-Ho algorithm, including the proof, see [19]. For an example of another procedure for generating a minimal realization, see [20].

To derive a completely controllable realization of a rational $m \times p$ transfer-function matrix $W(s)$ with $W(\infty) = 0$, we proceed in the following way.

Let $p(s)$ be the least common denominator of the mp entries of $W(s)$, with

$$p(s) = s^n + a_n s^{n-1} + \cdots + a_1 \tag{3.5.7}$$

Because of the definition of $p(s)$, the matrix $p(s)W(s)$ will be a matrix of polynomials in s, and because $W(\infty) = 0$, the highest degree polynomial occurring in $p(s)W(s)$ has degree less than n. Therefore, there exist constant

$m \times p$ matrices $B_1, B_2, \ldots, B_n$ with

$$p(s)W(s) = B_1 + B_2 s + \cdots + B_n s^{n-1} \tag{3.5.8}$$

Now we can state the main result.

Theorem 3.5.3. Let $W(s)$ be a rational $m \times p$ transfer-function matrix with $W(\infty) = 0$, and let $p(s)$ and $B_1, B_2, \ldots, B_n$ be constructed in the manner just described. Then a completely controllable realization for $W(s)$ is provided by

$$F = \begin{bmatrix} 0_p & I_p & 0_p & \cdots & 0_p \\ 0_p & 0_p & I_p & & 0_p \\ \vdots & & & \ddots & \vdots \\ 0_p & 0_p & 0_p & & I_p \\ -a_1 I_p & -a_2 I_p & -a_3 I_p & \cdots & -a_n I_p \end{bmatrix}$$

$$G = \begin{bmatrix} 0_p \\ 0_p \\ \vdots \\ 0_p \\ I_p \end{bmatrix} \qquad H = \begin{bmatrix} B'_1 \\ B'_2 \\ \vdots \\ B'_{n-1} \\ B'_n \end{bmatrix}$$

and a completely observable realization is provided by

$$F = \begin{bmatrix} 0_m & 0_m & \cdots & 0_m & -a_1 I_m \\ I_m & 0_m & & 0_m & -a_2 I_m \\ 0_m & I_m & & 0_m & -a_3 I_m \\ \vdots & & \ddots & & \vdots \\ 0_m & 0_m & \cdots & I_m & -a_n I_m \end{bmatrix} \qquad G = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_n \end{bmatrix} \qquad H = \begin{bmatrix} 0_m \\ 0_m \\ 0_m \\ \vdots \\ I_m \end{bmatrix}$$

where 0_q and I_q denote $q \times q$ zero and unit matrices, respectively.

The proof of this theorem is closely allied to the proof of the corresponding theorems for scalar $W(s)$ and, accordingly, will be omitted. Proof is requested in the problems. Notice that, in contrast to the theorems for scalar transfer function matrices, there is no statement in Theorem 3.5.3 concerning conditions for complete observability of the completely controllable realization, and complete controllability of the completely observable realization. Such a statement seems very difficult to give.

We now ask the reader to note very carefully the following points:

1. Theorem 3.5.3 establishes the result that *any* rational $W(s)$ with $W(\infty) < \infty$ possesses a realization—in fact, a completely controllable or completely observable realization. This in itself is a nontrivial result and, of course, justifies much of the discussion hitherto, in the sense that were there no such result, much of our discussion would have been valueless.
2. The characteristic polynomial of the F matrix in the completely controllable realization is

$$(s^n + a_n s^{n-1} + \cdots + a_1)^p = p(s)^p$$

Therefore, any eigenvalue of F must also be a pole of some element of $W(s)$, since $p(s)$ is the least common denominator of the entries of $W(s)$. It follows that *any eigenvalue of the F matrix in any minimal realization of $W(s)$ must be a pole of some element of $W(s)$*. (A proof is requested in the problems.) The converse is also true; i.e., *any pole of any element of $W(s)$ must be an eigenvalue of the F matrix in any minimal realization, in fact, in any realization at all.* This is immediate from the formula $W(s) = J + H'(sI - F)^{-1}G$, which shows that any singularity of any element of $W(s)$ must be a zero of $\det(sI - F)$.
3. Theorem 3.5.3 can be used to generate minimal realizations of a prescribed $W(s)$ by employing it in the first step of a two-step procedure. From $W(s)$ one first constructs a completely controllable or completely observable realization. Then, using the technique exhibited in an earlier section, we construct from either of these realizations a minimal realization. The existence of these techniques when noted in conjunction with remark 1 also shows that *any rational $W(s)$ with $W(\infty) < \infty$ possesses a minimal realization.*

Example 3.5.2 We provide a simple illustration of the preceding. Suppose that

$$W(s) = \begin{bmatrix} \dfrac{1}{s(s+1)} & \dfrac{2}{s+1} \\ \dfrac{2}{s+1} & \dfrac{1}{s+1} \end{bmatrix}$$

Evidently, $p(s) = s(s + 1)$ and there are two matrices B_i, viz.,

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$$

Following Theorem 3.5.3, we see that a completely controllable realization is provided by

$$F=\begin{bmatrix}0&0&1&0\\0&0&0&1\\0&0&-1&0\\0&0&0&-1\end{bmatrix}\qquad G=\begin{bmatrix}0&0\\0&0\\1&0\\0&1\end{bmatrix}\qquad H=\begin{bmatrix}1&0\\0&0\\0&2\\2&1\end{bmatrix}$$

Next, we check for complete observability by computing

$$[H\quad F'H\quad (F')^2H\quad (F')^3H]=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&2&1&-2&-1&2&1&-2\\2&1&-2&-1&2&1&-2&-1\end{bmatrix}$$

Evidently, this matrix has rank 3; therefore, the realization just computed is not minimal, though a realization of dimension 3 will be minimal. Accordingly, we seek a matrix T such that

$$TFT^{-1}=\begin{bmatrix}\hat{F}_{11}&0\\\hat{F}_{21}&\hat{F}_{22}\end{bmatrix}$$

$$(T^{-1})'H=\begin{bmatrix}\hat{H}_1\\0\end{bmatrix}$$

where $\hat{F}_{11}$ is 3×3 and $[\hat{F}_{11}, \hat{H}_1]$ is completely observable. If $\hat{G}_1$ and $\hat{G}_2$ are defined by

$$TG=\begin{bmatrix}\hat{G}_1\\\hat{G}_2\end{bmatrix}$$

it will then follow that $[\hat{F}_{11}, \hat{G}_1]$ is completely controllable, and that $[\hat{F}_{11}, \hat{G}_1, \hat{H}_1]$ is a minimal realization of $W(s)$.

We have discussed the computation of T for the dual problem of eliminating the uncontrollable part of a realization. It is easy to deduce from this procedure one applying here. We shall take for the first three columns of T' a basis for the space spanned by $\{H, F'H, (F')^2H, (F')^3H\}$ and for the remaining column any independent vector. This leads to

$$T'=\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

and thus

$$T=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}\qquad T^{-1}=\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

It follows that

$$TFT^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad TG = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad (T^{-1})'H = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 1 \\ 0 & 0 \end{bmatrix}$$

and thus a minimal realization of $W(s)$ is provided by

$$\hat{F}_{11} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \hat{G}_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \hat{H}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$$

Of course, we could equally well have formed a completely observable realization of $W(s)$ using Theorem 3.5.3 and then eliminated the uncontrollable part.

We shall now examine a procedure for obtaining a minimal realization directly, i.e., without having to cast out the uncontrollable or unobservable part of a realization obtained in an intermediate step.

Silverman-Ho Algorithm

Starting with an $m \times p$ rational matrix $W(s)$ with $W(\infty) = 0$, the algorithm proceeds by generating the Markov matrices $\{A_i\}$ via

$$W(s) = \frac{A_0}{s} + \frac{A_1}{s^2} + \frac{A_2}{s^3} + \cdots$$

and the Hankel matrices $\mathcal{H}_n$ by

$$\mathcal{H}_n = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ A_1 & A_2 & & A_n \\ \vdots & & & \vdots \\ A_{n-1} & A_n & \cdots & A_{2n-2} \end{bmatrix}$$

The rationality of $W(s)$ is sufficient to guarantee existence of a realization, though, as we have noted, this is a nontrivial fact. Since realizations exist, so do minimal realizations. We recall that if n_M is the dimension of a minimal realization, then rank $\mathcal{H}_n =$ rank $\mathcal{H}_{n_M}$ for all $n \geq n_M$. Thus, knowing the $\mathcal{H}_i$ but not knowing n_M, we can examine rank $\mathcal{H}_1$, rank $\mathcal{H}_2$, rank $\mathcal{H}_3$, and so on, and be assured that at some point the rank will cease to increase. In fact, one can show that if n_D is the degree of the least common denominator of the entries of $W(s)$, then $n_M \leq n_D \times \min(m, p)$. Thus *one can stop testing the ranks of the* $\mathcal{H}_i$ at, say, $i = n_D \times \min(m, p)$ in the knowledge that no further increase in the rank is possible.

Let r be the first integer for which rank $\mathcal{H}_n$ = rank $\mathcal{H}_r$ for all $n \geq r$. Let rank $\mathcal{H}_r = n_M$, which turns out to be the dimension of a minimal realization. The remaining steps in the algorithm are as follows (for a proof, see [19]):

1. Find nonsingular matrices P and Q such that

$$P\mathcal{H}_rQ = \begin{bmatrix} I_{n_M} & 0 \\ 0 & 0 \end{bmatrix}$$

(Standard procedures exist for this task. See, e.g., [4].)

2. Take

$$G = n_M \times p \text{ top left corner of } P\mathcal{H}_r$$
$$H' = m \times n_M \text{ top left corner of } \mathcal{H}_rQ$$
$$F = n_M \times n_M \text{ top left corner of } P(\sigma\mathcal{H}_r)Q$$

where

$$\sigma\mathcal{H}_r = \begin{bmatrix} A_1 & A_2 & \cdots & A_r \\ A_2 & A_3 & & A_{r+1} \\ \vdots & & & \vdots \\ A_r & A_{r+1} & \cdots & A_{2r-1} \end{bmatrix}$$

Example 3.5.3 Suppose that

$$W(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{s+1} \\ \dfrac{2}{s+1} & \dfrac{1}{s+1} \end{bmatrix}$$

It is easy to calculate that

$$A_0 = \lim_{s\to\infty} sW(s) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$A_1 = \lim_{s\to\infty} \{s[sW(s) - A_0]\} = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$$

$$A_2 = \lim_{s\to\infty} \{s^2[sW(s) - A_0 - A_1/s]\} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Furthermore,

$$A_1 = A_3 = A_5 = \cdots \qquad A_0 = A_2 = A_4 = \cdots$$

Then

$$\mathcal{H}_1 = A_0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \operatorname{rank} \mathcal{H}_1 = 2$$

$$\mathcal{H}_2 = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 2 & 1 & -2 & -1 \\ -1 & -2 & 1 & 2 \\ -2 & -1 & 2 & 1 \end{bmatrix} \qquad \operatorname{rank} \mathcal{H}_2 = 2$$

Furthermore,

$$\operatorname{rank} \mathcal{H}_i = 2$$

for all i. In terms of our earlier description of the algorithm, $r = 1$, and rank $\mathcal{H}_r = n_M = 2$. [Note that the bound $n_M \leq n_D \times \min(m, p)$ yields here that $n_M \leq 2$, so that the only matrices that need be formed are $\mathcal{H}_1$ and $\mathcal{H}_2$.] The next task is to find matrices P and Q so that

$$P\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Such are easy to find. One pair is

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

In accordance with the algorithm,

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$H' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

It is quickly verified that $H'(sI - F)^{-1}G$ is the prescribed $W(s)$, that $[F, G]$ is completely controllable, and $[F, H]$ completely observable.

Problem 3.5.1 Prove Theorem 3.5.2.

Problem 3.5.2 Suppose that G is an n vector and F an $n \times n$ matrix with $[F, G]$ completely controllable. Define T by

$$T^{-1} = [G\ FG \cdots F^{n-1}G]\begin{bmatrix} a_2 & a_3 & \cdots & a_n & 1 \\ a_3 & a_4 & & 1 & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ a_n & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where $|sI - F| = s^n + a_n s^{n-1} + \cdots + a_1$. Show that TFT^{-1} and TG have the form of the phase-variable realization of Theorem 3.5.1.

Problem 3.5.3 Construct two minimal realizations of

$$W(s) = \begin{bmatrix} \dfrac{2}{s+2} & \dfrac{1}{s+1} \\ \dfrac{1}{s+2} & \dfrac{2}{(s+1)(s+2)} \end{bmatrix}$$

by constructing first a completely controllable realization, and by constructing first a completely observable realization.

Problem 3.5.4 Prove Theorem 3.5.3.

Problem 3.5.5 Using the fact that the eigenvalues of F are the same as those of TFT^{-1}, and using the theorem dealing with the elimination of uncontrollable states, show that any eigenvalue of the F matrix in any minimal realization of a transfer-function matrix is also a pole of some element of that transfer-function matrix. [*Hint:* See remarks following Theorem 3.5.3.]

Problem 3.5.6 Using the Silverman-Ho algorithm, find a minimal realization of

$$W(s) = \begin{bmatrix} \dfrac{1}{s(s+1)} & \dfrac{2}{s+1} \\ \dfrac{2}{s+1} & \dfrac{1}{s+1} \end{bmatrix}$$

3.6 DEGREE OF A RATIONAL TRANSFER FUNCTION MATRIX

In this section we shall discuss the concept of the *degree of a rational transfer-function matrix* $W(s)$. If $W(s)$ has the property $W(\infty) < \infty$, then the degree of $W(s)$, written $\delta[W(s)]$ or $\delta[W]$ for short, is defined simply as the dimension of a minimal realization of $W(s)$; we shall, however, extend the definition of degree to consider those $W(s)$ for which $W(\infty) < \infty$ fails.

We have already explained how the degree of a $W(s)$ with $W(\infty) < \infty$ may be determined by examining the rank of Hankel matrices derived from the

Markov coefficients of $W(s)$. We shall, however, state a number of other properties of the degree concept, some of which offer alternative procedures for the computation of degree. A number of these will not be proved here, though they will be used in the sequel. We shall note the appropriate references (where proofs may be found) as we state each result.

The history of the degree concept is interesting; the concept originated in a paper by McMillan [21] dealing with a network-theory problem. McMillan posed the following question: given an impedance matrix $Z(s)$ of a network for which it is known that a passive synthesis exists using resistors, inductors, capacitors, transformers, and gyrators, what is the minimum number of energy storage elements, i.e., inductors and capacitors, required in a synthesis of $Z(s)$? McMillan termed this number the degree of $Z(s)$ and gave rules for its computation as well as various properties. In [20] *this network-theory-based definition was shown to be identical with a definition of degree as the dimension of a minimal realization of* $Z(s)$, in case $Z(\infty) < \infty$. This was an intuitively appealing result: in almost all cases, as we shall see, state-space equations of a network can be computed with the state vector entries being either a capacitor voltage or inductor current. Thus the result of [20] showed that, despite the evident constraints placed by passivity of a network on state-space equations derived from that network, minimal-dimension state-space equations could still be obtained.

Other network theoretic formulations of the degree concept have been given by Tellegen [22] and Duffin and Hazony [23]. Connection between these formulations and the definition of degree in terms of the dimension of a state-space realization appears also in [20].

We proceed now with a definition of the degree concept and a statement of some of its properties.

Definition of Degree. Let $W(s)$ be a matrix of real rational functions of s. If $W(\infty) < \infty$, the degree of $W(s)$, $\delta[W(s)]$ or $\delta[W]$, is defined as the dimension of a minimal realization of $W(s)$. If one or more elements of $W(s)$ has a pole at $s = \infty$, write

$$W(s) = W_{-1}(s) + W_1 s + W_2 s^2 + \cdots + W_r s^r \qquad (3.6.1)$$

where $W_{-1}(s)$ has $W_{-1}(\infty) < \infty$ and $W_1, W_2, \ldots, W_r$ are constant matrices. Set

$$V(s) = \frac{W_1}{s} + \frac{W_2}{s^2} + \cdots + \frac{W_r}{s^r} \qquad (3.6.2)$$

Then $\delta[W]$ is defined by

$$\delta[W] = \delta[W_{-1}] + \delta[V] \qquad (3.6.3)$$

Though transfer-function matrices of the form of (3.6.1) with $W_1, W_2, \ldots,$ nonzero occur very infrequently, it is worth noting that the impedance $Z(s)$ of a one-port passive network may well be of the form

$$Z(s) = sL + Z_{-1}(s) \tag{3.6.4}$$

if the network comprises an inductance L in series with another one-port network. In this case, the degree definition yields

$$\delta[Z] = 1 + \delta[Z_{-1}] \tag{3.6.5}$$

More generally, the impedance matrix of an m port may be of the form of (3.6.4), where L is a matrix. Then we obtain $\delta[Z] = \text{rank } L + \delta[Z_{-1}]$. This depends on the result that $\delta[sL] = \text{rank } L$, a proof of which is requested in the problems.

Note also from the above definition that if $p(s)$ is a scalar polynomial in s, $\delta[p(s)]$ coincides with the usual polynomial degree.

We shall now prove a number of properties.

Property 1. If M and N are constant matrices

$$\delta[MW(s)N] \leq \delta[W(s)] \tag{3.6.6}$$

Proof. For simplicity, assume that $W(\infty) < \infty$. Property 1 follows by observing that if $\{F, G, H, J\}$ is a *minimal* realization of $W(s)$, $\{F, GN, HM', MJN\}$ is a realization of $MW(s)N$ of dimension $\delta[W]$ and is *not necessarily minimal.* $\nabla\nabla\nabla$

As a special case of property 1, we have

Property 2. Let $W_1(s)$ be a submatrix of $W(s)$. Then

$$\delta[W_1] \leq \delta[W] \tag{3.6.7}$$

Next, we have

Property 3

$$|\delta[W_1] - \delta[W_2]| \leq \delta[W_1 + W_2] \leq \delta[W_1] + \delta[W_2] \tag{3.6.8}$$

Proof. For simplicity, assume that $W_1(\infty) < \infty$ and $W_2(\infty) < \infty$. The right-hand inequality in property 3 follows by noting that if we have separate minimal realizations of W_1 and W_2, simple 'paralleling' of inputs and outputs of these two realizations will provide a realization of $W_1 + W_2$, but this realization will not necessarily be minimal. More precisely, if $\{F_i, G_i, H_i, J_i\}$ is a minimal realization for $W_i(s)$, $i = 1, 2$, a realization of $W(s) = W_1(s)$

$+ W_2(s)$ is provided by

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix} \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \quad J = J_1 + J_2 \tag{3.6.9}$$

Proof of the left-hand inequality in (3.6.8) is requested in the problems. $\nabla\nabla\nabla$

Example 3.6.1 If $W_1(s) = 1/(s+1)$, $W_2(s) = 1/(s+2)$, then $\delta[W_1] = \delta[W_2] = 1$ and $\delta[W_1 + W_2] = 2$. If $W_1(s) = W_2(s) = 1/(s+1)$, then $\delta[W_1 + W_2] = 1$, and if $W_1(s) = -W_2(s) = 1/(s+1)$, then $\delta[W_1 + W_2] = 0$. This example shows that neither inequality in (3.6.8) need be satisfied with equality, but either may be.

An important special case of property 3 permitting replacement of the right-hand inequality in (3.6.8) by an equality is as follows.

Property 4. Suppose that the set of poles of elements of W_1 is disjoint from the set of poles of elements of W_2. Then

$$\delta[W_1 + W_2] = \delta[W_1] + \delta[W_2]$$

In proving this property we shall make use of a generalization of the well-known fact that if

$$\sum_{i=1}^{n} a_i e^{b_i t} = 0$$

for arbitrary a_i and b_i (real or complex), with $b_i \neq b_j$ for $i \neq j$, then $a_i = 0$ for all i. The generalization we shall use is an obvious one and is as follows: if

$$w_1' e^{F_1 t} G_1 + w_2' e^{F_2 t} G_2 = 0$$

for some vectors w_1 and w_2 and matrices F_1, G_1, F_2, and G_2, with no eigenvalue of F_2 the same as an eigenvalue of F_1, then

$$w_1' e^{F_1 t} G_1 = w_2' e^{F_2 t} G_2 = 0$$

Proof. Assume for simplicity that $W_1(\infty) < \infty$ and $W_2(\infty) < \infty$. Suppose that $\{F_i, G_i, H_i, J_i\}$, for $i = 1, 2$, define *minimal* realizations of $W_1(s)$ and $W_2(s)$. Let $\{F, G, H, J\}$ as given in (3.6.9) be a realization for $W(s) = W_1(s) + W_2(s)$. Suppose that $[F, G]$ is not completely controllable. Then there exists a constant nonzero vector w such that

$$w' e^{Ft} G = 0$$

Partitioning w comformably with F, it follows that

$$w_1'e^{F_1t}G_1 + w_2'e^{F_2t}G_2 = 0$$

Now because $\{F_i, G_i, H_i, J_i\}$ is a minimal realization for $W_i(s)$, the set of eigenvalues of F_i is contained in the set of poles of $W_i(s)$, as we showed in an earlier section. Therefore, the set of eigenvalues of F_1 is disjoint from the set of eigenvalues of F_2 and

$$w_1'e^{F_1t}G_1 = w_2'e^{F_2t}G_2 = 0$$

Because w is nonzero, at least one of w_1 and w_2 is nonzero. The complete controllability of $[F_i, G_i]$ is then contradicted. So our assumption that $[F, G]$ is not controllable is untenable; similarly, $[F, H]$ is completely observable, and the result follows. $\nabla\nabla\nabla$

An interesting application of this property is as follows. Suppose that a prescribed $W(s)$ is written as $W(s) = W_{-1}(s) + W_1s + W_2s^2 + \cdots + W_rs^r$ for constant matrices W_i, $i \geq 1$. We can speak of a generalized partial fraction expansion of $W(s)$ when we mean a decomposition of $W(s)$ as

$$W(s) = \frac{A_{11}}{(s - s_1)} + \cdots + \frac{A_{1r_1}}{(s - s_1)^{r_1}} + \frac{A_{21}}{(s - s_2)} + \cdots + \frac{A_{2r_2}}{(s - s_2)^{r_2}} + \cdots$$
$$+ \frac{A_{t1}}{(s - s_t)} + \cdots + \frac{A_{tr_t}}{(s - s_t)^{r_t}} + W_0 + W_1s + W_2s^2 + \cdots + W_rs^r$$

Let

$$V_i(s) = \frac{A_{i1}}{(s - s_i)} + \cdots + \frac{A_{ir_i}}{(s - s_i)^{r_i}}$$

and

$$V_\infty(s) = W_1s + \cdots + W_rs^r$$

We adopt the notation $\delta[W; s_i]$ to denote $\delta[V_i]$, the degree of that part of $W(s)$ with a pole at s_i. It is immediate then from property 4 that

Property 5

$$\delta[W(s)] = \sum_{s_i} \delta[W; s_i] \tag{3.6.10}$$

This property says that if a partial fraction expansion of $W(s)$ is known, the degree of $W(s)$ may be computed by computing the degree of certain summands in the partial fraction expansion.

Example 3.6.2 Suppose that

$$W(s) = \begin{bmatrix} \frac{2}{s+2} & \frac{1}{s+1} \\ \frac{1}{s+2} & \frac{2}{s+1} \end{bmatrix} = \begin{bmatrix} \frac{2}{s+2} & 0 \\ \frac{1}{s+2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{s+1} \\ 0 & \frac{2}{s+1} \end{bmatrix}$$

It is almost immediately obvious that the degree of each summand is one. For example, a realization of the first summand is

$$F = [-2] \qquad G = [1 \quad 0] \qquad H = [2 \quad 1]$$

and it is clearly minimal. It follows that $\delta[W] = 2$.

The next property has to do with products, rather than sums, of transfer-function matrices.

Property 6

$$\delta[W_1W_2] \leq \delta[W_1] + \delta[W_2] \tag{3.6.11}$$

Proof. Assume for simplicity that $W_1(\infty) < \infty$ and $W_2(\infty) < \infty$. The property follows by observing that if we have separate minimal realizations for W_1 and W_2, a cascade or series connection of the two realizations will yield a realization of W_1W_2, but this realization will not necessarily be minimal. More formally, suppose that $W_i(s)$ has minimal realization $\{F_i, G_i, H_i, J_i\}$, for $i = 1, 2$. Then it is readily verified that a set of state-space equations with transfer-function matrix $W_1(s)W_2(s)$ is provided by

$$\dot{x} = \begin{bmatrix} F_1 & G_1H_2' \\ 0 & F_2 \end{bmatrix} x + \begin{bmatrix} G_1J_2 \\ G_2 \end{bmatrix} u \tag{3.6.12}$$

$$y = [H_1' \quad J_1H_2']x + J_1J_2u \qquad \nabla\nabla\nabla$$

Example 3.6.3 If $W_1(s) = 1/(s+1)$ and $W_2(s) = 1/(s+2)$, then $\delta[W_1W_2] = 2 = \delta[W_1] + \delta[W_2]$. But if $W_1(s) = 1/(s+1)$ and $W_2(s) = (s+1)/(s+2)$, then $\delta[W_1W_2] = 1$.

For the case of a square $W(s)$ with $W(\infty)$ nonsingular, we can easily prove the following property. Actually, the property holds irrespective of the nonsingularity of $W(\infty)$.

Property 7. If $W(s)$ is square and nonsingular for almost all s,

$$\delta[W^{-1}] = \delta[W] \tag{3.6.13}$$

Proof. To prove this property for the case when $W(\infty)$ is nonsingular, suppose that $\{F, G, H, J\}$ is a minimal realization for $W(s)$. Then a realization for $W^{-1}(s)$ is provided by $\{F - GJ^{-1}H', GJ^{-1}, -H(J^{-1})', J^{-1}\}$. To see this, observe that

$$\begin{aligned}
&[J^{-1} - J^{-1}H'(sI - F + GJ^{-1}H')^{-1}GJ^{-1}][J + H'(sI - F)^{-1}G] \\
&= I - J^{-1}H'(sI - F + GJ^{-1}H')^{-1}G + J^{-1}H'(sI - F)^{-1}G \\
&\quad -J^{-1}H'(sI - F + GJ^{-1}H')^{-1}GJ^{-1}H'(sI - F)^{-1}G
\end{aligned}$$

The last term on the right side is

$$\begin{aligned}
&-J^{-1}H'(sI - F + GJ^{-1}H')^{-1}[(sI - F + GJ^{-1}H') \\
&\quad -(sI - F)](sI - F)^{-1}G \\
&= -J^{-1}H'(sI - F)^{-1}G + J^{-1}H'(sI - F + GJ^{-1}H')^{-1}G
\end{aligned}$$

and we then have

$$[J^{-1} - J^{-1}H'(sI - F + GJ^{-1}H')^{-1}GJ^{-1}][J + H'(sI - F)^{-1}G] = I \tag{3.6.14}$$

This proves that $\{F - GJ^{-1}H', GJ^{-1}, -H(J^{-1})', J^{-1}\}$ is a realization of $W^{-1}(s)$.

Since $[F, G]$ is completely controllable, $[F - GJ^{-1}H', G]$ and $[F - GJ^{-1}H', GJ^{-1}]$ are completely controllable, the first by a result proved in an earlier section of this chapter, the second by a trivial consequence of the rank condition. Similarly, the fact that $[F - GJ^{-1}H', -H(J^{-1})']$ is completely observable follows from the complete observability of $[F, H]$. Thus the realization $\{F - GJ^{-1}H', GJ^{-1}, -H(J^{-1})', J^{-1}\}$ of $W^{-1}(s)$ is minimal, and property 7 is established, at least when $W(\infty)$ is nonsingular. $\nabla\nabla\nabla$

An interesting consequence of property 7 of network theoretic importance is that *the scattering matrix and any immittance or hybrid matrix of a network have the same degree.* For example, to show that the impedance matrix $Z(s)$ and scattering matrix $S(s)$ have the same degree, we have that

$$S(s) = I - 2(I + Z(s))^{-1}$$

Then $\delta[S(s)] = \delta[(I + Z)^{-1}] = \delta[(I + Z)] = \delta[Z(s)]$.

Next, we wish to give two other characterizations of the degree concept. The first consists of the original scheme offered in [21] for computing degree and requires us to understand the notion of the *Smith canonical form of a matrix of polynomials.*

Smith canonical form [4]. Suppose that $V(s)$ is an $m \times p$ matrix of real polynomials in s with rank r. Then there exists a representation of $V(s)$, termed the Smith canonical form, as a product

$$V(s) = P(s)\Gamma(s)Q(s) \tag{3.6.15}$$

where $P(s)$ is $m \times m$, $\Gamma(s)$ is $m \times p$, and $Q(s)$ is $p \times p$. Further,

1. $P(s)$ and $Q(s)$ possess constant nonzero determinants.
2. $\Gamma(s)$ is of the form

$$\Gamma(s) = \begin{bmatrix} \gamma_1(s) & 0 & \cdots & & & & 0 \\ 0 & \gamma_2(s) & & & & & \cdot \\ \cdot & & \ddots & & & & \cdot \\ \cdot & & & \gamma_r(s) & & & \cdot \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & 0 & & & & & 0 \end{bmatrix} \tag{3.6.16}$$

written in shorthand notation as

$$\Gamma(s) = \text{diag}\,\{\gamma_1(s), \gamma_2(s), \ldots, \gamma_r(s)\}_{m \times p}$$

3. The $\gamma_i(s)$ are uniquely determined monic polynomials* with the property that $\gamma_i(s)$ divides $\gamma_{i+1}(s)$ for $i = 1, 2, \ldots, r - 1$; the polynomial $\gamma_i(s)$ is the ratio of the greatest common divisor of all $i \times i$ minors of $V(s)$ to the greatest common divisor of all $(i - 1) \times (i - 1)$ minors of $V(s)$. By convention, $\gamma_1(s)$ is the greatest common divisor of all entries of $V(s)$.

Techniques for computing $P(s)$, $\Gamma(s)$, and $Q(s)$ may be found in [4]. Notice

*A monic polynomial in s is one in which the coefficient of the highest power of s is one.

that $\Gamma(s)$ alone is claimed as being unique; there are always an infinity of $P(s)$ and $Q(s)$ that will serve in (3.6.15).

In [21] the definition of the Smith canonical form was extended to real rational matrices $W(s)$ in the following way.

Smith–McMillan canonical form. Let $W(s)$ be an $m \times p$ matrix of real rational functions of s, and let $p(s)$ be the least common denominator of the entries of $W(s)$. Then $V(s) = p(s)W(s)$ is a polynomial matrix and has a Smith canonical form representation $P(s)\Gamma(s)Q(s)$, as described above. With $\gamma_i(s)$ as defined above, define $\epsilon_i(s)$ and $\psi_i(s)$ by

$$\frac{\gamma_i(s)}{p(s)} = \frac{\epsilon_i(s)}{\psi_i(s)} \tag{3.6.17}$$

where $\epsilon_i(s)$ and $\psi_i(s)$ have no common factors and the $\psi_i(s)$ are monic polynomials. Define also

$$\begin{aligned} E(s) &= \operatorname{diag}\{\epsilon_1(s), \epsilon_2(s), \dots, \epsilon_r(s)\}_{m\times p} \\ \Psi(s) &= \operatorname{diag}\{\psi_1^{-1}(s), \psi_2^{-1}(s), \dots, \psi_r^{-1}(s)\}_{p\times p} \end{aligned} \tag{3.6.18}$$

Then there exists a representation of $W(s)$ (the Smith–McMillan canonical form) as

$$W(s) = P(s)E(s)\Psi(s)Q(s) \tag{3.6.19}$$

where $P(s)$ and $Q(s)$ have the properties as given in the description of the Smith canonical form, $E(s)$ and $\Psi(s)$ are unique, $\epsilon_i(s)$ divides $\epsilon_{i+1}(s)$ for $i = 1, 2, \dots, r-1$, and $\psi_{i+1}(s)$ divides $\psi_i(s)$ for $i = 1, 2, \dots, r-1$.

Finally, we have the following connection with $\delta[W(s)]$, proved in [20].

Property 8. Let $W(\infty) < \infty$ and let $\psi_1(s), \psi_2(s), \dots, \psi_r(s)$ be polynomials associated with $W(s)$ in a way described in the Smith–McMillan canonical form statement. Then

$$\delta[W(s)] = \sum_i \delta[\psi_i(s)] \tag{3.6.20}$$

[Recall that $\delta[\psi_i(s)]$ is the usual degree of the polynomial $\psi_i(s)$].

In case the inequality $W(\infty) < \infty$ fails, one can write

$$\delta[W(s)] = \sum \delta\left[\frac{\epsilon_i(s)}{\psi_i(s)}\right]$$

From this result, it is not hard to show that if W is invertible, $\delta[W^{-1}] = \delta[W]$ even if $W(\infty)$ is not finite and nonsingular. (See Problem 3.6.6.)

Notice that the right-hand side of (3.6.20) may be rewritten as $\delta[\prod_i \psi_i(s)]$, and that $\delta[W(s)]$ may therefore be computed if we know only $\psi(s) = \prod_i \psi_i(s)$, rather than the individual $\psi_i(s)$. A characterization of $\psi(s)$ is available as follows.

Property 9. With $W(\infty) < \infty$, with $\psi(s) = \prod_i \psi_i(s)$, and $\psi_i(s)$ as above,

$$\delta[W(s)] = \delta[\psi(s)] \tag{3.6.21}$$

and $\psi(s)$ is the least common denominator of all $\rho \times \rho$ minors of $W(s)$, where ρ takes on all values from 1 to $\min(m, p)$.

For a proof of this result, see [20]; a proof is also requested in the problems, the result following from the definitions of the Smith and Smith–McMillan canonical forms together with property 8.

Example 3.6.4 Consider

$$W(s) = \begin{bmatrix} \frac{2}{s+2} & \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{1}{s+2} & \frac{2}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix}$$

The least common denominator of all 1×1 minors is $(s+1)(s+2)$. Also, the denominators of the three 2×2 minors of $W(s)$ are

$$(s+1)(s+2)^2, \qquad (s+1)(s+2)^2, \qquad (s+1)^2(s+2)^2$$

Accordingly, the least common denominator of all minors of $W(s)$ is $(s+1)^2(s+2)^2$ and $\delta[W(s)] = 4$.

There is a close tie between property 9 and property 5, where we showed that

$$\delta[W(s)] = \sum_{s_i} \delta[W; s_i] \tag{3.6.10}$$

Recall that the summation is over the poles of entries of $W(s)$, and $\delta[W; s_i]$ is the degree of the sum $V_i(s)$ of those terms in the partial fraction expansion of $W(s)$ with a singularity at $s = s_i$.

The least common denominator of all $\rho \times \rho$ minors of $V_i(s)$ will be a power of $s - s_i$, and therefore the least common denominator of all $\rho \times \rho$ minors will simply be that denominator of highest degree. Since the degree of this denominator will also be the order of s_i as a pole of the minor, it follows that, by *property 9, $\delta[W; s_i]$ will be the maximum order that s_i possesses as a pole of any minor of $V_i(s)$.*

Because $V_j(s)$ for $j \neq i$ has no poles in common with $V_i(s)$, it follows that $\delta[W; s_i]$ will also be the maximum order that s_i possesses as a pole of any minor of $\sum_k V_k(s)$. Since $W(s) = \sum_k V_k(s)$, we have

> **Property 10.** $\delta[W; s_i]$ is the maximum order that s_i possesses as a pole of any minor of $W(s)$.

Example 3.6.5 Consider

$$W(s) = \begin{bmatrix} \dfrac{2}{s+2} & \dfrac{1}{s+1} \\ \dfrac{1}{s+2} & \dfrac{2}{(s+1)(s+2)} \end{bmatrix}$$

The poles of entries of $W(s)$ are -1 and -2. Observing that

$$\begin{aligned} \det W(s) &= \frac{4}{(s+1)(s+2)^2} - \frac{1}{(s+1)(s+2)} \\ &= \frac{2-s}{(s+1)(s+2)^2} \end{aligned}$$

it follows that -1 has a maximum order of 1 as a pole of any minor of $W(s)$, being a simple pole of two 1×1 minors, and of the only 2×2 minor. Also, -2 has a maximum order of 2 as a pole. Therefore, $\delta[W] = 3$.

In our subsequent discussion of synthesis problems, we shall be using many of the results on degree. McMillan's original definition of degree as the minimum number of energy storage elements in a network synthesis of an impedance of course provides an impetus for attempting to solve, via state-space procedures, the synthesis problem using a minimal number of reactive elements. This inevitably leads us into manipulations of minimal realizations of impedance and other matrices, which, in one sense, is fortunate: minimal realizations possess far more properties than nonminimal ones, and, as it turns out, these properties can play an important role in developing synthesis procedures.

Problem 3.6.1 Prove that $\delta[sL] = \text{rank } L$.

Problem 3.6.2 Show that $|\delta(W_1) - \delta(W_2)| \leq \delta(W_1 + W_2)$.

Problem 3.6.3 Suppose that W_1 is square and W_1^{-1} exists. Show that $\delta[W_1 W_2] \geq \delta[W_2] - \delta[W_1]$. Extend this result to the case when W_1 is square and W_1^{-1} does not exist, and to the case when W_1 is not square. (The extension represents a difficult problem.)

Problem 3.6.4 Prove property 9. [*Hint:* First fix p. Show that the greatest common divisor of all $p \times p$ minors of $p(s)W(s)$ is $\gamma_1(s)\gamma_2(s) \cdots \gamma_p(s)$. Noting that $p(s)^p$ is a common denominator of all $p \times p$ minors of $W(s)$, show that the least common denominator of all $p \times p$ minors of $W(s)$ is the numerator, after common factor cancellations, of $p(s)^p/\gamma_1(s)\gamma_2(s) \cdots \gamma_p(s)$. Evaluate this numerator, and complete the proof by letting p vary.]

Problem 3.6.5 Evaluate the degree of

$$W(s) = \frac{1}{s^4}\begin{bmatrix} s^3 - s^2 + 1 & 1 & -s^3 + s^2 - 2 \\ 1.5s + 1 & s + 1 & -1.5s - 2 \\ s^3 - 9s^2 - s + 1 & -s^2 + 1 & s^3 - s - 2 \end{bmatrix}$$

Problem 3.6.6 Suppose that $W(s)$ is a square rational matrix decomposed as $P(s)E(s)\Psi(s)Q(s)$ with $P(s)$ and $Q(s)$ possessing constant nonzero determinants, and with

$$E(s) = \text{diag}\,\{\epsilon_1(s), \epsilon_2(s), \ldots, \epsilon_m(s)\}$$
$$\Psi(s) = \text{diag}\,\{\psi_1^{-1}(s), \psi_2^{-1}(s), \ldots, \psi_m^{-1}(s)\}$$

Also, $\epsilon_i(s)$ divides $\epsilon_{i+1}(s)$ and $\psi_{i+1}(s)$ divides $\psi_i(s)$. As we know, $\delta[W(s)] = \sum \delta[\epsilon_i(s)/\psi_i(s)]$. Assuming $W(s)$ is nonsingular almost everywhere prove that

$$W^{-1}(s) = Q^{-1}(s)\Psi^{-1}(s)E^{-1}(s)P^{-1}(s)$$
$$= \hat{P}(s)\hat{E}(s)\hat{\Psi}(s)\hat{Q}(s)$$

where $\hat{P} = Q^{-1}R$, R being a nonsingular permutation matrix and $\hat{Q} = SP^{-1}$, S being a nonsingular permutation matrix, and with $\hat{P}, \hat{Q}, \hat{E}$, and $\hat{\Psi}$ possessing the same properties as P, Q, E, and Ψ. In particular,

$$\hat{E}(s) = \text{diag}\,\{\psi_m(s), \psi_{m-1}(s), \ldots, \psi_1(s)\}$$
$$\hat{\Psi}(s) = \text{diag}\,\{\epsilon_m^{-1}(s), \epsilon_{m-1}^{-1}(s), \ldots, \epsilon_1^{-1}(s)\}$$

Deduce that $\delta[W^{-1}] = \delta[W]$.

3.7 STABILITY

Our aim in this section is to make connection between those transfer-function-matrix properties and those state-space-equation properties associated with stability. We shall also state the lemma of Lyapunov, which provides a method for testing the stability properties of state-space equations.

As is learned in elementary courses, the stability properties of a linear system described by a transfer-function matrix may usually (the precise qualifications will be noted shortly) be inferred from the pole positions of entries of the transfer-function matrix. In fact, the following two statements should be well known.

1. Usually, if each entry of a rational transfer-function matrix $W(s)$ with $W(\infty) < \infty$ has all its poles in $\text{Re}[s] < 0$, the system with $W(s)$ as transfer-function matrix is such that bounded inputs will produce bounded outputs, and outputs associated with nonzero initial conditions will decay to zero.
2. Usually, if each entry of a rational transfer-function matrix $W(s)$ with $W(\infty) < \infty$ has all its poles in $\text{Re}[s] \leq 0$, with any pole on $\text{Re}[s] = 0$ being simple, the system with $W(s)$ as transfer-function matrix is such that outputs associated with nonzero initial conditions will not be unstable, but may not decay to zero.

The reason for the qualification in the above statement arises because the system with transfer-function matrix $W(s)$ may have state equations

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= H'x + Ju \end{aligned} \tag{3.7.1}$$

with $[F, G]$ not completely controllable and with uncontrollable states unstable. For example, the state-space equations

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1]x$$

have an associated transfer function $1/(s + 1)$, but x_1 is clearly unstable, and an unbounded output would result from an excitation $u(\cdot)$ commencing at $t = 0$ if $x_1(0)$ were nonzero. Even if $x_1(0)$ were zero, in practice one would find that bounded inputs would not produce bounded outputs. Thus, either on the grounds of inexactitude in practice or on the grounds that the bounded-input, bounded-output property is meant to apply for nonzero initial conditions, we run into difficulty in determining the stability of (3.7.1) simply from knowledge of the associated transfer function.

It is clearly dangerous to attempt to describe stability properties using transfer-function-matrix properties unless there is an associated state-space realization in mind. Description of stability properties using state-space-equation properties is however quite possible.

Theorem 3.7.1. Consider the state-space equations (3.7.1). If $\text{Re}\,\lambda_i[F] < 0$,* then bounded $u(\cdot)$ lead to bounded $y(\cdot)$, and the state $x(t)$ resulting from any nonzero initial state $x(t_0)$ will decay to zero under zero-input conditions. If $\text{Re}\,\lambda_i[F] \leq 0$, and pure imaginary eigenvalues are simple, or even occur only in 1

*This notation is shorthand for "the real parts of all eigenvalues of F are negative."

$\times$ 1 blocks in the Jordan form of F, then the state $x(t)$ resulting from any nonzero initial state will remain bounded for all time under zero-input conditions.

Proof. The equation relating $x(t)$ to $x(t_0)$ and $u(\cdot)$ is

$$x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)}Gu(\tau)\,d\tau$$

Suppose that Re $\lambda_i[F] < 0$. It is easy to prove that if $\|u(\tau)\| \le c_1$ for all τ and $\|x(t_0)\| \le c_2$, then $\|x(t)\| \le c_3(c_2, c_1)$ for all t; i.e., bounded $u(\cdot)$ lead to bounded $x(\cdot)$, the bound on $x(\cdot)$ depending purely on the bound on $u(\cdot)$ and the bound on $x(t_0)$. With $x(\cdot)$ and $u(\cdot)$ bounded, it is immediate from the second equation in (3.7.1) that $y(\cdot)$ is bounded. With $u(t) \equiv 0$, it is also trivial to see that $x(t)$ decays to zero for nonzero $x(t_0)$.

Suppose now that Re $\lambda_i[F] \le 0$, with pure imaginary eigenvalues occurring only in 1×1 blocks in the Jordan form of F. (If no pure imaginary eigenvalue is repeated, this will certainly be the case.) Let T be a matrix, in general complex, such that

$$TFT^{-1} = J$$

where J is the Jordan form of F. Then e^{Jt} is easily computed, and the condition that pure imaginary eigenvalues occur only in 1×1 blocks will guarantee that only diagonal entries of e^{Jt} will contain exponentials without negative real part exponents, and such diagonal entries will be of the form $e^{j\mu t}$ for some real μ, rather than $t^{\alpha}e^{j\mu t}$ for positive integer α and real μ. It follows that

$$e^{Ft} = T^{-1}e^{Jt}T$$

will be such that the only nondecaying parts of any entry will be of the form cos μt or sin μt. This guarantees that if $u(\cdot) \equiv 0$, $x(t)$ will remain bounded if $x(t_0)$ is bounded. $\nabla\nabla\nabla$

Example 3.7.1 If $F = \begin{bmatrix} -2 & -1 \\ +1 & -2 \end{bmatrix}$, the eigenvalues of F are $-2 \pm j$ and, accordingly, whatever G, H, and J may be in (3.7.1), bounded $u(\cdot)$ will produce bounded $y(\cdot)$, and nonzero initial state vectors will decay to zero under zero excitations. If $F = \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$ the eigenvalues become $\pm j$. The solution of $\dot{x} = Fx$, the homogeneous version of (3.7.1) is

$$x(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} x(0)$$

and, as expected, the effect of nonzero initial conditions will not die out.

Consider now for arbitrary positive δ

$$\begin{aligned}
\int_{n\delta}^{(n+1)\delta} \dot{V}\,dt &= -\int_{n\delta}^{(n+1)\delta} x'HH'x\,dt \\
&= -\int_{n\delta}^{(n+1)\delta} x'(0)e^{F't}HH'e^{Ft}x(0)\,dt \\
&= -x'(0)e^{F'n\delta}\int_0^{\delta} e^{F't}HH'e^{Ft}\,dt\,e^{Fn\delta}x(0) \\
&= -x'(0)e^{F'n\delta}Qe^{Fn\delta}x(0)
\end{aligned}$$

where Q is a matrix whose nonsingularity is guaranteed by the complete observability of $\{F, H\}$. Since

$$V(x(\overline{n+1}\delta)) = V(x(0)) + \int_0^{(n+1)\delta} \dot{V}\,dt$$

it follows from the existence of $\lim_{t\to\infty} V(x(t))$ and from the nonpositivity of $\dot{V}$ that

$$\int_{n\delta}^{(n+1)\delta} \dot{V}\,dt \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

That is,

$$x'(0)e^{F'n\delta}Qe^{Fn\delta}x(0) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

or

$$x'(n\delta)Qx(n\delta) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

Since Q is positive definite, it is immediate that $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$ and that F has eigenvalues satisfying the required constraint.

We comment that the reader familiar with Lyapunov stability theory will be able to shorten the above proof considerably; V is a Lyapunov function, being positive definite. Also, $\dot{V}$ is nonpositive, and the complete observability of $[F, H]$ guarantees that $\dot{V}$ is not identically zero unless $x(0) = 0$. An important theorem on Lyapunov functions (see, e.g., [25]) then immediately guarantees that $x(t) \longrightarrow 0$.

Next, we have to prove that if F is such that Re $\lambda_i[F] < 0$, then a positive definite P exists satisfying (3.7.3). We shall prove that the matrix

$$\Pi = \int_0^{\infty} e^{F't}HH'e^{Ft}\,dt \tag{3.7.6}$$

exists, is positive definite, and satisfies $\Pi F + F'\Pi = -HH'$. Then we shall prove uniqueness of the solution P of (3.7.3). Thus we shall be able to identify P with Π.

That Π exists follows from the eigenvalue restriction on F which, as we know, guarantees that every entry of e^{Ft} contains only decaying exponentials. That Π is positive definite follows by the complete observability of $[F, H]$. Also,

$$\begin{aligned}\Pi F + F'\Pi &= \int_0^\infty (e^{F't}HH'e^{Ft}F + F'e^{F't}HH'e^{Ft})\,dt \\ &= \int_0^\infty \frac{d}{dt}(e^{F't}HH'e^{Ft})\,dt \\ &= -HH'\end{aligned}$$

on using the fact that

$$\lim_{t\to\infty} e^{Ft} = 0$$

(This follows from the eigenvalue restriction.)

To prove uniqueness, suppose that there are two different solutions P_1 and P_2 of (3.7.3). Then

$$(P_1 - P_2)F + F'(P_1 - P_2) = 0$$

It follows that

$$\begin{aligned}0 &= e^{F't}(P_1 - P_2)Fe^{Ft} + e^{F't}F'(P_1 - P_2)e^{Ft} \\ &= \frac{d}{dt}[e^{F't}(P_1 - P_2)e^{Ft}]\end{aligned}$$

Since $e^{F't}(P_1 - P_2)e^{Ft}$ is constant, it follows by taking $t = 0$ that

$$P_1 - P_2 = e^{F't}(P_1 - P_2)e^{Ft}$$

Letting t approach infinity and using the eigenvalue property again, we obtain

$$P_1 = P_2$$

This is a contradiction, and thus uniqueness is established.

$\nabla\nabla\nabla$

The uniqueness part of the above proof is mildly misleading, since it would appear that the eigenvalue constraint on F is a critical requirement. This is not actually the case. In fact, the following theorem, which we shall have occasion to use later, is true. For the proof of the theorem, we refer the reader to [4].

Theorem 3.7.4. Suppose that the matrices A, B, and C are $n \times n$, $m \times m$, and $n \times m$, respectively. There exists a unique $n \times m$ matrix X satisfying the equation

$$AX + XB = C \tag{3.7.7}$$

if and only if $\lambda_i[A] + \lambda_j[B] \neq 0$ for any two eigenvalues of A and B.

The reader is urged to check how this theorem may be applied to guarantee uniqueness of the solution P of (3.7.3), under the constraint Re $\lambda_i[F] < 0$.

Theorem 3.7.3 does not in itself suggest that checking that Re $\lambda_i[F] < 0$ is an easy task computationally. The only technique suggested for computing P is via the integral (3.7.6), where, we recall, Π turns out to be the same as P. Obviously, use of this integral for computing P would be pointless if we merely wished to examine whether Re $\lambda_i[F] < 0$.

Fortunately, there is another procedure for obtaining P, which, coupled with a well-known test for positive definiteness, allows ready examination of the stability of F. [This procedure for computing P may also be used to compute the solution X of (3.7.7).] Equation (3.7.3) is nothing more than a highly organized way of writing down $\frac{1}{2}n(n+1)$ linear simultaneous scalar equations (where n is the dimension of F), with the unknowns consisting of the $\frac{1}{2}n(n+1)$ diagonal and superdiagonal entries of P. Procedures for solving such equations are of course well known; solvability is actually guaranteed by the eigenvalue constraint on F. Thus, if in a specific situation, (3.7.3) proves insolvable, then F is known to not have eigenvalues all with negative real parts.

Once P has been found, its positive-definite character may be checked by examining the signs of all leading principal minors. These are positive if and only if P is positive definite [4].

Observe that in applying the test, there is great freedom in the possible choice of H. A simple choice that can always be made is

$$HH' = I$$

It can easily be checked that any H satisfying this equation is such that $[F, H]$ is completely observable for any F.

Example 3.7.2 Suppose that

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

(The characteristic polynomial of F is easily found in this case, as $s^3 + 3s^2 + 4s + 2$, and can be verified to have negative real part zeros. However, we shall proceed via the lemma of Lyapunov.)

We shall take $HH' = I$, and solve

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & -4 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From these equations we have

$$\begin{aligned} -4p_{13} &= -1 \\ p_{11} - 4p_{13} - 2p_{23} &= 0 \\ p_{12} - 3p_{13} - 2p_{33} &= 0 \\ 2p_{12} - 4p_{22} &= -1 \\ p_{22} - 3p_{23} + p_{13} - 4p_{33} &= 0 \\ 2p_{23} - 6p_{33} &= -1 \end{aligned}$$

from which one can compute

$$p_{11} = \tfrac{19}{16} \qquad p_{12} = \tfrac{55}{48} \qquad p_{13} = \tfrac{1}{4}$$
$$p_{22} = \tfrac{79}{96} \qquad p_{23} = \tfrac{3}{32}$$
$$p_{33} = \tfrac{19}{96}$$

Tedious calculations can then be used to check that

$$p_{11} > 0 \qquad \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} > 0 \qquad \begin{vmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{vmatrix} > 0$$

This establishes stability.

A simple extension of the lemma of Lyapunov, which we shall have occasion to use, is the following. Suppose that $[F, H]$ is completely observable and L is an arbitrary matrix with n rows, n being the dimension of F. Then $\operatorname{Re} \lambda_i[F] < 0$ *if and only if there is a positive definite symmetric* P satisfying

$$PF + F'P = -HH' - LL' \tag{3.7.8}$$

Proof is requested in the problems.

As stated, the lemma of Lyapunov is concerned with conditions for $\text{Re}\,\lambda_i[F] < 0$ rather than $\text{Re}\,\lambda_i[F] \leq 0$. A complete generalization to cover this case of a nonstrict inequality is not straightforward; however, we can prove one result that will be of assistance.

Theorem 3.7.5. Given an $n \times n$ matrix F, if there exists a positive definite matrix P such that

$$PF + F'P = -HH' \tag{3.7.3}$$

with $H = 0$ permitted, then $\text{Re}\,\lambda_i[F] \leq 0$ and the Jordan form of F has no blocks of size greater than 1×1 with pure imaginary diagonal elements; equivalently, $\dot{x} = Fx$ is stable, in the sense that for any $x(0)$, the resulting $x(t)$ is bounded for all t.

Proof. Consider the function

$$V(x(t)) = x'(t)Px(t) \tag{3.7.5}$$

Observe that, as we have earlier calculated,

$$\dot{V} = -x'HH'x \leq 0$$

Thus V is nonnegative and monotonic decreasing. It follows that all entries of $x(t)$ are bounded for any initial $x(0)$. The Jordan form of F has the stated property for the following reasons. Since $x(t) = e^{Ft}x(0)$ is bounded for arbitrary $x(0)$, it must be true that e^{Ft} is bounded. Now the entries of e^{Ft} are made up of terms $a_i e^{\lambda_i t}$, where λ_i is an eigenvalue of F, and terms $a_i t^\alpha e^{\lambda_i t}$, $\alpha > 0$, if a Jordan block associated with λ_i is of size greater than 1×1. Boundedness of e^{Ft} therefore implies that no Jordan block can be of size greater than 1×1 if $\text{Re}\,\lambda_i = 0$. ∇ ∇ ∇

As an interesting sidelight to the above theorem, notice that if F has only pure imaginary eigenvalues with no repeated eigenvalue, the *only* H for which there exists a positive definite P satisfying (3.7.3) is $H = 0$. In that instance, negative definite P also satisfy (3.7.3), as is seen by replacing P by $-P$. To prove the claim that $H = 0$, observe that F must be such that

$$TFT^{-1} = [0] \dotplus \begin{bmatrix} 0 & -\mu_1 \\ \mu_1 & 0 \end{bmatrix} \dotplus \begin{bmatrix} 0 & -\mu_2 \\ \mu_2 & 0 \end{bmatrix} \dotplus \cdots$$

for some nonsingular T, with $\dotplus$ denoting direct sum, the first block possibly absent, and each μ_i real. Then (3.7.3) gives

$$[(T^{-1})'PT^{-1}](TFT^{-1}) + (TFT^{-1})'[(T^{-1})'PT^{-1}] = -[(T^{-1})'H][(T^{-1})'H]'$$

or

$$\hat{P}\hat{F} + \hat{F}'\hat{P} = -\hat{H}\hat{H}'$$

for a skew $\hat{F}$ and positive definite $\hat{P}$. Now

$$\operatorname{tr}[AB] = \operatorname{tr}[B'A'] = \operatorname{tr}[A'B']$$

for any A and B. Therefore

$$\begin{aligned}\operatorname{tr}[\hat{P}\hat{F}] &= \operatorname{tr}[\hat{F}'\hat{P}] = -\operatorname{tr}[\hat{F}\hat{P}] \qquad \text{by skewness}\\ &= -\operatorname{tr}[\hat{P}\hat{F}]\end{aligned}$$

Therefore $\operatorname{tr}[\hat{P}\hat{F}] = 0$ and so $\operatorname{tr}[\hat{H}\hat{H}'] = 0$. This can only be so if $\hat{H} = 0$, since $\hat{H}\hat{H}'$ is nonnegative definite, and the trace of a matrix is the sum of its eigenvalues. With $\hat{H} = 0$, it follows that $H = 0$.

Problem 3.7.1 Suppose that $\operatorname{Re} \lambda_i[F] < 0$, and that $x(t)$ is related to $x(t_0)$ and $u(\tau)$ for $t_0 \leq \tau \leq t$ by

$$x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-\tau)}Gu(\tau)\,d\tau$$

Show that $\|u(\tau)\| \leq c_1$ for all τ and $\|x(t_0)\| \leq c_2$ imply that $\|x(t)\| \leq c_3$, where c_3 is a constant depending solely on c_1 and c_2.

Problem 3.7.2 Discuss how the equation

$$PF + F'P = -HH'$$

might be solved if F is in diagonal form and $\operatorname{Re} \lambda_i[F] < 0$.

Problem 3.7.3 Show that if H is any matrix of n rows with $HH' = I$ and F is any $n \times n$ matrix, then $[F, H]$ is completely observable.

Problem 3.7.4 Suppose that $[F, H]$ is completely observable and L is an arbitrary matrix with n rows, n being the dimension of F. Prove that $\operatorname{Re} \lambda_i[F] < 0$ if and only if there is a positive definite symmetric P satisfying

$$PF + F'P = -HH' - LL'$$

Problem 3.7.5 Determine, using the lemma of Lyapunov, whether the eigenvalues of the following matrix have negative real parts:

$$F = \begin{bmatrix} -\frac{1}{2} & 5 & \frac{5}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -3 & -\frac{5}{2} \end{bmatrix}$$

Problem 3.7.6 Let σ be an arbitrary real number, and let $[F, H]$ be a completely observable pair. Prove that $\text{Re}\,\lambda_i[F] < -\sigma$ if and only if there exists a positive definite P such that

$$PF + F'P + 2\sigma P = -HH'$$

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Part III

NETWORK ANALYSIS

Two of the longest recognized problems of network theory are those of analysis and synthesis. In analysis problems, one generally starts with a description of a network in terms of its components or a circuit schematic, and from this description one deduces what sort of properties the network will exhibit in its use. In synthesis problems, one generally starts with a list of properties which it is desired that a network will have, and from this list one deduces a description of the network in terms of its components and a circuit schematic. The task of this part, which consists of one chapter, is to study the use of state-space equations in tackling the analysis problem.

vector y evanesce, and we are left with the problem of having to formulate a homogeneous equation $\dot{x} = Fx$.

At this point, we wish to issue a caution to the reader: given a network described in physical terms, and given the physical significance of the vectors u and y, i.e., the input or excitation and output or response vectors, *there is no unique state-space equation set describing the network*—in fact, as we shall shortly see, there may be no such set. The reason for nonuniqueness is fairly obvious: transformations of a state vector x according to $\hat{x} = Tx$ for nonsingular T yield as equally valid a set of state-space equations, with state-vector $\hat{x}$, as the equation set with state-vector x. Though this point is obvious, the older literature on the state-space analysis of networks sometimes suggests implicitly that the state-space equations are unique, and that the only valid state vector is one whose entries are either inductor currents or capacitor voltages.

In this chapter we shall discuss the derivation of a set of state-space equations of a network at three levels of complexity. At the first and lowest level, we shall show how equations may be set up if the network is simple enough to write down *Kirchhoff voltage law or current law equations* by inspection; this is the most common situation in practice. At the second level, we shall show how equations may be set up if a certain hybrid matrix exists; this is the next most common situation in practice. At the third and most complex level, we shall consider a generally applicable procedure that does not rely on the assumptions of the first- or second-level treatments; this is the least common situation in practice.

Generally speaking, all the methods of setting up state-space equations that we shall present *attempt* to take as the entries of the state vector, the inductor currents and capacitor voltages, and, having identified the entries of the state vector in physical terms, attempt to compute from the network the coefficient matrices of the state-space equations. The attempts are not always successful. In our discussions at the first level of complexity we shall try to indicate with examples the sorts of difficulties that can arise in attempting to formulate state-space equations this way.

The fundamental source of such difficulties has been known for some time; there are generally one of two factors operating:

1. The network contains a node with the only elements incident at the node comprising inductors and/or current generators. (More generally, the networks may contain a cut set,* the branches of which comprise inductors and/or current generators.)
2. The network contains a loop with the only branches in the loop comprising capacitors and/or voltage generators.

*If the reader is not familiar with the notion of a cut set, this point may be skipped over.

In case (2), for example, assuming an all-capacitor loop, it is evident that the capacitor voltages cannot take on independent values without violating Kirchhoff's voltage law, which demands that the sum of the capacitor voltages be zero. It is this sort of constraint on the potential entries of a state vector that may interfere with their actual use.

The treatment of difficulties of the sort mentioned can proceed with the aid of network topology. However, *if the networks concerned contain gyrators and transformers, the use of network topology methods involves enormous complexity.* Even a qualitative description of all possible difficulties becomes very difficult as soon as one proceeds past the two difficulties already noted. Accordingly, we shall avoid use of network topology methods, and pay little attention to trying to qualitatively and generally describe the difficulties in setting up state-space equations. We shall, however, present examples illustrating difficulties.

The earliest papers on the topic of generating state-space equations for networks [1–3] dealt almost solely with the unexcited case; i.e., they dealt with the derivation of the homogeneous equation

$$\dot{x} = Fx \tag{4.1.1}$$

[Actually, [2] and [3] did permit certain generators to be present, but the basic issue was still to derive (4.1.1).] More recent work [4–7] considers the derivation of the nonhomogeneous equation

$$\dot{x} = Fx + Gu \tag{4.1.2}$$

An extended treatment of the problem of generating the nonhomogeneous equation can be found in [8].

There are two features of most treatments dealing with the derivation of state-space equations that should be noted. First, the treatments depend heavily on notions of network topology. As already noted, our treatment will not involve network topology. Second, most treatments have difficulty coping with ideal transformers and gyrators. The popular approach to deal with the problem is to set up procedures for deriving state-space equations for networks containing controlled sources, and to then replace the ideal transformers and gyrators with controlled sources. Our treatment of *passive* networks does not require this, and by avoiding the introduction of element classes like controlled sources for which elements of the class may be active, we are able to get results that are probably sharper.

4.2 STATE-SPACE EQUATIONS FOR SIMPLE NETWORKS

Our approach in this section will be to state a general rule for the derivation of a set of state-space equations, and then to develop a number of examples illustrating application of the rule. Examples will also show difficulties with its application. The two following sections will consider situations in which the procedure of this section is unsatisfactory. Throughout this section we allow only resistor, inductor, capacitor, transformer, and gyrator elements in the networks under consideration.

The procedure for setting up state-space equations is as follows:

1. Identify each inductor current and each capacitor voltage with an entry of the state variable x. Naturally, different inductor currents and capacitor voltages correspond to different entries!
2. Using Kirchhoff's voltage law or current law, write down an equation involving each entry of $\dot{x}$, but not involving any entry of the output y or any variables other than entries of x and u.
3. Using Kirchhoff's voltage law or current law, write down an equation involving each entry of the output, but not involving any entry of $\dot{x}$ or any variables other than entries of x and u.
4. Organize the equations derived in step 2 into the form $\dot{x} = Fx + Gu$, and those derived in step 3 into the form $y = H'x + Ju$.

Example 4.2.1 Consider the network of Fig. 4.2.1. We identify the current I with u, and the voltage V with y. Also, following step 1, we take

$$x = \begin{bmatrix} V_{c_1} \\ V_{c_2} \end{bmatrix}$$

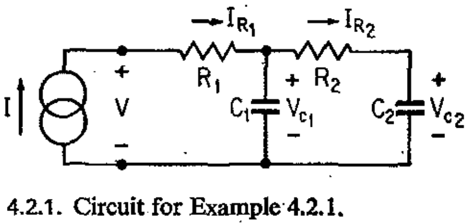


FIGURE 4.2.1. Circuit for Example 4.2.1.

Next, Kirchhoff's current law yields

$$\dot{V}_{c_1} = \frac{1}{C_1}(I_{R_1} - I_{R_2}) = \frac{1}{C_1}I - \frac{1}{C_1 R_2}(V_{c_1} - V_{c_2})$$

Note that the equation

$$\dot{V}_{c_1} = \frac{1}{C_1 R_1}(V - V_{c_1}) - \frac{1}{C_1 R_2}(V_{c_1} - V_{c_2})$$

though correct, is no help and does not conform with the requirements of step 2 since it involves the output variable V.

An equation for $\dot{V}_{c_2}$ is straightforward to obtain:

$$\dot{V}_{c_2} = \frac{1}{C_2} I_{R_2} = \frac{1}{C_2 R_2}(V_{c_1} - V_{c_2})$$

An equation for the output V is now required expressing V in terms of I, V_{c_1}, and V_{c_2} (see step 3). This is easy to obtain.

$$V = R_1 I + V_{c_1}$$

Combining the equations for $\dot{V}_{c_1}$, $\dot{V}_{c_2}$, and V together we have

$$\dot{x} = \begin{bmatrix} -\frac{1}{C_1 R_2} & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C_1} \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0]x + R_1 u$$

Example 4.2.2 Figure 4.2.2 shows a doubly terminated ladder network, with input variable V and output variable I_{R_2}. Thus $u = V$ and $y = I_{R_2}$. The state vector x we shall take to be

$$x = \begin{bmatrix} V_{c_1} \\ I_{L_1} \\ V_{c_2} \\ \cdot \\ \cdot \\ \cdot \\ V_{c_n} \end{bmatrix}$$

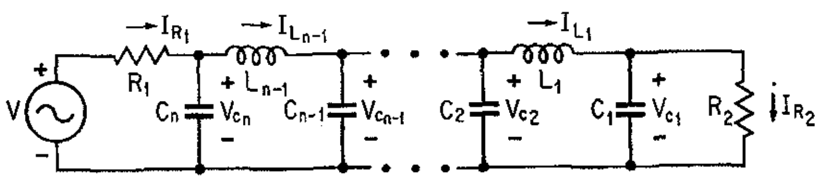


FIGURE 4.2.2. Ladder Network Discussed in Example 4.2.2.

Proceeding according to step 2, we have

$$\dot{V}_{c_1} = \frac{1}{C_1}(I_{L_1} - I_{R_2}) = \frac{1}{C_1} I_{L_1} - \frac{1}{C_1 R_2} V_{c_1}$$

$$\dot{I}_{L_1} = \frac{1}{L_1} V_{c_2} - \frac{1}{L_1} V_{c_1}$$

$$\dot{V}_{c_2} = \frac{1}{C_2} I_{L_2} - \frac{1}{C_2} I_{L_1}$$

$$\vdots$$

$$\dot{I}_{L_{n-1}} = \frac{1}{L_{n-1}} V_{c_n} - \frac{1}{L_{n-1}} V_{c_{n-1}}$$

$$\dot{V}_{c_n} = \frac{1}{C_n}(I_{R_1} - I_{L_{n-1}}) = -\frac{V_{c_n}}{C_n R_1} + \frac{V}{C_n R_1} - \frac{I_{L_{n-1}}}{C_n}.$$

Following step 3,

$$I_{R_2} = \frac{1}{R_2} V_{c_1}$$

State-space equations are thus

$$\dot{x} = \begin{bmatrix} -\frac{1}{C_1 R_2} & \frac{1}{C_1} & 0 & \cdots & & & 0 \\ -\frac{1}{L_1} & 0 & \frac{1}{L_1} & & & & \vdots \\ 0 & -\frac{1}{C_2} & 0 & & & & \\ \vdots & & & \ddots & & & \\ & & & & 0 & \frac{1}{C_{n-1}} & 0 \\ & & & & -\frac{1}{L_{n-1}} & 0 & \frac{1}{L_{n-1}} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{C_n} & -\frac{1}{C_n R_1} \end{bmatrix} \times x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ 0 \\ \frac{1}{C_n R_1} \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{1}{R_2} & 0 & \cdots & 0 \end{bmatrix} x$$

The above state-space equations of course are not unique; another set is provided by making the transformation

$$\hat{x} = \begin{bmatrix} C_1^{1/2}x_1 \\ L_1^{1/2}x_2 \\ C_2^{1/2}x_3 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

and the equations that result are

$$\dot{\hat{x}} = \begin{bmatrix} -\frac{1}{C_1R_2} & \frac{1}{\sqrt{L_1C_1}} & 0 & \cdots & & & 0 \\ -\frac{1}{\sqrt{L_1C_1}} & 0 & \frac{1}{\sqrt{L_1C_2}} & & & & \\ 0 & -\frac{1}{\sqrt{L_1C_2}} & 0 & & & & \\ \vdots & & & \ddots & & & \\ & & & & 0 & \frac{1}{\sqrt{L_{n-1}C_{n-1}}} & 0 \\ & & & & -\frac{1}{\sqrt{L_{n-1}C_{n-1}}} & 0 & \frac{1}{\sqrt{L_{n-1}C_n}} \\ & & & & 0 & -\frac{1}{\sqrt{L_{n-1}C_n}} & -\frac{1}{C_nR_1} \end{bmatrix}$$

$$\times \hat{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ \frac{1}{\sqrt{C_nR_1}} \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{1}{\sqrt{C_1R_1}} & 0 & \cdots & 0 \end{bmatrix} \hat{x}$$

State-space equations of these two forms and their connection with ladder networks have been investigated in [9]–[12].

Now we shall look at some difficulties that can arise with this approach.

Example 4.2.3 Consider the circuit of Fig. 4.2.3. By inspection we see that

$$V = RI + L\dot{I}$$

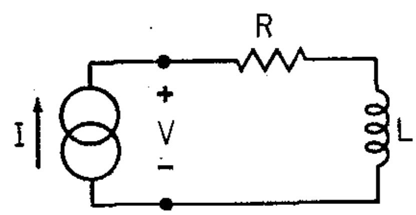


FIGURE 4.2.3. Network of Example 4.2.3 with Nonstandard State-Space Equation.

With excitation I and response V, there is obviously no way we can recover state-space equations of the standard form. Of course, if I were the response and V the excitation, we would have

$$\dot{I} = -\frac{R}{L}I + \frac{1}{L}V$$

which is of the standard form; but this is not the case, and a new form of state equation, if the name is still appropriate, has evolved. Again, consider the circuit of Fig. 4.2.4. For this circuit, we have

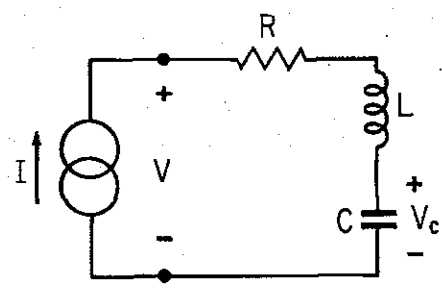


FIGURE 4.2.4. Second Network for Example 4.2.3.

$$\dot{V}_c = \frac{1}{C}I$$
$$V = V_c + RI + L\dot{I}$$

which is of the general form

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= H'x + Ju + E\dot{u} \end{aligned} \tag{4.2.1}$$

Again therefore, nonstandard equations have evolved.

One way around the apparent difficulty raised in the above example is simply to recognize (4.2.1) as a valid set of state-space equations in the same sense that

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= H'x + Ju \end{aligned} \tag{4.2.2}$$

is a valid set of state-space equations. Whereas the transfer-function matrix associated with (4.2.2) is

$$J + H'(sI - F)^{-1}G$$

the transfer-function matrix associated with (4.2.1) is

$$J + H'(sI - F)^{-1}G + sE$$

as may easily be checked. With $E \neq 0$, this transfer-function matrix has elements with a pole at $s = \infty$, in contrast to the transfer-function matrix of (4.2.2). Obviously, it must then be impossible for a set of state-space equations of the form of (4.2.1) to represent a transfer-function matrix representable by equations of the form (4.2.2), at least with $E \neq 0$.

Notice that although there are two energy storage elements in the circuit of Fig. 4.2.4, the state vector is of dimension 1. However, knowledge of both $V_c(0)$ and $I(0\text{-})$ is required to compute the response V; i.e., in a sense, I itself is a state vector entry. Likewise, if equations of the general form of (4.2.1) are derived from a circuit, it will be found that the dimension of the vector x will be less than the number of energy storage elements, with the difference equal to the rank of the matrix E.

A systematic procedure for dealing with circuits in which input derivatives tend to arise in the state-space equations appears in Section 4.4.

We return now to the presentation of additional examples suggesting difficulties with the ad hoc approach.

Example 4.2.4 Consider the circuit of Fig. 4.2.5. For this circuit,

$$C\dot{V}_c = I_{L_2} \qquad L_1\dot{I}_{L_1} = L_2\dot{I}_{L_2} + V_c \qquad I = I_{L_1} + I_{L_2} \qquad V = L_1\dot{I}_{L_1}$$

There are no other independent equations, and therefore state-space

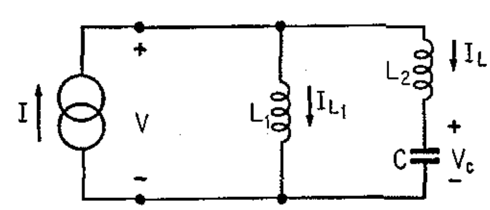


FIGURE 4.2.5. Network for Example 4.2.4.

equations can only be constructed with these equations. After a number of trials, the reader will soon convince himself that with

$$x = \begin{bmatrix} I_{L_1} \\ I_{L_2} \\ V_c \end{bmatrix} \qquad u = I$$

it is impossible to find F and G such that

$$\dot{x} = Fx + Gu$$

The best that can be done is

$$\frac{d}{dt}\begin{bmatrix} I_{L_1} \\ I_{L_2} \\ V_c \end{bmatrix} = \begin{bmatrix} 0 & 0 & (L_1 + L_2)^{-1} \\ 0 & 0 & -(L_1 + L_2)^{-1} \\ 0 & C^{-1} & 0 \end{bmatrix}\begin{bmatrix} I_{L_1} \\ I_{L_2} \\ V_c \end{bmatrix} + \begin{bmatrix} L_2(L_1 + L_2)^{-1} \\ L_1(L_1 + L_2)^{-1} \\ 0 \end{bmatrix}\dot{I}$$

The difficulty arises in the above instance really because of our insistence that the entries of the state vector correspond to inductor currents and capacitor voltages. As we shall see in Section 4.4, by dropping this assumption we can get state-space equations of a more usual form.

Another sort of difficulty arises when the circuit and its excitations are not necessarily well defined.

Example 4.2.5 Consider the circuit of Fig. 4.2.6. Because of the presence of the transformer, V_1 and V_2 cannot be chosen independently. Therefore, it is impossible to write down state-space equations with u a two-vector with entries V_1 and V_2.

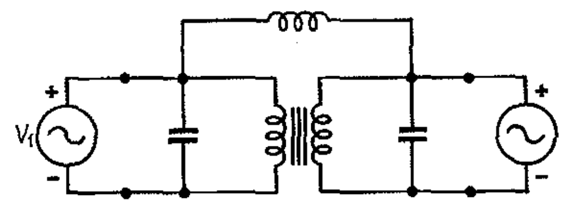


FIGURE 4.2-6. Network for Example 4.2.5. The Inputs V_1 and V_2 cannot be Chosen Independently.

There is no real way around this sort of difficulty, a general version of which will be noted in Section 4.4. Essentially, all that can be done is to disallow independent generators at ports where the desired excitation variable cannot in reality be independently specified. Thus, in the case of the above example, this would mean ceasing to think of V_2 as an independent generator. Of course, the solution is not to replace V_2 by a short circuit—this would mean that V_1 would have to be identically zero!

Next we wish to consider a circuit for which no problems arise, but which does illustrate an interesting feature.

Example 4.2.6 Consider the circuit of Fig. 4.2.7. We take

$$x = \begin{bmatrix} I_L \\ V_c \end{bmatrix} \qquad u = I \qquad y = V$$

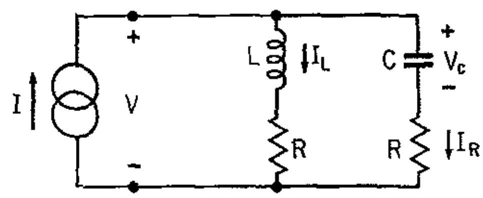


FIGURE 4.2.7. Network for Example 4.2.6.

By inspection of the circuit, we have

$$C\dot{V}_c = I - I_L \qquad \dot{V}_c = -\frac{1}{C}I_L + \frac{1}{C}I$$

Also,

$$\begin{aligned} L\dot{I}_L &= V_c + RI_R - RI_L \\ &= V_c + R(I - I_L) - RI_L \\ &= -2RI_L + V_c + RI \end{aligned}$$

or

$$\dot{I}_L = -\frac{2R}{L}I_L + \frac{1}{L}V_c + \frac{R}{L}I$$

Further,

$$V = V_c + RI_R = V_c + RI - RI_L$$

Thus the state-space equations are

$$\dot{x} = \begin{bmatrix} -\frac{2R}{L} & \frac{1}{L} \\ -\frac{1}{C} & 0 \end{bmatrix} x + \begin{bmatrix} \frac{R}{L} \\ \frac{1}{C} \end{bmatrix} I$$

$$y = [-R \quad 1]x + RI$$

Let us examine the controllability of this set of equations. We form

$$\begin{bmatrix} \frac{R}{L} & -\frac{2R^2}{L^2} + \frac{1}{LC} \\ \frac{1}{C} & -\frac{R}{LC} \end{bmatrix}$$

The determinant of this matrix is zero if and only if

$$R^2 = \frac{L}{C}$$

To check observability, we form

$$\begin{bmatrix} -R & \frac{2R^2}{L} - \frac{1}{C} \\ 1 & -\frac{R}{L} \end{bmatrix}$$

and the condition for lack of observability, obtained by setting the determinant of the matrix equal to zero, is also

$$R^2 = \frac{L}{C}$$

The transfer function associated with the state-space equations is readily found as

$$\frac{\left(\frac{1}{C} - \frac{R^2}{L}\right)s}{s^2 + \frac{2R}{L}s + \frac{1}{LC}} + R = \frac{Rs^2 + \left(\frac{1}{C} + \frac{R^2}{L}\right)s + \frac{R}{LC}}{s^2 + \frac{2R}{L}s + \frac{1}{LC}}$$

With $R^2 = L/C$, this transfer function becomes simply R. Under this constraint, the network of Fig. 4.2.7 is known as a constant resistance network.

The point illustrated by the above example is that *the ability to set up state-space equations is independent of the controllability and observability of the resulting equations.* The reader can check too that the difficulties we have noted in setting up state-space equations in earlier examples have not resulted from any properties associated with the controllability or observability notions.

Problem 4.2.1 Write down state-variable equations for the network of Fig. 4.2.8. The sources are a voltage generator at port 1 and a current generator at port

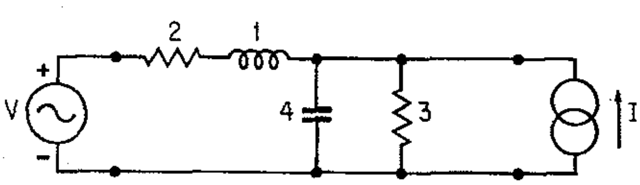


FIGURE 4.2.8. Network for Problem 4.2.1.

2. Compute from the state-space equations the transfer function relating the exciting voltage at port 1 to a response voltage at port 2, assuming that port 2 is open circuit.

Problem 4.2.2 Write down state-space equations describing the behavior of the network of Fig. 4.2.9. Verify that the eigenvalues of the F matrix are simple and pure imaginary. (*Note:* There are no excitations for the network.)

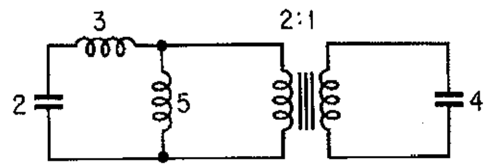


FIGURE 4.2.9. Network for Problem 4.2.2.

Problem 4.2.3 Write down state-space equations for the network of Fig. 4.2.10. Examine the controllability and observability properties and comment.

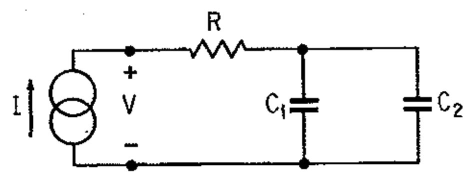


FIGURE 4.2.10. Network for Problem 4.2.3.

Problem 4.2.4 Write down state-space equations for the network of Fig. 4.2.11. If $V = V_0 \sin \omega t$ for $t \geq 0$, $V = 0$ for $t < 0$, $\omega \neq 1$, compute from the state-space equations initial conditions for the circuit that will guarantee that I will be purely sinusoidal, containing no transient term.

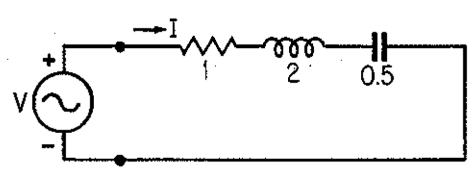


FIGURE 4.2.11. Network for Problem 4.2.4.

Problem 4.2.5 Write down state-space equations for the circuit of Fig. 4.2.12. Can you show that with $L = CR^2$, the circuit is a constant resistance network; i.e., $V = IR$?

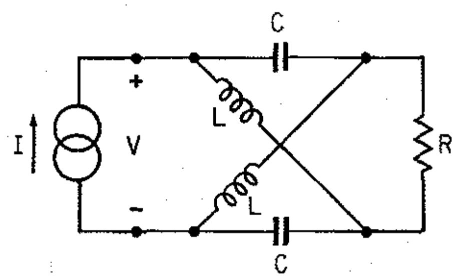


FIGURE 4.2.12. Network for Problem 4.2.5.

Problem 4.2.6 Consider the network of Fig. 4.2.13, which might have resulted from a classical Brune synthesis [13] of an impedance function. Obtain equations successively for V_{c_3} and I_{L_3}, V_{c_2} and I_{L_2}, and V_{c_1} and I_{L_1}. Write down a set of state-space equations with state vector x given by

$$x' = [C_1 V_{c_1} \quad C_2 V_{c_2} \quad C_3 V_{c_3} \quad L_1 I_{L_1} \quad L_2 I_{L_2} \quad L_3 I_{L_3}]$$

Note the interesting structure of the F matrix.

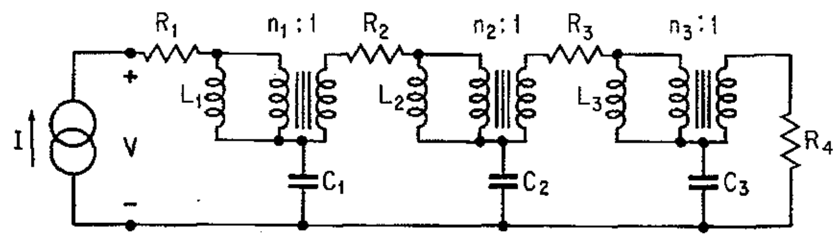


FIGURE 4.2.13. Network for Problem 4.2.6.

4.3 STATE-SPACE EQUATIONS VIA REACTANCE EXTRACTION—SIMPLE VERSION

In this section we outline a somewhat more systematic procedure than that of the previous section for the derivation of state-space equations for passive networks. As before, the key idea is still to identify the entries of the state vector with capacitor voltages and inductor currents. However, we replace the essentially ad hoc method of obtaining state equations by one in which the *hybrid matrix of a memoryless or nondynamic network* is computed. (A memoryless or nondynamic network is one that contains no inductor or capacitor elements, but may contain resistors, transformers, and gyrators.) The procedure breaks down if the hybrid matrix is not computable; in this instance, the more complex procedure of the next section must be used.

As before, we assume that the input u and output y of the state-space equations are identified physically, in the sense that each entry of u and y is known to be a port voltage or current. However, we shall constrain u and y further in this and the next section; in what proves to be an eventual simplification in notation and in development of the theory, we shall assume that if a network port is excited by a voltage source (current source), the current (voltage) at that port will be a response; further, we shall assume that if the current (voltage) at a port is a response, then the port is excited with a voltage (current) generator. *If it turns out that we can still generate the state-space equations, there is no loss of generality in making this assumption.* To see this, consider a situation in which we are given a two-port network, with port 1 excited by a voltage source and the network response (with port 2 short-circuited) comprising the current at port 2 (see Fig. 4.3.1a). In effect,

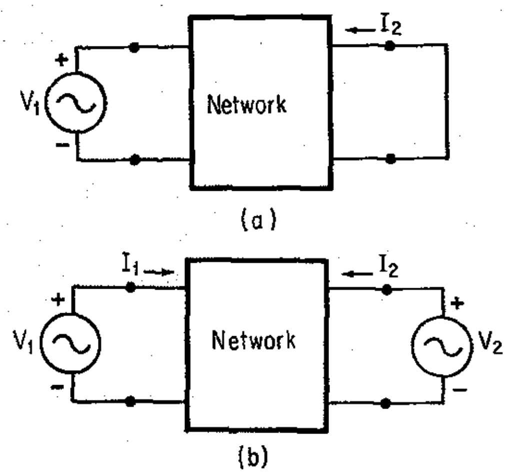


FIGURE 4.3.1. Introduction of Excitation at a Port where Response is Measured.

all we desire is state-space equations with the input u equal to V_1 and output y equal to the response I_2. According to our assumption, instead we shall seek to generate state-space equations with the input u a two-vector, with entries V_1 and V_2, and the output y a two-vector, with entries I_1 and I_2 (see Fig. 4.3.1b). Assuming that we obtain such equations, it is immediate that embedded within these equations are equations with input V_1 and output I_2. Thus assume that we derive

$$\dot{x} = Fx + [G_1 \quad G_2]\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} H_1' \\ H_2' \end{bmatrix} x \tag{4.3.1}$$

Then certainly, with $V_2 = 0$, we can write

$$\begin{aligned} \dot{x} &= Fx + G_1V_1 \\ I_2 &= H_2'x \end{aligned} \tag{4.3.2}$$

More generally the process of inserting new excitation variables proceeds as follows. Suppose that at port j there is initially prescribed a response that is a current, but no voltage excitation is prescribed. The port will be short circuited and the response will be the current through this short circuit; an *excitation is introduced by replacing the short circuit with a voltage generator.* If at port j there is initially prescribed a response that is a voltage, but no current excitation, we note that the port must initially be open circuited. An excitation is introduced by connecting a current generator across the port. Note that if there is initially prescribed a response that is a current through some component, we conceive of the current as being through a short circuit in series with the component, and introduce a voltage generator in this short circuit (see Fig. 4.3.2a). Likewise, if a response is a voltage across a component, we conceive of an open-circuit port being in parallel with the component, and introduce a current generator across this open circuit (see Fig. 4.3.2b).

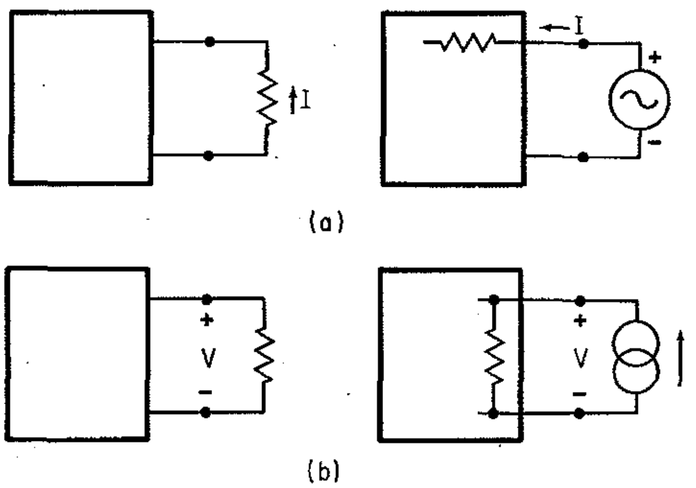


FIGURE 4.3.2. Procedure for Dealing with Response Associated with a Component.

When an excitation has been defined for every port, the definition of response variables at every port is of course immediate.

What our assumption demands is that we require, at least temporarily, a full complement of inputs and outputs; i.e., every port current and voltage is either an entry of the input u or output y; subsequent to the derivation of the state-space equations, certain inputs and outputs can be discarded if desired.

We shall also assume that the ith components u_i and y_i are associated with the ith port of the network. This ensures, for example, that $u'y$ is the instantaneous power flow into the network. Having identified the state-space *equation input* u and output y in the manner just described, the next step is to extract all the reactive elements, i.e., inductors and capacitors, from the prescribed network. This amounts to redrawing the prescribed network N as a cascade connection of a memoryless network N_r terminated in inductors and capacitors (see Fig. 4.3.3). If N has m ports and n reactive elements,

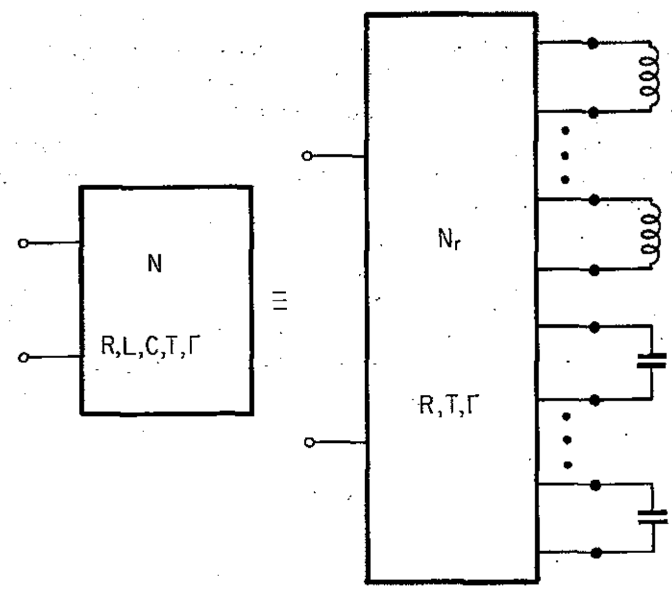


FIGURE 4.3.3. Illustration of Reactance Extraction.

then N_r will have $(m + n)$ ports, at n of which there will be inductor or capacitor terminations. This is the procedure of *reactance extraction.*

Suppose that there are n_l inductors $L_1, L_2, \ldots, L_{n_l}$ in N, and n_c capacitors $C_1, C_2, \ldots, C_{n_c}$. Of course, $n_l + n_c = n$. We shall take as the state vector

$$x = [I_{L_1} \quad I_{L_2} \quad \cdots \quad I_{L_{n_l}} \quad V_{c_1} \quad V_{c_2} \quad \cdots \quad V_{c_{n_c}}]' \tag{4.3.3}$$

The sign convention is shown in Fig. 4.3.4 and is set up so that each inductor current is positive when the port current is positive and each capacitor voltage is positive when the port voltage is positive.

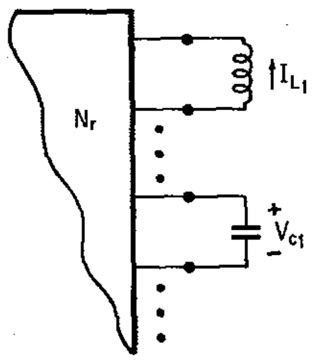


FIGURE 4.3.4. Definition of State Vector Components.

To deduce state equations, we form a certain hybrid matrix for N_r, assumed stripped of its terminations. If the matrix does not exist, then the procedure of this section cannot be used, and a modification, described in the next section, is required.

To specify the hybrid matrix for N_r, we need to specify the excitation and response variables. The first m excitation variables coincide with the entries of the input vector u; i.e., the excitation variables for the left-hand m ports of N_r are the same as those for the m ports of N. The next n_l excitation variables, corresponding to those ports of N_r shown terminated in inductors in Fig. 4.3.3, are all currents; we shall denote the vector of such currents by I_2. Finally, the remaining n_c ports of N_r will be assumed to be excited by voltages, the vector of which will be denoted by V_3. These excitations are shown in Fig. 4.3.5.

The response variables will be denoted by $\hat{y}$, an m vector of responses at the first m ports; by V_2, an n_l vector of responses at the next n_l ports, the ones shown as inductor terminated in Fig. 4.3.3; and by I_3, an n_c vector of responses at the remaining n_c ports. Notice that with the inductor and capacitor terminations, $\hat{y}$ has to be the same as y, since y is the response vector for N when u is the excitation, and N_r with terminations is the same as N. But if N_r is considered alone, without the terminations, the response at the first m ports will not be the same as y, and so we have used a different symbol for the response.

The equation relating the defined excitation and response variables for N_r is

$$\begin{bmatrix} \hat{y} \\ V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} u \\ I_2 \\ V_3 \end{bmatrix} \tag{4.3.4}$$

where

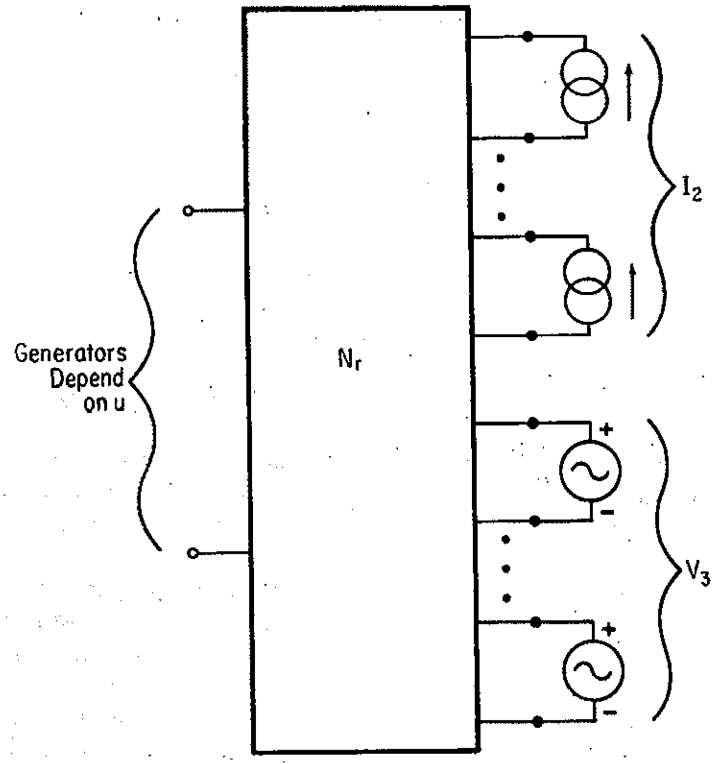


FIGURE 4.3.5. Excitations of N_r.

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \tag{4.3.5}$$

is the hybrid matrix of N_r, partitioned conformably with the excitations.

In principle, the determination of M, assuming that it exists, is a straightforward circuit analysis task. If N_r is exceedingly complex, analysis techniques based on topological results may be called into play if desired, although they are not always necessary. In this sense, the technique being presented for the derivation of state-space equations could be thought of as demanding notions of network topology for its implementation. But certainly, application of network topology could not be regarded as an *intrinsic* part of the procedure. We shall make no further comments here or in the examples regarding the computation of M, in view of the generally easy nature of the task.

Now we can derive state-space equations from our knowledge of M. Recall that our aim is to describe N. We have a description of N_r provided by (4.3.4), and we have a definition of the state vector provided by (4.3.3); so we need to consider what happens to our description of N_r when terminations are introduced. When N_r is terminated in inductors and capacitors, the following relation is forced between I_2 and V_2.

$$V_2 = -\begin{bmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_{n_l} \end{bmatrix} \begin{bmatrix} \dot{I}_{L_1} \\ \dot{I}_{L_2} \\ \vdots \\ \dot{I}_{L_{n_l}} \end{bmatrix} = -\mathcal{L}\dot{I}_2 \tag{4.3.6}$$

where $\mathcal{L} = \operatorname{diag}[L_1, L_2, \ldots, L_{n_l}]$ and we have used the fact that termination of the ith port of N_r in L_i identifies the ith entry of I_2 with I_{L_i}. The minus sign arises because of the sign convention applying to excitations of N_r. Similarly, the following relation holds between I_3 and V_3.

$$-I_3 = \begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_{n_c} \end{bmatrix} \begin{bmatrix} \dot{V}_{c1} \\ \dot{V}_{c2} \\ \vdots \\ \dot{V}_{cn_c} \end{bmatrix} = \mathcal{C}\dot{V}_3 \tag{4.3.7}$$

where $\mathcal{C} = \operatorname{diag}[C_1, C_2, \ldots, C_{n_c}]$. The minus sign again arises because of the sign convention applying to excitations of N_r.

Now we substitute for V_2 and I_3 in (4.3.4) to obtain an equation reflecting the performance of N_r when it is terminated in inductors and capacitors. We can therefore replace $\hat{y}$ by y in this equation, which becomes

$$\begin{bmatrix} y \\ -\mathcal{L}\dot{I}_2 \\ -\mathcal{C}\dot{V}_3 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} u \\ I_2 \\ V_3 \end{bmatrix}$$

or

$$\begin{bmatrix} y \\ \dot{I}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ -\mathcal{L}^{-1}M_{21} & -\mathcal{L}^{-1}M_{22} & -\mathcal{L}^{-1}M_{23} \\ -\mathcal{C}^{-1}M_{31} & -\mathcal{C}^{-1}M_{32} & -\mathcal{C}^{-1}M_{33} \end{bmatrix} \begin{bmatrix} u \\ I_2 \\ V_3 \end{bmatrix} \tag{4.3.8}$$

Finally, we use the definition (4.3.3) of the state vector x. Recalling that each entry I_2 is the same as an inductor current when N_r is terminated, while each entry of V_3 is the same as a capacitor voltage, we obtain from (4.3.3) and (4.3.8)

$$\begin{bmatrix} y \\ \dot{x} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ -\mathcal{L}^{-1}M_{21} & -\mathcal{L}^{-1}M_{22} & -\mathcal{L}^{-1}M_{23} \\ -\mathcal{C}^{-1}M_{31} & -\mathcal{C}^{-1}M_{32} & -\mathcal{C}^{-1}M_{33} \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix} \tag{4.3.9}$$

Now identify

$$F = \begin{bmatrix} -\mathcal{L}^{-1}M_{22} & -\mathcal{L}^{-1}M_{23} \\ -\mathcal{C}^{-1}M_{32} & -\mathcal{C}^{-1}M_{33} \end{bmatrix} \qquad G = \begin{bmatrix} -\mathcal{L}^{-1}M_{21} \\ -\mathcal{C}^{-1}M_{31} \end{bmatrix}$$
$$H' = [M_{12} \quad M_{13}] \qquad J = M_{11} \tag{4.3.10}$$

Equations (4.3.9) become simply

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= H'x + Ju \end{aligned} \tag{4.3.11}$$

This completes our statement of the procedure for deducing (4.3.11). Let us summarize it.

1. Ensure that every port current and voltage of N_r is either an excitation or response, with the ith entry of u and ith entry of y denoting the excitation and response at the ith port.
2. Perform a reactance extraction to generate a nondynamic network N_r (see Fig. 4.3.3). Assign the entries of the state variable as inductor currents and capacitor voltages, with the sign convention shown in Fig. 4.3.4.
3. Compute a hybrid matrix for N_r, with excitation variables comprising the entries of u, together with the currents at inductively terminated ports and voltages at capacitively terminated ports [see (4.3.4)].
4. With $\mathcal{L} = \text{diag}\,[L_1, L_2, \ldots, L_{n_l}]$ and $\mathcal{C} = \text{diag}\,[C_1, C_2, \ldots, C_{n_c}]$, define the matrices of the state-space equations of N by (4.3.10).
5. If step 3 fails, in the sense that no hybrid matrix exists with the required excitation and response variables, the method fails.

Example 4.3.1 Consider the network of Fig. 4.3.6a. This network is redrawn showing a reactance extraction in Fig. 4.3.6b. The associated nondynamic network N_r is shown in Fig. 4.3.6c with sources corresponding to the correct excitation variables. Notice that because no inductors are present, I_2 evanesces. With two capacitors, V_3 becomes a two-vector.

The hybrid matrix of the network of Fig. 4.3.6c is easily found. We have

$$\begin{bmatrix} y \\ (I_3)_1 \\ (I_3)_2 \end{bmatrix} = \begin{bmatrix} R_1 & 1 & 0 \\ -1 & \frac{1}{R_2} & -\frac{1}{R_2} \\ 0 & -\frac{1}{R_2} & \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} u \\ (V_3)_1 \\ (V_3)_2 \end{bmatrix}$$

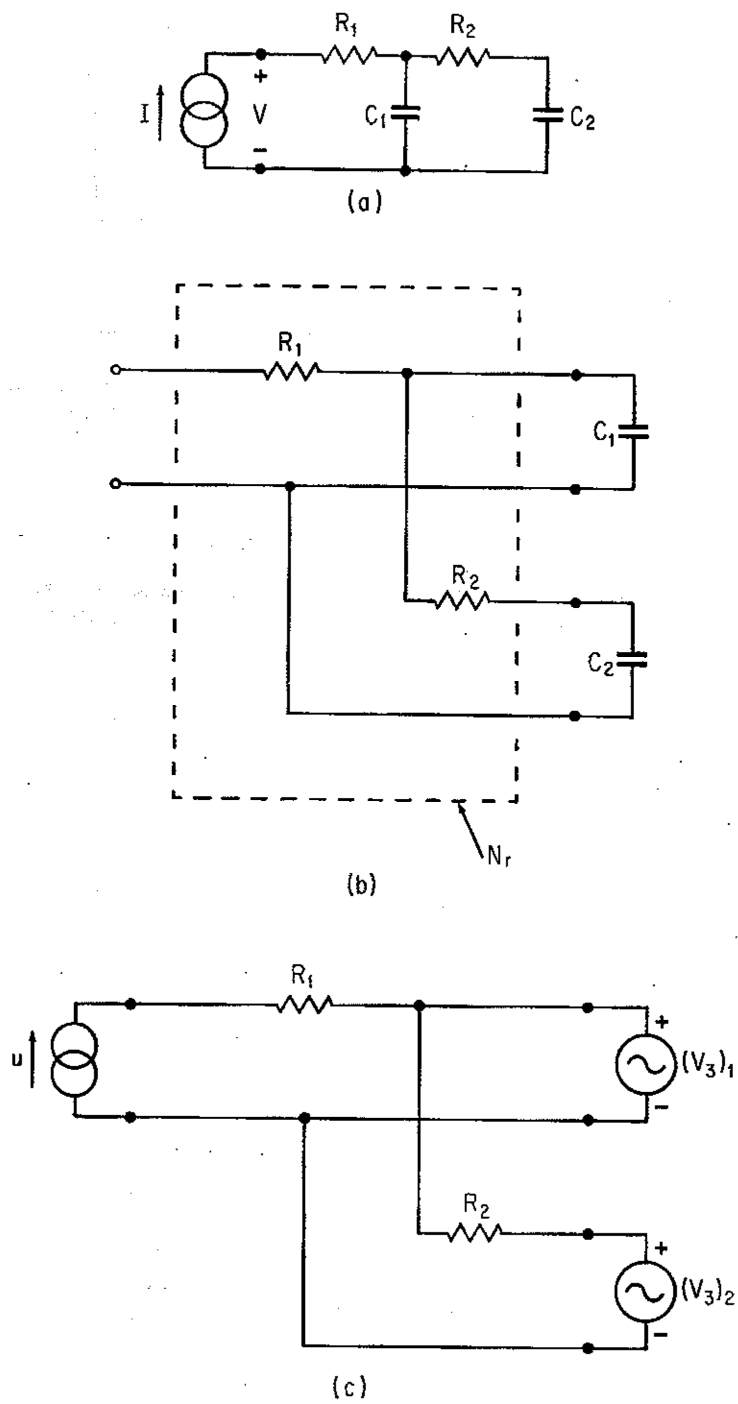
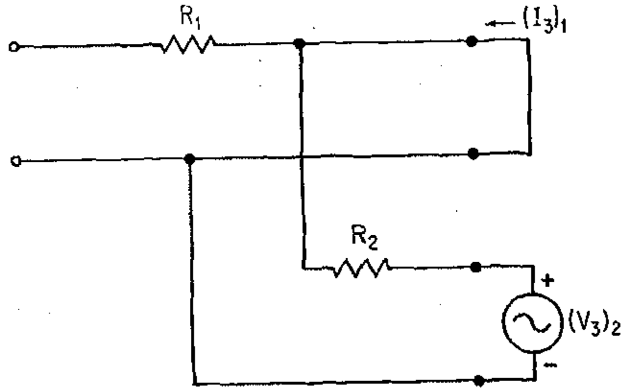


FIGURE 4.3.6. Network for Example 4.3.1.



(d)

FIGURE 4.3.6. (cont.)

To derive, for example, the 2–3 entry of this matrix, we replace u by an open circuit and $(V_3)_2$ by a short circuit (see Fig. 4.3.6d). We readily derive $(I_3)_1 = -(V_3)_2/R_2$.

In terms of the block matrices of (4.3.4), we have

$$M_{11} = R_1 \qquad M_{13} = [1 \quad 0] \qquad M_{31} = [-1 \quad 0]'$$

$$M_{33} = \begin{bmatrix} \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_2} \end{bmatrix}$$

with the remaining matrices evanescing. The matrix $\mathcal{C}$ is of course

$$\mathcal{C} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

and, applying the formulas of (4.3.10), we obtain

$$F = \begin{bmatrix} -\frac{1}{C_1R_2} & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix} \qquad G = \begin{bmatrix} \frac{1}{C_1} \\ 0 \end{bmatrix}$$

$$H' = [1 \quad 0] \qquad J = R_1$$

That is, the state-space equations are

$$\dot{x} = \begin{bmatrix} -\frac{1}{C_1R_2} & \frac{1}{C_1R_2} \\ \frac{1}{C_2R_2} & -\frac{1}{C_2R_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C_1} \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 0]x + R_1 u$$

We obtained the same result in the last section.

Example 4.3.2 We consider the circuit shown in Fig. 4.3.7a. The excitation u is a two-vector comprised of the voltage at the left-hand port and current at the right-hand port, while the response y consists, of course, of the input current at the left-hand port and the voltage at the right-hand port. The effect of reactance extraction is shown in Fig. 4.3.7b, and the excitation variables used in computing the hybrid matrix of N_r are shown in Fig. 4.3.7c. It is straightforward to obtain from Fig. 4.3.7c the equation

$$\begin{bmatrix} (\hat{y})_1 \\ (\hat{y})_2 \\ V_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ 0 & -1 & 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} V = (u)_1 \\ I = (u)_2 \\ I_2 \\ V_3 \end{bmatrix}$$

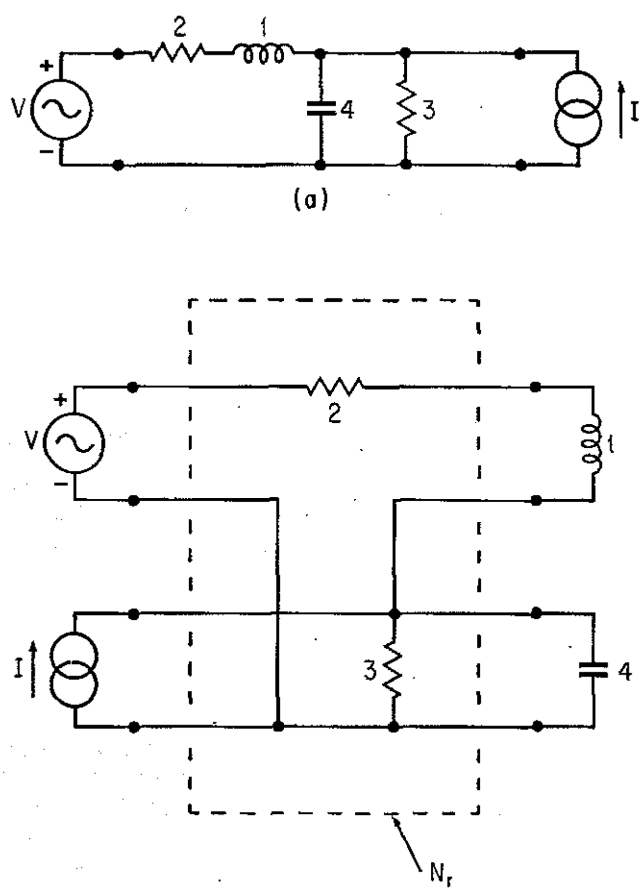


FIGURE 4.3.7 Network for Example 4.3.2.

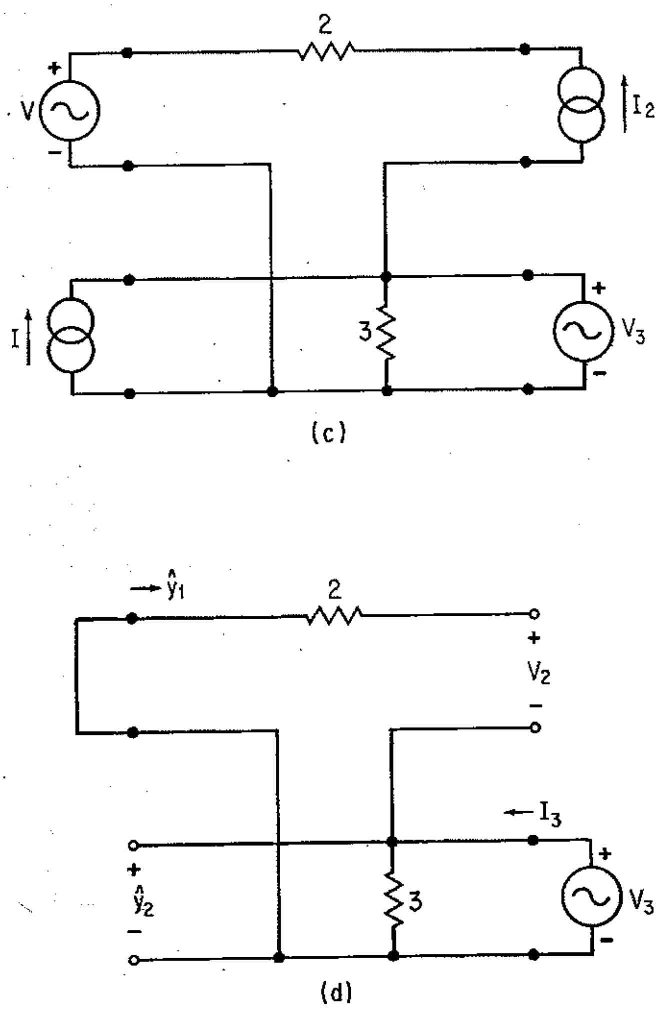


FIGURE 4.3.7 (cont.)

The last column of the hybrid matrix is computed, for example, by forcing $V = I = I_2 = 0$ (see Fig. 4.3.7d) and evaluating the resulting $(\hat{y})_1$, etc. In this case, the quantities M_{11}, M_{12}, etc., of Eq. (4.3.4) are

$$M_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad M_{12} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad M_{13} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$M_{21} = [1 \quad 0] \qquad M_{22} = [2] \qquad M_{23} = [-1]$$

$$M_{31} = [0 \quad -1] \qquad M_{32} = [1] \qquad M_{33} = [\tfrac{1}{3}]$$

The matrix $\mathcal{L}$ is simply [1] and $\mathcal{C}$ is simply [4]. Using (4.3.10), the state-space equations become

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -\frac{1}{4} & -\frac{1}{12} \end{bmatrix} x + \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x$$

Let us now consider some situations in which this procedure does not work.

Example 4.3.3 Consider the circuit of Fig. 4.3.8a. As we noted in the last section, this circuit is not describable by state-space equations of the form of (4.3.11). Since this is the only form of state-space equations the method of this section is capable of generating, we would expect that the method would fall down. Indeed, this is the case. Figure 4.3.8b shows the associated

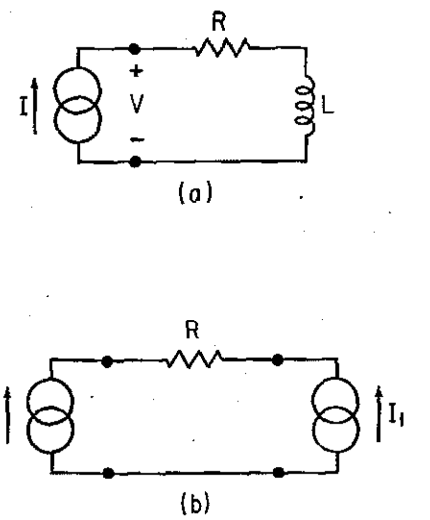


FIGURE 4.3.8. Network for Example 4.3.3. I and I_1 cannot be taken as Independent Excitations.

nondynamic network together with the sources needed to define the hybrid matrix. It is obvious from the figure that

$$I + I_1 = 0$$

which precludes the formation of the desired hybrid matrix. Existence of the hybrid matrix would imply that I and I_1 can be selected independently.

Example 4.3.4 Consider the network of Fig. 4.3.9a. Obviously, the natural way to analyze this network is to replace the two capacitors by a single one of value

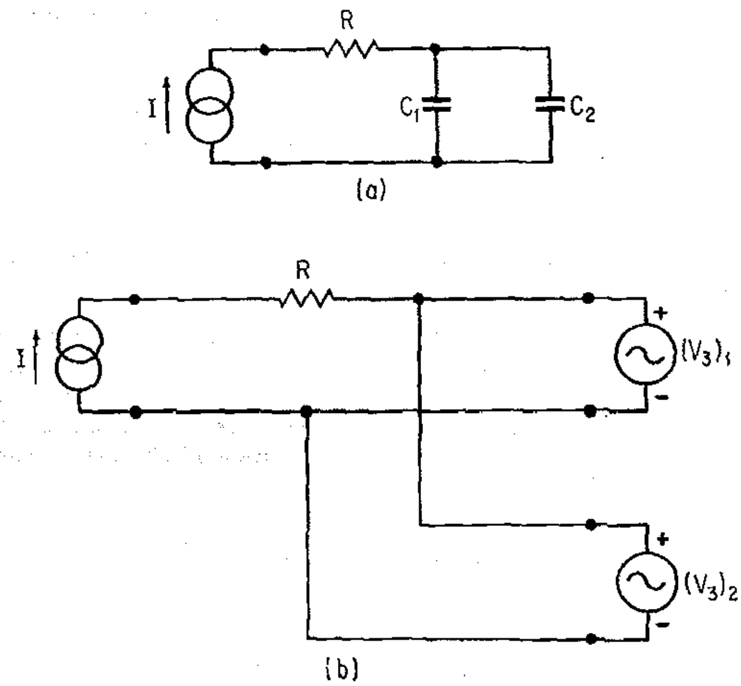


FIGURE 4.3.9. Network for Example 4.3.4. $(V_3)_1$ and $(V_3)_2$ are not Independent.

$C_1 + C_2$. Let us suppose however that we do *not* do this, and attempt to apply the technique of this section. The associated nondynamic network is shown in Fig. 4.3.9b, together with the excitation variables used for computing the hybrid matrix.

Evidently, the circuit forces $(V_3)_1 = (V_3)_2$, and so the hybrid matrix cannot be formed, for its existence would imply the ability to select $(V_3)_1$ and $(V_3)_2$ independently.

The difficulties described in Examples 4.3.3 and 4.3.4 and difficulties in related situations are resolvable using the techniques of the next section.

Problem 4.3.1 Find a homogeneous state equation for the circuit of Fig. 4.3.10 using the procedure of this section.

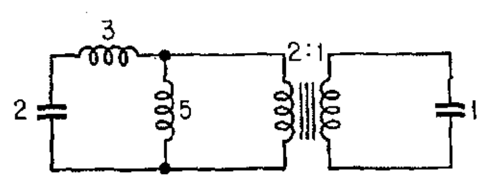


FIGURE 4.3.10. Network for Problem 4.3.1.

Problem 4.3.2 Find state-space equations for the circuit of Fig. 4.3.11 using the procedure of this section.

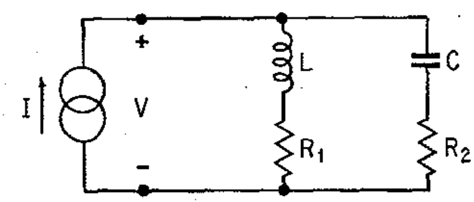


FIGURE 4.3.11. Network for Problem 4.3.2.

Problem 4.3.3 Any inductor of L henries is equivalent to a transformer of turns-ratio $\sqrt{L}:1$ terminated at its secondary port in a 1-H inductor. A similar statement holds for capacitors. This means that before carrying out a reactance extraction, one can convert all inductors to 1 H and capacitors to 1 F, absorbing the transformers into the nondynamic network. What are the state-space equations resulting from such a procedure, and what is the coordinate transformation linking the state variable in these equations with the state variable in the equations derived in the text?

Problem 4.3.4 Find state-space equations for the circuit of Fig. 4.3.12 using the procedure of this section. Compare the result with the worked example of the preceding section. (Notice that the input variable is V and the output variable I_R; an extension of the number of inputs and outputs is therefore required, as explained in the text.)

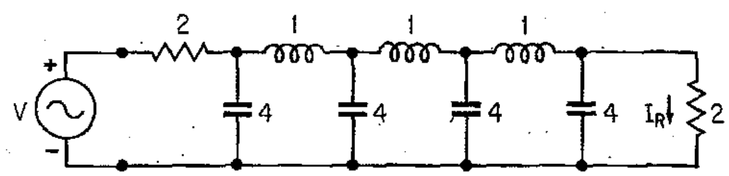


FIGURE 4.3.12. Network for Problem 4.3.4.

4.4 STATE-SPACE EQUATIONS VIA REACTANCE EXTRACTION—THE GENERAL CASE*

Our aim in this section is to present a procedure for generating state-space equations of an m-port network N that is free from the difficulties associated with the two approaches given hitherto. The procedure constitutes

*This section may be omitted at a first reading.

a variation of that given in Section 4.3. We retain the same assumption concerning the input u and output y appearing in the state-space equations: every port current and voltage must be an entry of either u or y, with the ith entry of u and of y corresponding to quantities associated with the ith port. Again, we carry out a reactance extraction to form a nondynamic network N_r from the prescribed network N. Now, however, we consider the situation in which that hybrid matrix of N_r cannot be formed which is associated with the particular set of responses and excitations described in Section 4.3. The technique to be used in this case is to form a new set of excitation and response variables such that the new hybrid matrix for N_r does exist. The passivity of N_r implies an important property of this hybrid matrix that enables construction of the state-space equations.

We shall break up our treatment in this section into several subsections, as follows:

1. Proof of some preliminary lemmas, using the passivity of the nondynamic network N_r.
2. Selection of the first m scalar excitation and response variables required to generate a hybrid matrix for N_r. Here, m is the number of ports of N.
3. Selection of the remaining n scalar excitation and response variables required to generate a hybrid matrix for N_r. Here, n is the number of reactive elements in N.
4. Proof of hybrid-matrix existence.
5. Construction of the state-space equations from the hybrid matrix.

Preliminary Lemmas

Lemma 4.4.1. Every m-port network possesses at least one hybrid matrix.*

For the proof of this lemma, see [14]. The lemma says that there exists at least one choice of excitation and response variable for the network for which a hybrid matrix may be defined, but it does not say what this choice is—indeed, choices that work in the sense of allowing definition of a hybrid matrix will vary from network to network.

We shall use Lemma 4.4.1 only to prove Lemma 4.4.3. Lemma 4.4.3 will be of direct use in developing the procedure for setting up state-space equations. Lemma 4.4.2 will be used both in proving Lemma 4.4.3 and in the course of the procedure for setting up state-space equations.

Lemma 4.4.2. Let $M(s)$ be the hybrid matrix of a multiport network (the usual conventions apply). Suppose that M is partitioned as

*Note: This lemma does not extend to networks that may contain active elements.

$$M(s) = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \tag{4.4.1}$$

Then if $M_{11}(s) \equiv 0$,

$$M_{12}(s) = -M'_{21}(-s) \tag{4.4.2}$$

Proof. The passivity property guarantees that on the $j\omega$ axis $M + M'^*$ is nonnegative definite or, with $M_{11} = 0$, that

$$\begin{bmatrix} 0 & M_{12} + M'^*_{21} \\ M_{21} + M'^*_{12} & M_{22} + M'^*_{22} \end{bmatrix} \geq 0$$

for all ω such that $j\omega$ is not a pole of any element of $M(s)$. Now suppose that (4.4.2) fails for s equal to some $j\omega_0$. Then $M_{12} + M'^*_{21}$ has a nonzero element, m say, that occurs in the ith row and the jth column of $M + M'^*$. Since $M + M'^*$ is nonnegative definite, the minor

$$\begin{bmatrix} (M + M'^*)_{ii} & (M + M'^*)_{ij} \\ (M + M'^*)_{ji} & (M + M'^*)_{jj} \end{bmatrix} = \begin{bmatrix} 0 & m \\ m^* & (M + M'^*)_{jj} \end{bmatrix} = -|m|^2$$

is nonnegative. Hence $m = 0$. Therefore, (4.4.2) holds for all $j\omega$ and therefore for all* s. ∇ ∇ ∇

Lemma 4.4.3. Let $\hat{N}$ be a $(1 + m_1 + m_2)$-port network. Suppose that the input impedance and Thévenin equivalent voltage at port 1 are zero whatever the *excitations* at ports $2, 3, \ldots, 1 + m_1$, and whatever the *terminations* at ports $2 + m_1, 3 + m_1, \ldots, 1 + m_1 + m_2$. Then the input impedance and Thévenin equivalent voltage at port 1 will be zero whatever the excitations at ports $2, 3, \ldots, 1 + m_1 + m_2$. Conversely, if the input impedance and Thévenin equivalent voltage at port 1 are zero for arbitrary excitations at ports $2, 3, \ldots, 1 + m_1 + m_2$, they will be zero for arbitrary terminations *or* excitations at these ports.

Before proving this lemma, we shall make several comments. First, the lemma has to do with conditions for a port to look like a short circuit and to be decoupled in an obvious sense from all other ports. Second, the lemma is saying that if a certain performance is observed for arbitrary terminations, it will be observed for arbitrary excitations, and conversely. Third, $m_1 = 0$

*Strictly speaking, we should exclude the finite set of s that are poles of elements of M_{12}.

or $m_2 = 0$ are permitted. If $m_1 = 0$, the lemma says that if zero input impedance is observed for arbitrary terminations on ports other than the first, the first port is essentially disconnected from the remainder. Fourth, the lemma need not hold if $\hat{N}$ is active. Thus suppose that $\hat{N}$ is a two port with an impedance matrix that is not positive real:

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

It is easily checked that the impedance seen at port 1 is zero irrespective of the value of a terminating impedance at port 2. But if port 2 is excited, a nonzero Thévenin voltage will be observed at port 1.

Proof of Lemma 4.4.3. By Lemma 4.4.1, $\hat{N}$ possesses a hybrid matrix, which we shall partition as

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \tag{4.4.3}$$

where M_{11} is 1×1, M_{22} is $m_1 \times m_1$, and M_{33} is $m_2 \times m_2$. Each M_{ij} in general will be a function of the complex variable s, but this fact is irrelevant here.

We do not require knowledge of whether voltage or current excitations are applied at ports beyond the first in defining M. However, we do need the fact that M can be defined assuming a current excitation at port 1. To see this, we argue as follows. By Lemma 4.4.1, there exists an M with either a voltage or current excitation assumed at port 1. Suppose that it is a voltage excitation. Set excitations at ports 2 through $1 + m_1$ to zero, and—by the use of short-circuit or open-circuit terminations, as appropriate—set all excitations at ports $2 + m_1$ through $1 + m_1 + m_2$ to zero. Then M_{11} will be the input admittance at port 1. By the lemma statement, $M_{11} = \infty$, which is not allowable. Hence M exists only with a current excitation assumed at port 1.

Suppose now that M in (4.4.3) is derived on the basis of there being a current excitation at port 1; then the above argument shows that $M_{11} = 0$. It follows by Lemma 4.4.2 that

$$M_{12} = -M_{21}'^{*} \qquad M_{13} = -M_{31}'^{*}$$

Since the Thévenin equivalent voltage at port 1 is zero irrespective of the excitations at ports 2 through $1 + m_1$, it follows that

$M_{12} = 0$. Thus M looks like

$$\begin{bmatrix} 0 & 0 & M_{13} \\ 0 & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

and we have

$$\begin{bmatrix} v_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & M_{13} \\ 0 & M_{22} & M_{23} \\ -M_{13}'^{*} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} i_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{4.4.4}$$

as the equation describing $\hat{N}$ (with E standing for excitation, R for response). Let us set $E_2 = 0$, and terminate ports $2 + m_1$ through $1 + m_1 + m_2$ so that

$$R_3 = -kE_3$$

for some arbitrary positive constant k. (This amounts to passive termination of each port in k ohms or k mhos.) Then (4.4.4) implies that

$$v_1 = M_{13}(kI + M_{33})^{-1}M_{13}'^{*}i_1$$

The inverse always exists because M_{33} is a passive hybrid matrix in its own right and kI is positive definite. By the lemma statement, the input impedance is zero, and so $M_{13} = 0$. Therefore, $\hat{N}$ is described by

$$\begin{bmatrix} v_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_{22} & M_{23} \\ 0 & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} i_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{4.4.5}$$

From this equation it is immediate that for arbitrary E_2 and E_3, the input impedance and Thévenin equivalent voltage at port 1 are zero; i.e., port 1 is internally short circuited and decoupled from the remainder of the network.

The converse contained in the lemma statement follows easily. We argue as before that M can only be defined with a current excitation at port 1 and that $M_{11} = 0$. That M_{12} and M_{13} are zero follows from the fact that arbitrary excitations at all ports lead to zero Thévenin voltage, and that M_{21} and M_{31} are zero follows from $M_{12} = -M_{21}'^{*}$ and $M_{13} = -M_{31}'^{*}$. Thus (4.4.5) is valid. It is immediate from this equation that replacement of any number of arbitrary excitations by arbitrary terminations will not affect the zero input impedance and zero Thévenin voltage properties. $\nabla\nabla\nabla$

The dual of Lemma 4.4.3 is easy to state; the proof of the dual lemma requires but simple variations on the proof of Lemma 4.4.3 and will not be stated.

Lemma 4.4.4. Let $\hat{N}$ be a $(1 + m_1 + m_2)$-port network. Suppose that the input admittance and Thévenin equivalent current at port 1 are zero whatever the *excitations* at ports $2, 3, \ldots, 1 + m_1$, and whatever the *terminations* at ports $2 + m_1, \ldots, 1 + m_1 + m_2$. Then the input admittance and Thévenin equivalent current are zero whatever the excitations at ports $2, 3, \ldots, 1 + m_1 + m_2$. Conversely, if the input admittance and Thévenin equivalent current at port 1 are zero for arbitrary excitations at ports $2, 3, \ldots, 1 + m_1 + m_2$, they will be zero for arbitrary excitations *or* terminations at these ports.

The final lemma we require is a generalization of Lemmas 4.4.3 and 4.4.4.

Lemma 4.4.5. Let $\hat{N}$ be a $(p + m_1 + m_2)$-port network, and let $w_1, w_2, \ldots, w_p$ be variables associated with ports 1 through p; each w_i can be either a current or voltage. Let $v = \sum_{i=1}^{p} \alpha_i w_i$ for some constants α_i. Then if the network $\hat{N}$ constrains* $v \equiv 0$ whatever the excitations at ports $p + 1, p + 2, \ldots, p + m_1$ and whatever the terminations at ports $p + m_1 + 1, p + m_1 + 2, \ldots, p + m_1 + m_2$, then $v \equiv 0$ for arbitrary excitations or terminations at these ports.

Proof. We reduce the problem to one permitting application of Lemma 4.4.3. From $\hat{N}$ we construct a network $\bar{N}$ as follows. If w_i is a current, we cascade a unit gyrator with port i of $\hat{N}$. For the network $\bar{N}$ comprising $\hat{N}$ with gyrators at some of its ports, every w_i will be a port voltage. Next, connect the secondary ports of a $1 \times p$ multiport transformer to the first p ports of the network $\bar{N}$, with the turns-ratio matrix of the transformer chosen so that the primary voltage v is $\sum \alpha_i w_i$. The network $\bar{N}$ with the

*The word constrain here is used roughly in the following sense. Some internal feature of the network forces a condition to hold, and if sources are connected in such a way as to apparently prevent the condition from holding, the responses will not be well defined. Examples of constraints are: a short circuit, which constrains a port voltage to be zero, and the constraint on the port voltages of a two-port transformer, expressible in terms of the turns ratio.

transformer connected will be denoted by $\tilde{N}$. Then $\tilde{N}$ is a $(1 + m_1 + m_2)$-port network such that the voltage at port 1 is constrained to be zero by $\hat{N}$, for arbitrary excitations at ports $2, \ldots, 1 + m_1$ *and arbitrary terminations at ports* $2 + m_1, 3 + m_1, \ldots, 1 + m_1 + m_2$. To say that the voltage is constrained to be zero is the same as to say that both Thévenin voltage and input impedance are zero under the stated conditions. By Lemma 4.4.3, it follows that $v \equiv 0$ for arbitrary excitations at all ports of $\tilde{N}$ past the first, and thus at all ports of $\hat{N}$ past the pth. The first part of Lemma 4.4.5 is therefore proved. The second part follows similarly.

$\nabla \nabla \nabla$

Now we return to the development of state-space equations for a prescribed m-port network N. The associated nondynamic network is N_r and is an $(m + n)$ port. The first order of business is to define excitation and response variables for N_r that will enable definition of a hybrid matrix for N_r.

Selection of the First m Excitation and Response Variables

Suppose that there are m network ports at which excitations *for the prescribed network* N are applied and responses are measured. As in the previous section, we suppose that if initially excitation and response variables are not both associated with each one of the m ports, then the set of excitation and response variables is expanded so that one of each sort of variable is associated with each port. Suppose that *before* the number of exciting variables is expanded to one at each port, and likewise for the number of response variables, there are m' ports at which exciting variables are present. By renumbering the ports if necessary, we may suppose that these m' ports are port 1, port 2, ..., port m'. The initially prescribed exciting variables are $u_1, u_2, \ldots, u_{m'}$. The quantities $y_{m'+1}, \ldots, y_m$ must all be initially prescribed as response variables, as well as possibly some of $y_1, \ldots, y_{m'}$. (If not, then one or more of ports $m' + 1$ through m would have initially prescribed neither an excitation nor a response variable, and we could therefore avoid consideration of this port.) Note also that, for the moment, the variables mentioned are associated with N, not the network N_r obtained from N by reactance extraction.

As earlier described, we expand the exciting variables of N to become $u_1, u_2, \ldots, u_m$ and the response variables to become $y_1, y_2, \ldots, y_m$. Thus if $y_{m'+1}$ is a current, it will be a current through a short-circuited port; in place of the short circuit, we place a voltage generator $u_{m'+1}$. Or, if $y_{m'+1}$ is a voltage, it will be a voltage at an open-circuited port, and at this port we connect a current generator $u_{m'+1}$. We proceed similarly for $y_{m'+2}, \ldots, y_m$.

Now we commence the selection process for the excitation and response variables for N_r. We shall denote excitation variables for N_r by $e_1, e_2, \ldots$, and response variables by $r_1, r_2, \ldots$, where each e_i and r_i is a scalar. *In general, we shall identify each e_i with one of $u_1, u_2, \ldots, u_m$ and each r_i with one of $y_1, y_2, \ldots, y_m$.* The reasoning suggesting these identifications and an indication of when the identifications cannot be made will occupy our attention for the remainder of this subsection.

The reader is warned that the subsequent arguments concerning the choice of excitation and response variables are lengthy and intricate; however, they are not deep conceptually.

We first choose

$$e_1 = u_1 \tag{4.4.6}$$

subject to one proviso. The network N_r must not be such that it constrains u_1 to be identically zero irrespective of the terminations (or, by Lemmas 4.4.3 and 4.4.4, excitations) at the ports of N_r other than the first port. If u_1 is a voltage, the constraint $u_1 \equiv 0$ irrespective of terminations on ports of N_r is equivalent to the input impedance at port 1 being zero, i.e., port 1 looking like a short circuit; if u_1 is a current, the constraint is equivalent to port 1 looking like an open circuit.* By Lemmas 4.4.3 and 4.4.4, the constraint would then apply irrespective of excitations at the remaining ports of N_r, and port 1 of N_r would be decoupled from the remaining ports, either as a short circuit or an open circuit. *In either case, we could never obtain a nonzero response at the port.*

In other words, the network N_r essentially constrains not just the permissible excitation to be zero, but also the achievable responses to be zero, irrespective of the excitations at other ports of N_r. In this case, the port is not worth worrying about for further calculations. We can forget about its presence, assigning neither excitation nor response variables to it for the purpose of computing a hybrid matrix of N_r. We note too *that the originally posed problem of finding state-space equations for N was ill posed in the sense that the exciting variable u_1 could not be freely chosen;* thus, to state-space equations derived for N on the basis of neglect of the offending port, we need to append the constraint equations $u_1 \equiv y_1 \equiv 0$.

As a notational convention, we shall assume that if now, or at any stage in the procedure, we agree to neglect a port, then we shall renumber the ports, advancing their numbering (and the numbering of their excitation and response variables) by 1. Thus if it turned out that $u_1 \equiv 0$, we would renumber port 2 as port 1, port 3 as port 2, ... Normally, port m' would be

*One could perhaps argue that if port 1 looks like an open circuit, one could allow u_1 to be a nonzero current if *infinite* responses are permitted. To avoid this sort of problem, we are adopting the convention that finite excitations that produce infinite responses are not permitted.

renumbered as port $m' - 1$ and port m as port $m - 1$. We shall, however, assume that the numbers m', m, and n are always redefined in any port renumbering so that m' is the number of exciting variables originally prescribed for N after deletions, m is the number of exciting variables in N after the initial expansion and after deletions, and n is the number of ports of N_r that are reactively terminated in constructing N, again after deletions.

With this convention it follows that (4.4.6) will always be used to assign the first exciting variable of N_r, though perhaps after a number of port deletions.

Next we consider u_1 and u_2 and ask the question: Is there a relation of the form

$$\alpha_1 e_1(\cdot) + \alpha_2 u_2(\cdot) \equiv 0 \qquad \alpha_2 \neq 0 \tag{4.4.7}$$

that holds for arbitrary $e_1(\cdot)$ independently of the terminations on all but the first two ports of N_r? Note that the existence of a relation like (4.4.7) can be checked by analysis of N_r. Note also that a relation like (4.4.7) for $\alpha_2 = 0$ has already been ruled out; for if $\alpha_2 = 0$, then $u_2(\cdot)$ is unconstrained, and we would have $u_1(\cdot)$ identically zero for arbitrary excitations or terminations at ports of N_r past the first.

Two possibilities exist in (4.4.7); either $\alpha_1 = 0$ or $\alpha_1 \neq 0$. In the former case, we have $u_2 \equiv 0$ for arbitrary excitation on port 1 and arbitrary terminations on ports of N_r other than the first and second. Lemmas 4.4.3 and 4.4.4 then allow us to conclude by an argument already noted that port 2 is either an open circuit or a short circuit, disconnected from the remainder of the network, and that $y_2 \equiv 0$ is the only possible response. In this case we delete port 2 from those under consideration. If $\alpha_1 \neq 0$, suppose that N_r is terminated so as to produce the original network N. Then for the original network N we would have Eq. (4.4.7) holding; this would imply that *for the original network N, u_1 and u_2 could not reasonably be taken as independent excitations, and the problem of generating state-space equations would be ill posed.* Rather than bothering with a technique to deal with this situation, let us, quite reasonably, disallow it. (Of course, if $u_2 \equiv 0$ is constrained, this too implies that the original problem is ill posed. This constraint is however slightly easier and perhaps worth taking the trouble to deal with, since it is probably more likely to occur than the second sort of ill-posed problem just noted.)

Thus, either (4.4.7) does not hold, or, after deletion of one or more ports, it does not hold. We take

$$e_2 = u_2 \tag{4.4.8}$$

and recognize that no relation of the form

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) \equiv 0 \tag{4.4.9}$$

is forced to hold for some α_1, α_2 (not both zero) and arbitrary *terminations* at ports past the first two. By Lemma 4.4.5, no relation of this form is forced to hold for arbitrary excitations at ports past the first two.*

Next, we consider u_3 and ask if a relation of the form

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \alpha_3 u_3(\cdot) \equiv 0 \qquad \alpha_3 \neq 0 \tag{4.4.10}$$

can hold for arbitrary e_1, e_2 and arbitrary terminations on ports past the first three. [Note that $\alpha_3 = 0$ is impossible, by the remarks associated with (4.4.9).] We conclude that either port 3 is a decoupled short circuit or open circuit, in which case we reject it from consideration; or that the problem of finding state-space equations for N is ill posed, and we disallow this; or that (4.4.10) does not hold. In this case, we set

$$e_3 = u_3 \tag{4.4.11}$$

secure in the knowledge that no relation of the form

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \alpha_3 e_3(\cdot) \equiv 0 \tag{4.4.12}$$

is forced to hold for some α_1, α_2, α_3, not all zero, irrespective of terminations (or, by Lemma 4.4.5, excitations) on ports past the first three.

Clearly, we can proceed in this manner and set

$$e_4 = u_4 \cdots e_{m'} = u_{m'} \tag{4.4.13}$$

with the obvious generalizations of (4.4.12) and the associated remarks.

If $m' = m$, we are through with this part of the procedure. But if not, we now consider $e_1, e_2, \ldots, e_{m'}$ and $u_{m'+1}$ and ask the question: Is there a relation of the form

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \cdots + \alpha_{m'+1} u_{m'+1}(\cdot) \equiv 0 \qquad \alpha_{m'+1} \neq 0 \tag{4.4.14}$$

that holds for arbitrary $e_1(\cdot), \ldots, e_{m'}(\cdot)$ independently of the terminations on all but the first $m' + 1$ ports of N_r? There are two answers to this question, depending whether $\alpha_1, \ldots, \alpha_{m'}$ are all zero or not. If $\alpha_1, \ldots, \alpha_{m'}$ are all zero, the equation becomes

$$u_{m'+1} \equiv 0 \tag{4.4.15}$$

and, by a familiar argument, we can dispense with further consideration of this port. So we now consider what happens when one or more of $\alpha_1, \ldots, \alpha_{m'}$ are nonzero. In this case, we can still argue that the problem of generating

*The fact that (4.4.9) is not forced to hold means that $y_1(\cdot)$ and $y_2(\cdot)$ are finite when $e_1(\cdot)$ and $e_2(\cdot)$ are finite, by our earlier convention.

state-space equations for N is ill posed. For from (4.4.14), it follows that $u_{m'+1}(\cdot)$ is uniquely determined by $u_1(\cdot), u_2(\cdot), \ldots, u_{m'}(\cdot)$. An *independent* generator, for the sake of argument a voltage generator if $u_{m'+1}$ is a voltage, cannot be connected at port $m' + 1$, since the voltage at this port is required to be

$$u_{m'+1}(\cdot) = -\frac{\alpha_1}{\alpha_{m'+1}}u_1(\cdot) - \cdots - \frac{\alpha_{m'}}{\alpha_{m'+1}}u_{m'}(\cdot) \tag{4.4.16}$$

Because of structural features of N_r, a voltage equal to the right-hand side of (4.4.16) must be developed at port $m' + 1$, irrespective of the excitations at ports 1 through m' and the terminations at ports after the first $m' + 1$.

If port $m' + 1$ of N_r is short circuited, we see from (4.4.16) that the relation

$$-\frac{\alpha_1}{\alpha_{m'+1}}u_1(\cdot) - \cdots - \frac{\alpha_{m'}}{\alpha_{m'+1}}u_{m'}(\cdot) = 0 \tag{4.4.17}$$

will hold with not all of $\alpha_1, \ldots, \alpha_{m'}$ nonzero. Equation (4.4.17) holds irrespective of the terminations on ports of N_r past the first $m' + 1$. Accordingly, suppose that the inductor and capacitor terminations are used that will yield the original network N. Equation (4.4.17) will then apply for N, under the conditions that port $m' + 1$ is short circuited, that excitations at ports past the first $m' + 1$ are arbitrary, and that $y_{m'+1}$ is a current.

Now recall that $y_{m'+1}$ is the first response variable for which $u_{m'+1}$ was not originally listed as an excitation variable for N; $u_1, \ldots, u_{m'}$ were the original excitation variables, $u_{m'+1}, \ldots, u_m$ were added ones. *It follows, as explained in more detail in the previous section, that the originally posed problem corresponds to forcing $u_{m'+1} = \cdots = u_m = 0$ in the problem with the added excitation variables.* Therefore, Eq. (4.4.17) holds when $u_1, \ldots, u_{m'}$ are the only excitation variables for N, since it holds under the conditions of $u_{m'+1} = 0$ and arbitrary $u_{m'+2}, \ldots, u_{m'}$. Since the equation says that the originally listed excitations of N, naturally desired independent, are not really independent, it means that *the originally posed problem of finding state-space equations for N relating prescribed excitations and responses is not well posed. Again, we shall disallow this situation,* on reasonable grounds.

Thus either (4.4.14) does not hold, or after deletion of one or more ports, it does not hold. We take

$$e_{m'+1} = u_{m'+1} \tag{4.4.18}$$

and we are guaranteed that there exist no constants $\alpha_1, \alpha_2, \ldots, \alpha_{m'+1}$, not all zero, such that the equation

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \cdots + \alpha_{m'+1}e_{m'+1}(\cdot) \equiv 0 \tag{4.4.19}$$

is forced to hold irrespective of the terminations or excitations at ports beyond the first $m' + 1$.

Next we ask if it is possible for a relation of the following form to hold:

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \cdots + \alpha_{m'+2} u_{m'+2}(\cdot) \equiv 0 \qquad \alpha_{m'+2} \neq 0 \tag{4.4.20}$$

The relation of course is supposed to hold for arbitrary $e_1(\cdot)$ through $e_{m'+1}(\cdot)$ independently of the terminations on all but the first $m' + 2$ ports of N_r.

Following an earlier argument, we can show that if any of $\alpha_1, \alpha_2, \ldots, \alpha_{m'}$ is nonzero, then the original problem of finding state-space equations for N with the initially given excitations $u_1, u_2, \ldots, u_{m'}$ is ill posed. Accordingly, we suppose that $\alpha_1 = \alpha_2 = \cdots = \alpha_{m'} = 0$. Also, if $\alpha_{m'+1} = 0$, it is easy to argue that both $u_{m'+2}$ and $y_{m'+2}$ must be identically zero, and port $m' + 2$ can be omitted from further consideration. Hence we are left with the possibility

$$\alpha_{m'+1} e_{m'+1}(\cdot) + \alpha_{m'+2} u_{m'+2}(\cdot) \equiv 0 \tag{4.4.21}$$

for nonzero $\alpha_{m'+1}$ and $\alpha_{m'+2}$. We can show that now port $m' + 1$ and port $m' + 2$ should *both* be neglected; the argument is as follows. Consider N, with excitations $u_1, u_2, \ldots, u_{m'}$ but with zero excitations applied at ports $m' + 1$, $m' + 3, \ldots, m$. (If there were also zero excitation applied at port $m' + 2$, this would correspond to the initially given situation before the list of excitation variables was expanded to include $u_{m'+1}, \ldots, u_m$.) Equation (4.4.21) implies that $u_{m'+2} \equiv 0$; i.e., if $u_{m'+2}$ is a voltage, the voltage at port $m' + 2$ of N (in fact N_r) is constrained to be zero if $u_{m'+1} = e_{m'+1}$ is zero—*irrespective of the presence of any external short circuit at port* $m' + 2$. The voltage $u_{m'+2}$ will be zero whatever termination is employed at port $m' + 2$, and therefore the current $y_{m'+2} = 0$. Likewise, if $u_{m'+2}$ is a current, the voltage $y_{m'+2}$ at port $m' + 2$ will be zero irrespective of the termination at port $m' + 2$, so long as $u_{m'+1}$ is identically zero.

Summing up, if N is terminated so as to force $u_{m'+1}, \ldots, u_m$ to equal zero, then $y_{m'+2}$ will equal zero.

Accordingly, we can drop port $m' + 2$ from further consideration. As far as N_r is concerned, no excitation or response will be prescribed for port $m' + 2$, and the port will not even be thought of as a port of N_r. Then, when state-space equations are derived for N, we can adjoin to them the equation $y_{m'+2} \equiv 0$.

The above argument is symmetric with respect to ports $m' + 1$ and $m' + 2$. It follows that when N is excited with only the *originally* specified set of variables, $y_{m'+1} \equiv 0$. Again, we drop port $m' + 1$ of N_r from further consideration* and simply adjoin the equation $y_{m'+1} \equiv 0$ to the state-space equations of N.

*It might be thought that one could drop either port $m' + 1$ or $m' + 2$ from consideration, but not both. However, the fact that N is excited with only the originally specified set of variables implies that $e_{m'+1} = 0$ and $e_{m'+2} = 0$, which imply $y_{m'+2} = 0$ and $y_{m'+1} = 0$, independently of all other excitations.

Conditions under which (4.4.20) can hold are, in summary,

1. When N is excited only by its originally specified excitation variable $u_1, u_2, \ldots, u_{m'}$, then $y_{m'+1} \equiv y_{m'+2} \equiv 0$ and ports $m'+1$ and $m'+2$ need be given no further consideration, or
2. The problem of finding state-space equations for N is ill posed.

We proceed as before, ruling out case 2, rejecting ports $m'+1$ and $m'+2$ in case 1, and renumbering the ports, but retaining the number m to denote the number of ports at which excitations are applied or responses measured. When this is done, we choose

$$e_{m'+2} = u_{m'+2} \tag{4.4.22}$$

In a similar manner, ruling out the possibility of the original problem being ill posed and rejecting from consideration ports of N at which the response will be identically zero (with consequential renumbering of the ports and redefinition of m), we take

$$e_{m'+3} = u_{m'+3} \cdots e_m = u_m \tag{4.4.23}$$

Of course, we also take

$$r_i = y_i \qquad i = 1, 2, \ldots, m \tag{4.4.24}$$

In addition, we know that $e_1, e_2, \ldots, e_m$ are independent in the following sense: *there exist no constants $\alpha_1, \ldots, \alpha_m$, not all zero, such that a relation of the form*

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \cdots + \alpha_m e_m(\cdot) \equiv 0 \tag{4.4.25}$$

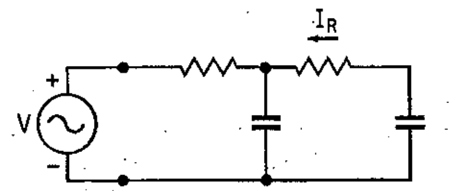
is forced to hold independently of the terminations or, by Lemma 4.4.5, the excitations at ports of N_r beyond the first m.

The rule for assigning excitation and response variables to the first m ports of N_r is simple, the only complication being that some ports are rejected from consideration. Excluding this complication, excitation variables for N_r are chosen to agree with the extended set of excitation variables $u_1, u_2, \ldots, u_m$ of N, and response variables are assigned in the obvious way.

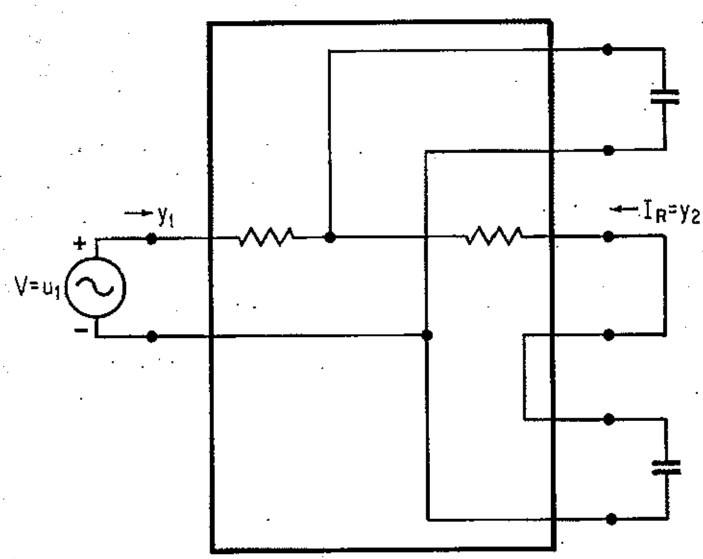
Example 4.4.1 Consider the network of Fig. 4.4.1a. The initially prescribed excitation variable is V, and the initially prescribed response variable is I_R. The circuit is redrawn in Fig. 4.4.1b to exhibit the effect of reactance extraction and the effect of assigning each response variable to a port.

We take first

$$e_1 = u_1$$



(a)



(b)

FIGURE 4.4.1. Network for Example 4.4.1.

Next, we introduce a voltage generator u_2 at the short-circuited port with current $I_R = y_2$. It is easy to check that u_1 and u_2 are independent, so we take

$$e_2 = u_2$$

Figure 4.4.2 shows a circuit for which the excitations u_1 and u_2 cannot be chosen independently. The problem of finding state-space equations for the circuit with these excitations is therefore ill posed.

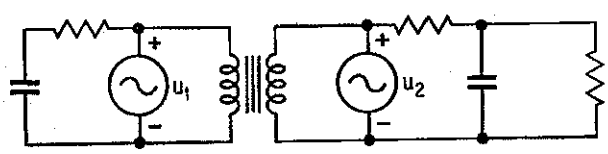


FIGURE 4.4.2. State-Space Equations Cannot Be Found with u_1 and u_2 as Excitations.

Selection of Remaining Excitation and Response Variables

The nondynamic network N_r resulting from reactance extraction will be assumed to have $m + n$ ports, with inductor and capacitor terminations at the last n ports yielding N. We assume that excitation variables $e_1, e_2, \ldots, e_m$ have been chosen for the first m ports of N_r to agree with the (extended) set of excitation variables of N, viz., $u_1, u_2, \ldots, u_m$. We shall now explain how to choose $e_{m+1}, e_{m+2}, \ldots$, etc.

In our procedure, we shall have occasion to renumber the ports and perhaps to eliminate some from consideration. If some are eliminated, we assume, as explained earlier, that n is redefined.

Selection of e_{m+1} begins by choosing any port from among the remaining n ports of N_r; suppose that in producing N from N_r this port is inductively terminated. Let $\hat{i}$ denote the current at this port and $\hat{v}$ the voltage. If there is no relation of the form

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_m e_m(\cdot) + \alpha_{m+1}\hat{i}(\cdot) \equiv 0 \qquad \alpha_{m+1} \neq 0 \tag{4.4.26}$$

that holds independently of $e_1, e_2, \ldots, e_m$ and of the terminations on ports other than the first m and the port under consideration, we select

$$e_{m+1} = \hat{i} \tag{4.4.27}$$

A similar procedure is followed in case the port is capacitively terminated. The quantities $\hat{i}$ and $\hat{v}$ are as before, but now we vary (4.4.26). If there is no relation of the form

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_m e_m(\cdot) + \alpha_{m+1}\hat{v}(\cdot) \equiv 0 \qquad \alpha_{m+1} \neq 0 \tag{4.4.28}$$

that holds independently of $e_1, e_2, \ldots, e_m$ and of the terminations on ports other than the first m and the port under consideration, we select

$$e_{m+1} = \hat{v} \tag{4.4.29}$$

Several points should be noted. First, in (4.4.27) and (4.4.29), e_{m+1} agrees with the excitation that we chose in the last section. Second, as the notation implies, in case (4.4.26) or (4.4.28) do not hold, the port under consideration becomes the $(m + 1)$th port. Third, in the event that e_{m+1} is selected as described above, we are assured that there do not exist constraints $\alpha_1, \alpha_2, \ldots, \alpha_{m+1}$, not all zero, for which a relation

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \cdots + \alpha_{m+1} e_{m+1}(\cdot) \equiv 0 \tag{4.4.30}$$

is forced to hold irrespective of the terminations or excitations on all but the first $m + 1$ ports of N_r. [Failure of (4.4.26) or (4.4.28) assures us of the nonexistence of the α_i if $\alpha_{m+1} \neq 0$. If $\alpha_{m+1} = 0$, e_{m+1} is free, and the termination on port $m + 1$ can be considered arbitrary. In that case, the material of the last subsection guarantees nonexistence of the α_i.]

Now let us consider what happens if (4.4.26) or (4.4.28) holds. For the moment, we shall leave the excitation variable of the port under consideration unassigned and select a different port from those to which excitation variables have not been assigned—always assuming that we have not exhausted the number of ports. Thus we obtain a new candidate for port $(m + 1)$. We treat it just like the first candidate; if in constructing N it is inductively terminated, we ask whether there is a relation like (4.4.26), and, if not, we assign e_{m+1} as the port current. Likewise, if in constructing N the port is capacitively terminated, we ask whether there is a relation like (4.4.28). If not, then we assign e_{m+1} as the port voltage.

If the equivalent of (4.4.26) or (4.4.28) is satisfied, then we select another candidate port from among those not yet considered and proceed as for the first two candidates.

Eventually, one of two things happens.

1. One candidate is successful, and the excitation e_{m+1} is assigned as a current in the case of a port terminated in an inductor when N is constructed, and a voltage in the contrary case.
2. No candidate is successful.

Let us leave case 2 for the moment, and assume that case 1 applies. Note that no relation of the form (4.4.30) is forced to hold for constants α_i, not all zero, and arbitrary terminations (or excitations) on the remaining ports. If case 1 applies, we again go through the candidate selection process and attempt to locate a port with the following properties. If the port is inductively terminated when N is constructed from N_r, there must not exist a relation of the form

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_{m+1} e_{m+1}(\cdot) + \alpha_{m+2}\hat{i}(\cdot) \equiv 0 \qquad \alpha_{m+2} \neq 0 \tag{4.4.31}$$

(where $\hat{i}$ is the port current) holding for all $e_1(\cdot)$ through $e_{m+1}(\cdot)$ independently of the terminations on ports other than ports 1 through $(m + 1)$ and the port under consideration. If (4.4.31) does not hold, we select

$$e_{m+2} = \hat{i} \tag{4.4.32}$$

If the candidate port is capacitively terminated in the construction of N, the property desired is that there exist no relation of the form

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_{m+1} e_{m+1}(\cdot) + \alpha_{m+2}\hat{v}(\cdot) \equiv 0 \qquad \alpha_{m+2} \neq 0 \tag{4.4.33}$$

(where $\hat{v}$ is the port voltage) holding for all $e_1(\cdot)$ through $e_{m+1}(\cdot)$ independently of the terminations on ports other than ports 1 through $(m + 1)$ and the port under consideration. If (4.4.33) does not hold, of course we select

$$e_{m+2} = \hat{v}(\cdot) \tag{4.4.34}$$

Assuming that we find a successful candidate, we number this port as $m + 2$. We note that e_{m+2} agrees with the excitation chosen in the last section, and that there cannot exist constants α_1 through α_{m+2}, not all zero, for which a relation of the form

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \cdots + \alpha_{m+2} e_{m+2}(\cdot) \equiv 0 \tag{4.4.35}$$

is forced to hold irrespective of terminations (or excitations) on ports beyond the first $m + 2$.

Obviously in this search procedure, one of two things again must happen:

1. One candidate is successful, and the excitation e_{m+2} is assigned appropriately.
2. No candidate is successful.

Assuming case 1 again, we could proceed to try to assign e_{m+3}, e_{m+4}, etc. *At some step we shall have assigned all possible ports, or it will be impossible to find a successful candidate.*

Now suppose that after combining together the first m ports to which excitation and response variables were assigned and the ports to which excitation and response variables are assigned by the procedures just described, a total of $l < m + n$ ports has assigned variables. [Thus there will exist no constants $\alpha_1, \alpha_2, \ldots, \alpha_l$, not all zero, such that

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_l e_l(\cdot) \equiv 0 \tag{4.4.36}$$

is forced to hold irrespective of the terminations or excitations at those $m + n - l$ ports for which no excitation variables have been specified.] Moreover, let $\hat{i}$ and $\hat{v}$ denote the current and voltage at any one of these remaining $m + n - l$ ports. If the port is inductively terminated in constructing N, there will exist constants $\alpha_1, \ldots, \alpha_{l+1}$ such that for arbitrary e_1 through e_l

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_l e_l(\cdot) + \alpha_{l+1}\hat{i}(\cdot) \equiv 0 \qquad \alpha_{l+1} \neq 0 \tag{4.4.37}$$

irrespective of the terminations or excitations at the other of the last $m + n - l$ ports. (If this were not the case, then this port would be a successful candidate in the sense already described.) Likewise, if the port is capacitively terminated in the construction of N, there will exist constants $\beta_1, \ldots, \beta_{l+1}$ such that for arbitrary e_1 through e_l

$$\beta_1 e_1(\cdot) + \cdots + \beta_l e_l(\cdot) + \beta_{l+1}\hat{v}(\cdot) \equiv 0 \qquad \beta_{l+1} \neq 0 \tag{4.4.38}$$

irrespective of the terminations or excitations at the other of the last $m + n - l$ ports.

Suppose for the sake of argument that (4.4.37) applies. We claim that one of two things can happen:

1. $\hat{i}(\cdot) \equiv \hat{v}(\cdot) \equiv 0$ for all $e_1(\cdot)$ through $e_l(\cdot)$ and all terminations on the remaining ports. In this case, we give no further consideration to the port and do not assign excitation and response variables to it.
2. Equation (4.4.38) will *not* hold. In this case, we assign

$$e_{l+1} = \hat{v} \tag{4.4.39}$$

To establish the claim, let us suppose that (4.4.37) and (4.4.38) do hold simultaneously; we shall prove that $\hat{i} \equiv \hat{v} \equiv 0$. Suppose that the port is terminated in a resistance R, forcing $\hat{v} = -R\hat{i}$. Then we must have

$$\frac{1}{\beta_{l+1}}[\beta_1 e_1(\cdot) + \cdots + \beta_l e_l(\cdot)] = -\frac{R}{\alpha_{l+1}}[\alpha_1 e_1(\cdot) + \cdots + \alpha_l e_l(\cdot)] \tag{4.4.40}$$

This equation holds for all $e_1(\cdot)$ through $e_l(\cdot)$ and all terminations on ports other than the first l and that port terminated in R; the equation also holds for all R, which implies that

$$\beta_1 e_1(\cdot) + \cdots + \beta_l e_l(\cdot) = \alpha_1 e_1(\cdot) + \cdots + \alpha_l e_l(\cdot) = 0 \tag{4.4.41}$$

This equation holds for all $e_1(\cdot)$ through $e_l(\cdot)$, all terminations on ports other than the first l and that port terminated in R, and for all resistive (and therefore all) terminations on the port terminated in R; i.e., the equation holds for all terminations on ports other than the first l. This is a contradiction unless $\beta_i = \alpha_j = 0$ for all i, j [see (4.4.36) and associated remarks]. Thus (4.4.37) and (4.4.38) will only hold simultaneously if

$$\hat{i}(\cdot) \equiv 0 \qquad \hat{v}(\cdot) \equiv 0 \tag{4.4.42}$$

irrespective of $e_1(\cdot)$ through $e_l(\cdot)$ and port terminations on other than the first l ports and the port under consideration. The claim is therefore established.

If (4.4.42) holds, the reason for rejecting consideration of the port with these constraints is obvious. Notice that the presence of a terminating inductor at the port when N is constructed is irrelevant; if the inductor were removed, or replaced by any other component, the relation between the input and output variables of N would be unchanged.

The above arguments have been based on an assumption that the port under consideration is inductively terminated in the construction of N. Obviously, the only variation in the case of capacitor termination is to say that either (4.4.42) holds (in which case the port is rejected from consideration) or (4.4.38) holds but (4.4.37) does not hold, and we set

$$e_{l+1} = \hat{i} \tag{4.4.43}$$

Clearly, if not all ports are rejected through condition (4.4.42), we will assign $e_{l+1}(\cdot)$. Further, since if $e_{l+1} = \hat{v}$, Eq. (4.4.38) does not hold, and if $e_{l+1} = \hat{i}$, Eq. (4.4.37) does not hold, it follows that there do not exist constants $\alpha_1, \alpha_2, \ldots, \alpha_{l+1}$, not all zero, such that

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_{l+1} e_{l+1}(\cdot) \equiv 0 \tag{4.4.44}$$

[Of course, (4.4.37) only guarantees nonexistence with $\alpha_{l+1} \neq 0$. But the nonexistence of constants $\alpha_1, \ldots, \alpha_l$ satisfying (4.4.36) for arbitrary excitation at port $l + 1$ means that (4.4.44) cannot hold if $\alpha_{l+1} = 0$.]

Next we must select e_{l+2}. The procedure is much the same. Select a port from those for which no excitation variables have been assigned, and suppose for the sake of argument that it is inductively terminated when N is constructed from N_r. Suppose that the port current is $\hat{i}$ and port voltage $\hat{v}$. There exist constants $\alpha_1, \ldots, \alpha_{l+2}$ such that a relation of the form

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_{l+1} e_{l+1}(\cdot) + \alpha_{l+2}\hat{i}(\cdot) \equiv 0 \qquad \alpha_{l+2} \neq 0 \tag{4.4.45}$$

is forced to hold for arbitrary terminations on ports other than the first $l + 1$ and the port under consideration. [Actually, $\alpha_{l+1} = 0$ in (4.4.45), for the port under consideration must have failed to be a "successful candidate" for port $l + 1$, and the way it would have failed would have been through (4.4.45) holding without the $\alpha_{l+1}e_{l+1}(\cdot)$ term.] One of two possibilities must be true:

1. $\hat{i}(\cdot) = \hat{v}(\cdot) \equiv 0$ for all $e_1(\cdot)$ through $e_{l+1}(\cdot)$ and all terminations on the remaining ports. In this case, we give no further consideration to the port and do not assign excitation and response variables to it.
2. There do not exist constants $\beta_1, \beta_2, \ldots, \beta_{l+2}$, not all zero, such that a relation of the form

$$\beta_1 e_1(\cdot) + \cdots + \beta_{l+1} e_{l+1}(\cdot) + \beta_{l+2}\hat{v}(\cdot) \equiv 0 \tag{4.4.46}$$

is forced to hold irrespective of the terminations or excitations on ports other than the first $l+1$ and the port under consideration. In this case, we take

$$e_{l+2} = \hat{v} \tag{4.4.47}$$

The argument establishing this claim is the same as the argument establishing the claim in respect of the selection of an excitation variable for port $l+1$.

Of course, if the port under consideration is capacitively terminated in forming N, either possibility 1 above holds or we can take

$$e_{l+2} = \hat{i} \tag{4.4.48}$$

For either sort of termination, so long as possibility 1 does not hold, it is guaranteed that there do not exist constants $\alpha_1, \alpha_2, \ldots, \alpha_{l+2}$ such that the relation

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \cdots + \alpha_{l+2} e_{l+2}(\cdot) \equiv 0 \tag{4.4.49}$$

is forced to hold independently of the terminations or excitations on ports past the first $l+2$.

Clearly, we can continue in this fashion and assign response and excitation variables to all the remaining ports (other than those rejected from consideration).

Notice that the first $l-m$ of the last n ports of N_r, i.e., those terminated in reactances to yield N, will have excitation variables agreeing with those chosen in the last section—in fact they will have "natural" excitation variables. The last $m+n-l$ ports will have the opposite excitation variables to those which the last section suggested should have been chosen.

Notice also that *the response variable r_{l+i}, $1 \leq i \leq m+n-l$, of any of the last $m+n-l$ ports would have been the natural excitation variable but for the fact that after the selection of $e_{m+1}, \ldots, e_l$, we were faced with equations of the form*

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_l e_l(\cdot) + \alpha_{l+i} r_{l+i}(\cdot) \equiv 0 \qquad \alpha_{l+i} \neq 0 \tag{4.4.50}$$

[Equations (4.4.37) and (4.4.45), where in the latter equation $\alpha_{l+1} = 0$ as we noted, are examples of this equation.] Equation (4.4.50) of course holds for all terminations and therefore all excitations at ports other than ports 1 through l and port $l+i$, so it still holds after the selection of $e_{l+1}, e_{l+2}, \ldots, e_{m+n}$.

The last point to note (which is the whole point of the preceding development) is that $e_1(\cdot)$ through $e_{m+n}(\cdot)$ have the property that there do not exist constants $\alpha_1, \alpha_2, \ldots, \alpha_{m+n}$, not all zero, such that the relation

$$\alpha_1 e_1(\cdot) + \alpha_2 e_2(\cdot) + \cdots + \alpha_{m+n} e_{m+n}(\cdot) \equiv 0 \tag{4.4.51}$$

is forced to hold. In other words, *$e_1(\cdot)$ through $e_{m+n}(\cdot)$ may always be arbitrary and independently assigned.*

Example 4.4.2 Consider the circuit of Fig. 4.4.3a with excitation V and response I. This is redrawn to exhibit a reactance extraction in Fig. 4.4.3b, and port variables for the associated nondynamic network are shown in Fig. 4.4.3c.

We would first assign $e_1 = V$, then $e_2 = V_2$ (this being the desirable excitation, as a capacitor termination is used to form the circuit of Fig. 4.4.3a). Next we would take $e_3 = V_3$ (this also being the desirable excitation). We would like to take $e_4 = V_4$, but we observe the relation

$$e_3 - V_4 \equiv 0$$

which holds independently of e_1, e_2, and e_3. Therefore, we must take

$$e_4 = I_4$$

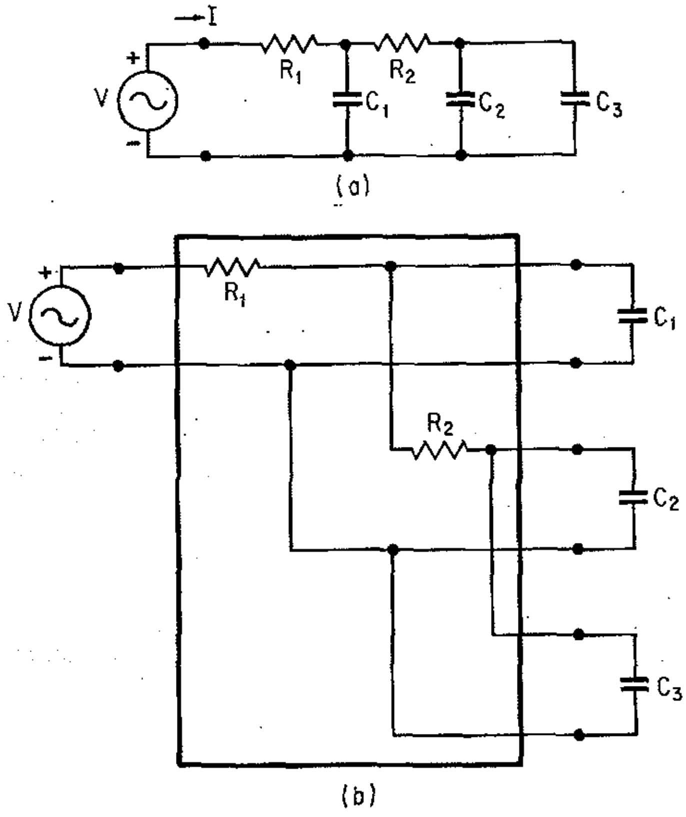
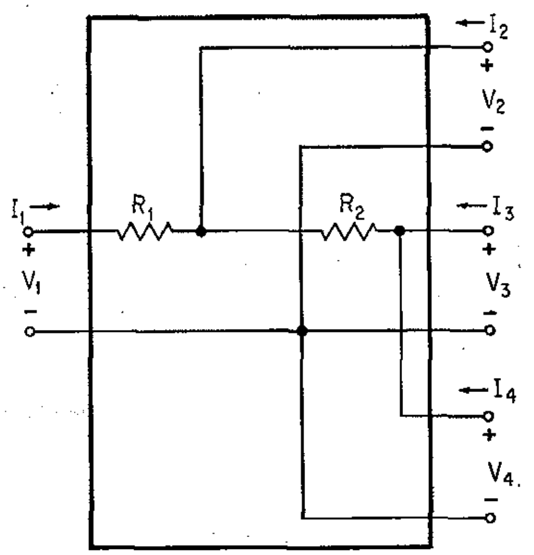


FIGURE 4.4.3. Network for Example 4.4.2.



(c)

FIGURE 4.4.3. (cont.)

Notice that there is no constraint relation of the form

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 \equiv 0$$

Hybrid-Matrix Existence

We have so far shown how to select excitation variables $e_1(\cdot)$ through $e_{m+n}(\cdot)$ and corresponding response variables $r_1(\cdot)$ through $r_{m+n}(\cdot)$ for the nondynamic network N_r. We claim that with the choice of variables made, a hybrid matrix M exists for N_r. The argument establishing existence is based on the nonexistence of constants $\alpha_1, \alpha_2, \ldots, \alpha_{m+n}$ such that a relation of the form

$$\alpha_1 e_1(\cdot) + \cdots + \alpha_{m+n} e_{m+n}(\cdot) \equiv 0 \qquad (4.4.51)$$

is forced to hold. This nonexistence implies that arbitrary choice of (finite) $e_1(\cdot)$ through $e_{m+n}(\cdot)$ lead to finite $r_1(\cdot)$ through $r_{m+n}(\cdot)$. It is then clear how to define the hybrid matrix M; we take

$$m_{ij} = \frac{r_i(\cdot)}{e_j(\cdot)} \qquad \text{with } e_k = 0,\ k \neq j \qquad (4.4.52)$$

and m_{ij} is guaranteed to be finite; i.e., the hybrid matrix exists.

Example 4.4.3 Consider the network discussed in Example 4.4.2 and depicted in Fig. 4.4.3. In Example 4.4.2 we identified

$$e_1 = V_1 \qquad e_2 = V_2 \qquad e_3 = V_3 \qquad e_4 = I_4$$

Obviously

$$r_1 = I_1 \qquad r_2 = I_2 \qquad r_3 = I_3 \qquad r_4 = V_4$$

and it is easy to verify that

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} R_1^{-1} & -R_1^{-1} & 0 & 0 \\ -R_1^{-1} & R_1^{-1} + R_2^{-1} & -R_2^{-1} & 0 \\ 0 & -R_2^{-1} & R_2^{-1} & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ I_4 \end{bmatrix}$$

Key Property of the Hybrid Matrix

Now we prove a property of the hybrid matrix that is essential in establishing the state equations. We make use of the passivity of the nondynamic network N_r in proving the property.

We shall define a partitioning of the hybrid matrix in a fairly natural way. (This partitioning will differ from a more complex one to be used in setting up the state-space equations.) The hybrid matrix is $(m + n) \times (m + n)$; roughly speaking, the first l excitation variables are desirable, and the last $m + n - l$ are not. Let us partition the hybrid matrix M as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{4.4.53}$$

where M_{11} is $l \times l$, and write

$$\begin{bmatrix} R_d \\ R_u \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} E_d \\ E_u \end{bmatrix}$$

where E_d is the vector $(e_1, e_2, \ldots, e_l)'$, etc. From (4.4.50) we see that each entry of R_u is a linear combination only of entries of E_d; i.e.,

$$M_{22} = 0 \tag{4.4.54}$$

Since M is the hybrid matrix of a passive network, it follows by Lemma 4.4.2 that

$$M_{12} = -M_{21}' \tag{4.4.55}$$

Notice that (4.4.54) and (4.4.55) hold for Example 4.4.3.

Generation of State-Space Equations from the Hybrid Matrix

Finally, we are in sight of our goal. To set up the state-space equations, a somewhat complicated partitioning of the hybrid matrix M is required. This is best described in terms of the partitioning of the excitation vector $(e_1, e_2, \ldots, e_{m+n})'$, the end result of the partitioning being illustrated in Fig. 4.4.4. The first m entries of this vector are, and will now be denoted

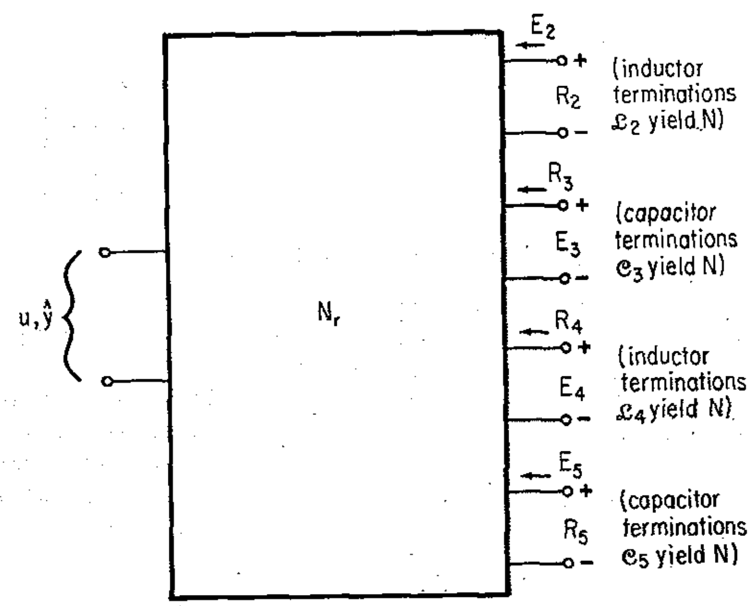


FIGURE 4.4.4. Selection of Excitation Variables Prior to Generating State-Space Equations.

by, u, where u is the vector of (the extended set of) inputs of N. It will be recalled that the next $l - m$ entries correspond to "desirable" selections of excitations, so that in the case of an inductively terminated port of N_r, the excitation is a current. Without loss of generality, suppose that the $l - m$ ports under consideration are renumbered so that all those ports that are inductively terminated in constructing N from N_r occur first, i.e., as ports $m + 1, m + 2, \ldots$, and that those ports which are capacitively terminated in constructing N from N_r occur second, i.e., as ports $\ldots l - 1, l$. The excitation variables for the first group of ports will be denoted by the vector E_2 and for the second group by E_3. Note that E_2 is a vector of currents and E_3 a vector of voltages.

There remain $m + n - l$ ports—those where the excitations were "undesirable" selections. Without loss of generality, suppose that the ports are renumbered so that all those ports inductively terminated in constructing N from N_r occur first, and those capacitively terminated occur second. The excitation variables for the first group of ports will be denoted by the vector E_4 and for the second group by E_5. Note that E_4 is a vector of voltages and E_5 a vector of currents.

Suppose that the response vectors corresponding to u and E_2 through E_5 are $\hat{y}$ and R_2 through R_5. The hybrid matrix M is partitioned conformably with e and r, so that

$$\begin{bmatrix} \hat{y} \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} \\ -M'_{14} & -M'_{24} & -M'_{34} & 0 & 0 \\ -M'_{15} & -M'_{25} & -M'_{35} & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{bmatrix} \tag{4.4.56}$$

Notice that we have used the structural property of the hybrid matrix M proved in the last subsection and summarized in (4.4.54) and (4.4.55). Note, however, that the partitioning of M applying in these equations differs from the new partitioning of M.

Now we define $\mathcal{L}_2$, $\mathcal{C}_3$, $\mathcal{L}_4$, and $\mathcal{C}_5$ as diagonal matrices whose diagonal entries are the values, respectively, of the inductors in N terminating the ports of N_r with excitation E_2, the capacitors in N terminating the ports of N_r with excitation E_3, the inductors in N terminating the ports of N_r with excitation E_4, and the capacitors in N terminating the ports of N_r with excitation E_5.

When N is formed from N_r, the following constraints on the port voltages and currents of N_r are forced. (Note the sign conventions shown in Fig. 4.4.4.)

$$R_2 = -\mathcal{L}_2\dot{E}_2 \qquad R_3 = -\mathcal{C}_3\dot{E}_3 \qquad E_4 = -\mathcal{L}_4\dot{R}_4 \qquad E_5 = -\mathcal{C}_5\dot{R}_5 \tag{4.4.57}$$

When these are combined with (4.4.56) and $\hat{y}$ is identified with y, the output of N, we have

$$\begin{bmatrix} y \\ \mathcal{L}_2\dot{E}_2 \\ \mathcal{C}_3\dot{E}_3 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ -M_{21} & -M_{22} & -M_{23} \\ -M_{31} & -M_{32} & -M_{33} \end{bmatrix} \begin{bmatrix} u \\ E_2 \\ E_3 \end{bmatrix} + \begin{bmatrix} M_{14} & M_{15} \\ -M_{24} & -M_{25} \\ -M_{34} & -M_{35} \end{bmatrix} \begin{bmatrix} -\mathcal{L}_4 & 0 \\ 0 & -\mathcal{C}_5 \end{bmatrix} \begin{bmatrix} \dot{R}_4 \\ \dot{R}_5 \end{bmatrix} \tag{4.4.58}$$

and

$$\begin{bmatrix} R_4 \\ R_5 \end{bmatrix} = -\begin{bmatrix} M_{14}' & M_{24}' & M_{34}' \\ M_{15}' & M_{25}' & M_{35}' \end{bmatrix} \begin{bmatrix} u \\ E_2 \\ E_3 \end{bmatrix} \tag{4.4.59}$$

Using (4.4.59) in (4.4.58), it follows that

$$\begin{aligned} &\left\{ \begin{bmatrix} \mathcal{L}_2 & 0 \\ 0 & \mathcal{C}_3 \end{bmatrix} + \begin{bmatrix} M_{24} & M_{25} \\ M_{34} & M_{35} \end{bmatrix} \begin{bmatrix} \mathcal{L}_4 & 0 \\ 0 & \mathcal{C}_5 \end{bmatrix} \begin{bmatrix} M_{24} & M_{25} \\ M_{34} & M_{35} \end{bmatrix}' \right\} \begin{bmatrix} \dot{E}_2 \\ \dot{E}_3 \end{bmatrix} \\ &= \begin{bmatrix} -M_{22} & -M_{23} \\ -M_{32} & -M_{33} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} + \begin{bmatrix} -M_{21} \\ -M_{31} \end{bmatrix} u \\ &\quad + \begin{bmatrix} M_{24} & M_{25} \\ M_{34} & M_{35} \end{bmatrix} \begin{bmatrix} \mathcal{L}_4 & 0 \\ 0 & \mathcal{C}_5 \end{bmatrix} \begin{bmatrix} -M_{14}' \\ -M_{15}' \end{bmatrix} \dot{u} \end{aligned} \tag{4.4.60}$$

and

$$\begin{aligned} y &= [M_{12} \quad M_{13}] \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} + M_{11} u \\ &\quad + [M_{14} \quad M_{15}] \begin{bmatrix} \mathcal{L}_4 & 0 \\ 0 & \mathcal{C}_5 \end{bmatrix} \begin{bmatrix} M_{14}' \\ M_{15}' \end{bmatrix} \dot{u} \\ &\quad + [M_{14} \quad M_{15}] \begin{bmatrix} \mathcal{L}_4 & 0 \\ 0 & \mathcal{C}_5 \end{bmatrix} \begin{bmatrix} M_{24} & M_{25} \\ M_{34} & M_{35} \end{bmatrix}' \begin{bmatrix} \dot{E}_2 \\ \dot{E}_3 \end{bmatrix} \end{aligned} \tag{4.4.61}$$

Equations (4.4.60) and (4.4.61) are almost, but not quite, the final state equations. We wish to note several points concerning these equations.

First, the coefficient matrix of $[\dot{E}_2' \quad \dot{E}_3']'$ in (4.4.60) is of the form $A + BCB'$, where A and C are positive definite. Therefore, this matrix is invertible, and we can solve (4.4.60) for $[\dot{E}_2' \quad \dot{E}_3']'$.

Second, the input u enters into (4.4.60) both directly and in differentiated form. Equation (4.4.60) is not therefore the usual form of the state-space equation, even if solved for $[\dot{E}_2' \quad \dot{E}_3']'$.

Third, the entries of E_2 are inductor currents and the entries of E_3 are capacitor voltages in N. Also, the entries of R_4 are inductor currents and those of R_5 are capacitor voltages, though R_4 and R_5 do not appear explicitly in (4.4.60). We see from (4.4.59), though, that R_4 and R_5 are linear combinations of u, E_2, and E_3.

Fourth, (4.4.61) shows that, in general, y depends linearly on E_2, E_3, u, and $\dot{u}$. [From (4.4.60) it follows that the apparent dependence of y on $\dot{E}_2$ and $\dot{E}_3$ in (4.4.61) can be eliminated, since $\dot{E}_2$ and $\dot{E}_3$ depend on E_2, E_3, u, and $\dot{u}$.]

To simplify notation, let us now rewrite (4.4.60) in the form

$$D_1\dot{x} = D_2x + D_3u - D_4\dot{D}_5\dot{u} \tag{4.4.62}$$

where the definition of all quantities is obvious, save D_5, which is $[M_{14} \quad M_{15}]'$. We rewrite Eq. (4.4.61) in the form

$$y = D_6x + D_7u + D_5'D_8D_5\dot{u} + D_5'D_4'\dot{x} \tag{4.4.63}$$

Now define a new variable

$$\hat{x} = x + D_1^{-1}D_4D_5u \tag{4.4.64}$$

The reason for introducing this variable is to eliminate the $\dot{u}$ terms in (4.4.62). Of course, $\hat{x}$ is the new state variable. When (4.4.64) is substituted into (4.4.62) and (4.4.63), there results

$$\begin{aligned} \dot{\hat{x}} &= D_1^{-1}D_2\hat{x} + D_1^{-1}(D_3 - D_2D_1^{-1}D_4D_5)u \\ y &= (D_6 + D_5'D_4'D_1^{-1}D_2)\hat{x} \\ &\quad + [D_7 - D_6D_1^{-1}D_4D_5 + D_5'D_4'D_1^{-1}(D_3 - D_2D_1^{-1}D_4D_5)]u \\ &\quad + (D_5'D_8D_5 - D_5'D_4'D_1^{-1}D_4D_5)\dot{u} \end{aligned} \tag{4.4.65}$$

Equations (4.4.65) constitute a set of state-space equations for N. As noted in an earlier section, a term involving $\dot{u}$ in the second of (4.4.65) may be inescapable.

Example 4.4.4 Consider the circuit of Fig. 4.4.5a with I as the input and V as the output variable. A reactance extraction is exhibited in Fig. 4.4.5b. The exciting variables for the hybrid matrix of the nondynamic network are chosen as follows. First, $e_1 = I$. This is obvious. Second, $e_2 = V_2$. Next, we might try $e_3 = I_3$, the inductor current, but then we would have $e_1 + e_3 \equiv 0$. Also, we might try $e_3 = V_4$, but then we would have $e_2 - e_3 \equiv 0$. So we take $e_3 = V_3$, an "undesirable" selection since port 3 is terminated in an inductor in constructing the circuit of Fig. 4.4.5a. Also, we make the "undesirable" selection $e_4 = I_4$.

The hybrid matrix is readily found, and

$$\begin{bmatrix} V \\ I_2 \\ I_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & \frac{1}{3} & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I \\ V_2 \\ V_3 \\ I_4 \end{bmatrix}$$

As required, the bottom right 2×2 submatrix is zero, and the lower left and upper right 2×2 submatrix are the negative transpose of each other.

The appropriate version of Eq. (4.4.58) is derived by setting $V_3 = -2\dot{I}_3$, $I_2 = -\dot{V}_2$, and $I_4 = -\dot{V}_4$. The result is

$$\begin{bmatrix} V \\ \dot{V}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} I \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{I}_3 \\ \dot{V}_4 \end{bmatrix}$$

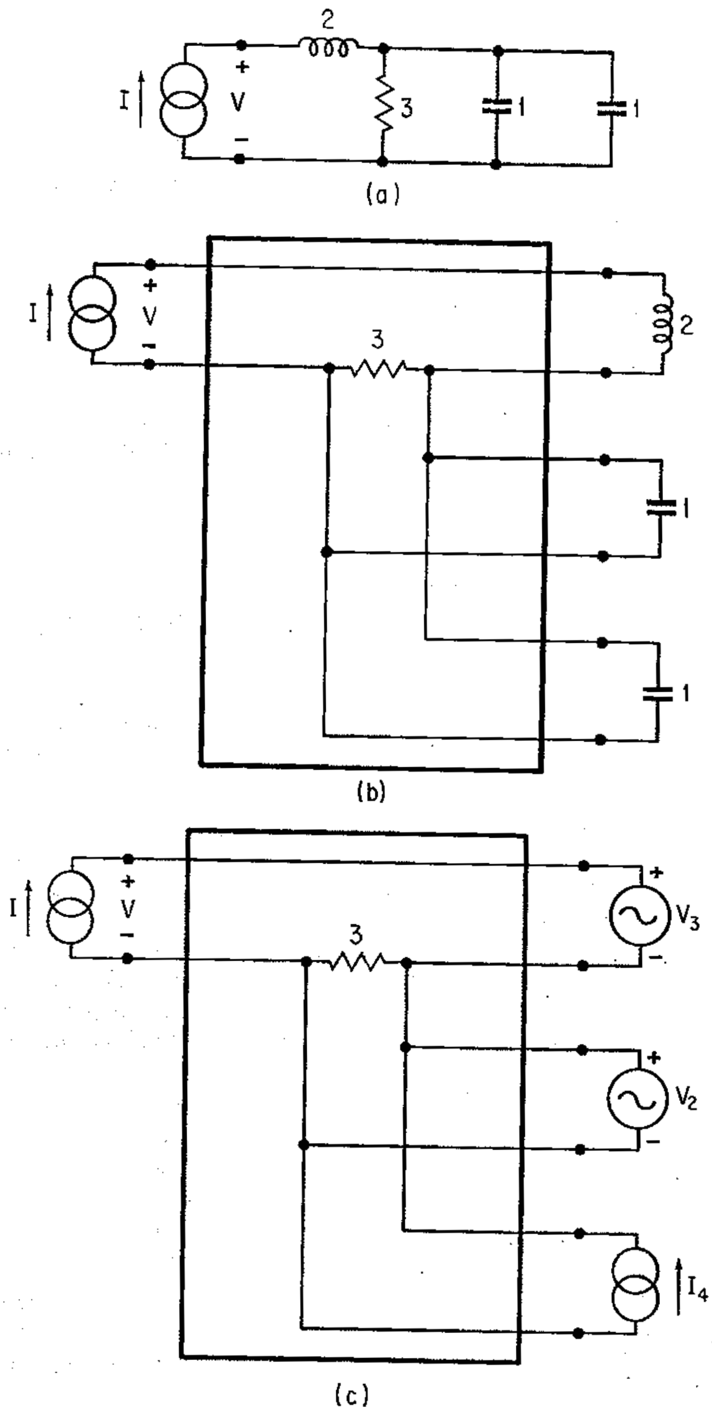


FIGURE 4.4.5. Network for Example 4.4.4.

Similarly, from (4.4.59) we obtain

$$\begin{bmatrix} I_3 \\ V_4 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} I \\ V_2 \end{bmatrix} = \begin{bmatrix} -I \\ V_2 \end{bmatrix}$$

(These equations are self-evident from Fig. 4.4.5c.) Combining these two equations and replacing I by u and V by y, we have

$$\dot{V}_2 = -\tfrac{1}{6}V_2 + \tfrac{1}{2}u$$
$$y = V_2 + 2\dot{u}$$

Properties of the State-Space Equations

Now we wish to note some important points arising out of the development of (4.4.65). In the problems of this section, proof is requested of the following fact.

Theorem 4.4.1. With the definitions

$$D_1 = \begin{bmatrix} \mathcal{L}_2 & 0 \\ 0 & \mathcal{C}_3 \end{bmatrix} + \begin{bmatrix} M_{24} & M_{25} \\ M_{34} & M_{35} \end{bmatrix}\begin{bmatrix} \mathcal{L}_4 & 0 \\ 0 & \mathcal{C}_5 \end{bmatrix}\begin{bmatrix} M_{24} & M_{25} \\ M_{34} & M_{35} \end{bmatrix}'$$

$$D_4 = \begin{bmatrix} M_{24} & M_{25} \\ M_{34} & M_{35} \end{bmatrix}\begin{bmatrix} \mathcal{L}_4 & 0 \\ 0 & \mathcal{C}_5 \end{bmatrix} \qquad D_8 = \begin{bmatrix} \mathcal{L}_4 & 0 \\ 0 & \mathcal{C}_5 \end{bmatrix}$$

and with $\mathcal{L}_2$, $\mathcal{C}_3$, $\mathcal{L}_4$, and $\mathcal{C}_5$ positive definite, the matrix

$$D_9 = D_8 - D_4' D_1^{-1} D_4$$

is positive definite symmetric.

Notice that the coefficient of $\dot{u}$ in the expression for y in (4.4.65) is $D_5' D_9 D_5$. Thus the following facts should be clear.

1. The coefficient of $\dot{u}$ in the expression for y in (4.4.65) is nonnegative definite symmetric.
2. From (4.4.65), y is independent of $\dot{u}$ if and only if $D_5 = [M_{14} \;\; M_{15}]' = 0$.
3. If $D_5 = 0$, no change of variable from x to $\hat{x}$ is necessary [see (4.4.64)]; Eq. (4.4.62) for x contains no $\dot{u}$ term; M_{14} and M_{15} are zero by definition of D_5; and the inductor currents R_4 and capacitor voltages R_5 depend [see (4.4.59)] only on E_2 and E_3 and not on u.
4. It is invalid to attempt to define the response of N for initial conditions E_2, E_3, R_4, and R_5 and excitation $u(\cdot)$ which do not satisfy (4.4.59). The initial values of the inductor currents E_2 and capacitor voltages E_3 can be arbitrary. The initial values of R_4 and R_5 depend only on E_2 and E_3 and not on u if $D_5 = 0$.

The following theorem is an interesting consequence of points 2 and 3.

Theorem 4.4.2. Consider an m-port network N with port excitation vector u and port response vector y defining a hybrid matrix $M(s)$ of N via

$$Y(s) = M(s)U(s)$$

Then if $M(\infty) < \infty$, there exists a set of state-space equations for N with the state vector consisting of inductor currents and capacitor voltages.

Proof. If $M(\infty) < \infty$, then we can set up equations of the form of (4.4.62) and (4.4.63) with the entries of x, being entries of E_2 and E_3, consisting solely of inductor currents and capacitor voltages. Using the transformation (4.4.64), we may obtain (4.4.65). But with $M(\infty) < \infty$ it follows that y must be independent of $\dot{u}$ and thus that $D_5 = 0$. Hence $\hat{x} = x$ and $\hat{x}$ is a state vector of the required form. $\nabla \nabla \nabla$

From the mere existence of state-variables equations of the form of (4.4.65), which is a nontrivial fact, we can also establish the following important result.

Theorem 4.4.3. Let $M(s)$ be a passive hybrid matrix, and let N be any passive network with $M(s)$ as hybrid matrix, perhaps obtained by synthesis. Then the sum of the number of inductors and capacitors in N is at least $\delta[M(s)]$.

Proof. Let (4.4.65) be state-space equations derived for N. Reference to the way these equations were derived will show that the dimension of x is the sum of the dimensions of E_2 and E_3 in (4.4.58), and that $D'_5 = [M_{14} \;\; M_{15}]$ has a number of columns equal to the sum of the dimensions of E_4 and E_5. Since the sum of the dimensions of E_2, E_3, E_4, and E_5 is the number of reactive elements in N (excluding perhaps some associated with deleted ports of N_r), the number of reactive elements in N is bounded below by

$$\text{dimension of } x + \text{number of columns of } D'_5$$

Now observe, using (4.4.65) and the definition of D_9, that the residue matrix associated with any pole at infinity of elements of $M(s)$ is $D'_5D_9D_5$. By the definition of degree, it follows that

$$\begin{aligned}\delta[M(s)] &\le \text{dimension of } x + \text{rank } D'_5D_9D_5\\ &\le \text{dimension of } x + \text{number of columns of } D'_5\end{aligned}$$

and the result follows. $\nabla \nabla \nabla$

Problem 4.4.1 Prove Theorem 4.4.1.

Problem 4.4.2 Let N be a passive network synthesizing a prescribed passive scattering matrix $S(s)$. Show that the number of capacitors and inductors in N is at least $\delta[S(s)]$.

Problem 4.4.3 Let N be a passive network of m ports, with the transfer-function matrix relating excitations at m_1 of these ports to responses at m_2 of these ports denoted by $W(s)$. Show that the number of capacitors and inductors in N is at least $\delta[W(s)]$.

Problem 4.4.4 Find state-space equations for the network of Fig. 4.4.6. Take all components to have unit values.

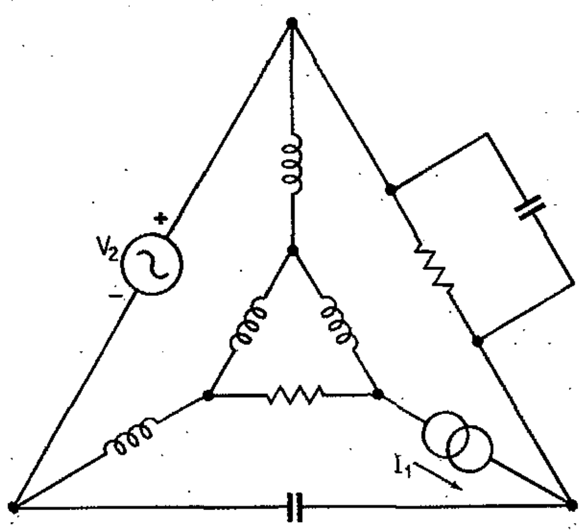


FIGURE 4.4.6. Network for Problem 4.4.4.

Problem 4.4.5 Suppose that in a prescribed network with external sources each capacitor and ideal voltage source is replaced by a like element with ϵ-ohms series resistance, and each inductor and ideal current source is replaced by a like element with ϵ-mhos shunt conductance. Show that state-space equations can be derived by the method of this section with all excitation variables of the hybrid matrix being assigned "desirably." What happens when $\epsilon \longrightarrow 0$?

4.5 STATE-SPACE EQUATIONS FOR ACTIVE NETWORKS

Up to this point of the chapter, our aim has been to derive state-space equations of passive networks, the elements in which are resistors, capacitors, inductors, transformers, and gyrators. Now we wish to extend this set of elements to include *controlled sources;* any of the four types, current-to-current, current-to-voltage, voltage-to-current, or voltage-to-voltage are permitted.

The reader will have observed, particularly in Section 4.4, how much passivity has been used in the schemes described hitherto for setting up state-space equations. It should therefore come as no surprise that while state-space equations can always be set up for a passive network (unless there is some fairly obvious form of ill posing), this is not the case for networks containing active elements. One startling illustration of this point is provided with the aid of the *nullator.* A nullator is a two-terminal device that may be constructed with controlled sources, as shown in Fig. 4.5.1a. (Note that a negative resistor is constructible with controlled sources, as in Fig. 4.5.1b.)

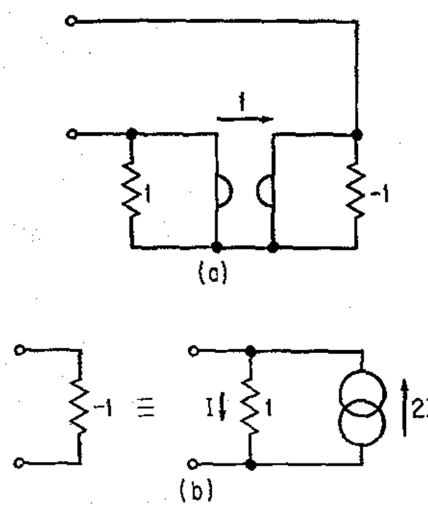


FIGURE 4.5.1. (a) The Nullator and (b) the Simulation of a Negative Resistor.

Its port current and port voltage are simultaneously always zero [15], which means that it looks simultaneously like an open circuit and a short circuit. Now consider the circuits of Fig. 4.5.2. These circuits can support neither a nonzero or independent voltage nor a nonzero or independent current source, and it is obvious that state-space equations cannot be found.

Results on the existence of state-space equations of active networks are not extensive, though some are available (see, e.g., [16]). In this section we shall avoid the existence question largely, concentrating on presenting ad hoc approaches to the development of equations. We can attempt a state-

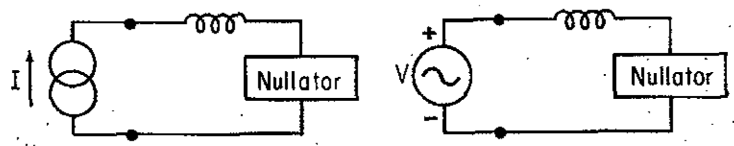


FIGURE 4.5.2. Circuits for which no State-Space Equations Exist

space equation derivation at any of the three levels of complexity applicable for passive networks. Thus if the network is particularly simple, straightforward inspection and application of Kirchhoff's laws will enable us to write down a differential equation for each inductor current and capacitor voltage and an equation relating the output to the inductor currents, capacitor voltages, and the input. A simple illustration follows.

Example 4.5.1 Consider the circuit of Fig. 4.5.3. The input variable is V_i, the output variable V_o. Notice that I_R is neither an input nor a state variable; so it

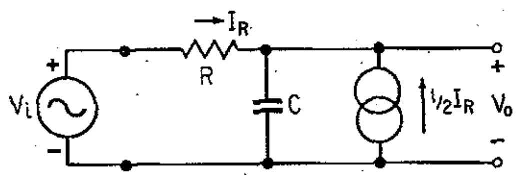


FIGURE 4.5.3. Network for Example 4.5.1.

is necessary to express it in terms of the input and state. The state x is obviously the capacitor voltage V_c, which we can choose to have the same sign as V_o. It is immediate from the circuit that

$$C\dot{V}_c = I_R + \frac{1}{2}I_R = \frac{3}{2}I_R = \frac{3}{2R}(V_i - V_c)$$

or

$$\dot{x} = -\frac{1\cdot5}{RC}x + \frac{1\cdot5}{RC}V_i$$

Also,

$$V_o = x$$

State-space equation derivation at the second level of complexity, if possible, proceeds in the same way for active networks as passive networks, the nondynamic network containing resistors, transformers, gyrators, and controlled sources. Rather than spelling out the procedure in detail, we shall describe state-space equation derivation at the third level of complexity. This includes derivation at the second level of complexity as a special case.

Equation Derivation at the Third Level of Complexity

The procedure we shall describe may or may not work; nevertheless, if the procedure does not work, it is unlikely that state-space equations can be found. The procedure falls into three main steps:

1. A reactance extraction is performed on a prescribed network N to obtain a nondynamic network N_r, which contains resistors, transformers, gyrators, and controlled sources.
2. A modified form of hybrid matrix is constructed for N_r. (The modification will be noted shortly.)
3. State-space equations for N are constructed using the modified hybrid matrix of N_r.

Step 1 is of course quite straightforward. It is at step 2 that we run into the first set of difficulties. For example, because N_r is not passive it may possess no hybrid-matrix description at all. Let us however point out how we try to form a hybrid matrix for N_r. As we shall see, it sometimes proves convenient to form only a submatrix of a hybrid matrix, this being the modification referred to in step 2.

Suppose that N is an m port, and that the originally prescribed set of excitations is $u_1, u_2, \ldots, u_{m_1}$ and the originally prescribed set of responses is $y_{m_2+1}, y_{m_2+2}, \ldots, y_m$. Notice that $m_2 \leq m_1$, for otherwise port $m_1 + 1$ and perhaps other ports as well would have neither response nor excitation assigned to them, and therefore they would be dropped from further consideration.

If N were passive, we would assign, or attempt to assign, the first m excitation variables for N_r by extending the m_1 excitations of N to a full m, and then identifying e_i, the excitation at the ith port of N_r, with u_i. However, here we merely set

$$e_i = u_i \qquad i = 1, 2, \ldots, m_1 \tag{4.5.1}$$

with the response r_i at the ith port of N_r defined by

$$r_i = y_i \qquad i = m_2 + 1, \ldots, m \tag{4.5.2}$$

Instead of seeking a full hybrid matrix for N_r, we seek a submatrix of it only; disallowing consideration of excitations e_{m_1+1} through e_m means that we delete columns $m_1 + 1$ through m of the full hybrid matrix of N_r, and disallowing consideration of responses r_1 through r_{m_2} means that we delete rows 1 through m_2 of the full hybrid matrix of N_r. The remaining rows and columns define the appropriate submatrix.

Now suppose that N contains n reactive elements. We follow the procedures of the last section in attempting to assign excitation and response variables to N_r.* Thus, for as many ports as possible, we attempt to identify the excitation of the port with a current if the port is inductively terminated and with a voltage if the port is capacitively terminated in forming N from N_r. Of course, the identification is not permitted should there exist a relationship of linear dependence between the proposed excitation variable and those already selected, the relationship holding independently of the terminations at ports with unassigned excitation variables.

In the event that no further excitation variables can be assigned in the desired way, we then attempt to assign them in the opposite of the desired way; for example, if a port of N_r is inductively terminated when N is constructed and independent current excitation is not possible, then we try an independent voltage excitation. For passive networks, this procedure, as we have shown, *will always result in an appropriate excitation. For active* networks, it may not. Not every active network possesses a hybrid matrix, which implies that for some active networks neither a voltage nor current excitation will be acceptable at one or more ports.

If neither the desirable nor undesirable allocation of excitation variables is possible, we must abandon our goal of forming a hybrid matrix or of forming state-space equations by the method being described.

Assume therefore that all excitation variables can be assigned; response variables for the last n ports of N_r are assigned in the obvious manner as the duals of the excitation variables. The excitation variables are assumed to be free of any constraints, so that we can argue the existence of a modified hybrid matrix M, actually a submatrix of a hybrid matrix, for which

$$\begin{bmatrix} r_{m_1+1} \\ \cdot \\ \cdot \\ \cdot \\ r_m \\ \hline r_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ r_l \\ \hline r_{l+1} \\ \cdot \\ \cdot \\ \cdot \\ r_{m+n} \end{bmatrix} = \left[\begin{array}{c|c|c} M_{11} & M_{12} & M_{13} \\ \hline M_{21} & M_{22} & M_{23} \\ \hline M_{31} & M_{32} & 0 \end{array}\right] \begin{bmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_{m_1} \\ \hline e_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ e_l \\ \hline e_{l+1} \\ \cdot \\ \cdot \\ \cdot \\ e_{m+n} \end{bmatrix} \tag{4.5.3}$$

*For the reader who has not studied the last section, the short summary here will probably suffice.

In this equation, M is shown in partitioned form with the partitioning *conforming* with that shown in the excitation and response variables. The integer l defines the last port at which assignation of an excitation variable as a *desired* variable is permissible. Notice that M_{33} is zero, the argument being the same as in the last section: at ports $l+1$ through $m+n$ the desirable excitation depends on the excitations e_1 through e_l. This desirable excitation becomes an actual response, and so each of r_{l+1} through r_{m+n} depends only on e_1 through e_l. Note also that because N_r is not passive, we *cannot* conclude that $M_{13} = -M'_{31}$ or $M_{23} = -M'_{32}$.

Example 4.5.2 Consider the circuit of Fig. 4.5.4a. The circuit is redrawn in Fig. 4.5.4b to exhibit a reactance extraction. We describe the generation of M with the aid of Fig. 4.5.4c.

Obviously, we identify $e_1 = V_i$ and $r_2 = V_o$. We do not assign e_2 or r_1. Next we assign excitation variables at the ports of N_r which are reactively terminated in forming the original circuit.

Assignation of $e_3 = V_3$ and $e_4 = V_4$ causes no problems, and these are desirable assignations. We would like to set $e_5 = V_5$, but this is not possible since

$$e_3 = e_4 + V_5$$

So we try to set $e_5 = I_5$. This works; i.e., there is no linear relation involving e_1, e_3, e_4, and the new e_5. The response variables are $r_3 = I_3$, $r_4 = I_4$, and $r_5 = V_5$. Conventional analysis yields

$$\begin{bmatrix} V_0 \\ I_3 \\ I_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -R_4^{-1} & R_1^{-1} + R_3^{-1} + R_4^{-1} + g_m & -R_3^{-1} & -1 \\ 0 & -R_3^{-1} - g_m & R_2^{-1} + R_3^{-1} & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_i \\ V_3 \\ V_4 \\ I_5 \end{bmatrix}$$

Step 3 of the procedure for obtaining state-space equations requires us to proceed from the hybrid matrix to the equations. The technique follows that of the last section; since some differences arise, including possible inability to form the equations, we shall now outline the procedure.

The starting point is (4.5.3), which we shall rewrite in the form

$$\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{4.5.4}$$

where the definitions of R_1, R_2, etc., should be obvious from (4.5.3).

We reiterate the following points: the entries of E_1 and R_1 do not necessarily correspond to the same set of ports, and the entries of E_2 constitute "desirable" excitation variables, while those of E_3 constitute "undesirable" excitation variables.

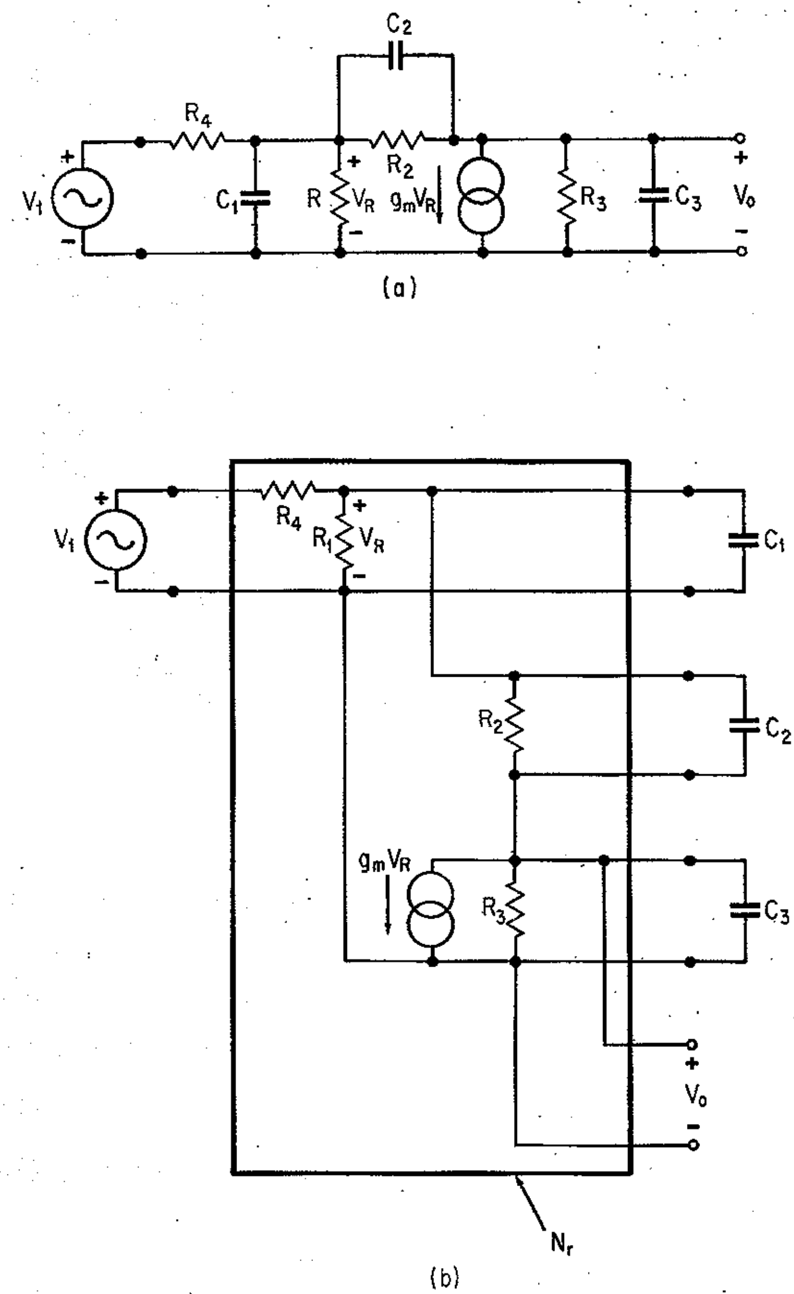
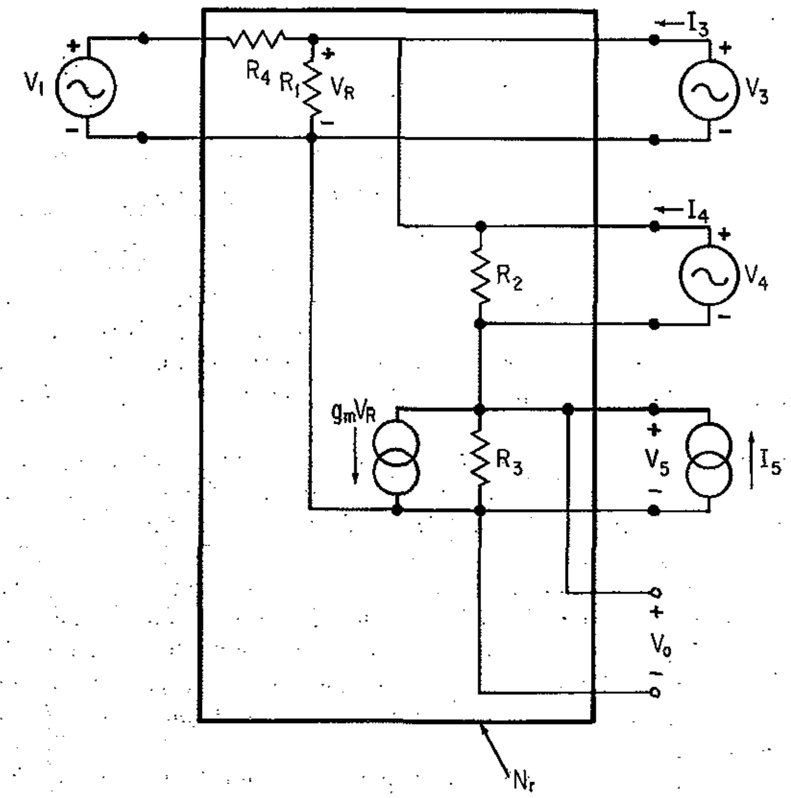


FIGURE 4.5.4. Network for Examples 4.5.2 and 4.5.3. The Network Models a Transistor.



(c)

FIGURE 4.5.4 (cont.)

When N_r is terminated in the appropriate reactive elements that yield N, this amounts to demanding

$$R_2 = -\mathcal{B}_2\dot{E}_2 \tag{4.5.5}$$

where $\mathcal{B}_2$ is a diagonal matrix of inductor or capacitor values, and also

$$E_3 = -\mathcal{B}_3\dot{R}_3 \tag{4.5.6}$$

where $\mathcal{B}_3$ is the same sort of matrix as $\mathcal{B}_2$. Using (4.5.5) and (4.5.6), (4.5.4) becomes

$$\begin{bmatrix} R_1 \\ -\mathcal{B}_2\dot{E}_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ -\mathcal{B}_3\dot{R}_3 \end{bmatrix} \tag{4.5.7}$$

In particular,

$$R_3 = M_{31}E_1 + M_{32}E_2 \tag{4.5.8}$$

so that, again from (4.5.7),

$$\begin{aligned} -\mathcal{B}_2\dot{E}_2 &= M_{22}E_2 + M_{21}E_1 - M_{23}\mathcal{B}_3M_{32}\dot{E}_2 - M_{23}\mathcal{B}_3M_{31}\dot{E}_1 \\ R_1 &= M_{11}E_1 + M_{12}E_2 - M_{13}\mathcal{B}_3M_{31}\dot{E}_1 - M_{13}\mathcal{B}_3M_{32}\dot{E}_2 \end{aligned} \tag{4.5.9}$$

Consider the first equation in (4.5.9). This can be "solved" for $\dot{E}_2$ if and only if $\mathcal{B}_2 - M_{23}\mathcal{B}_3M_{32}$ is nonsingular. For passive N_r, this is guaranteed by the fact that $\mathcal{B}_2$ and $\mathcal{B}_3$ are positive definite and that $M_{23} = -M_{32}'$ (the latter following because $M_{33} = 0$). *No such guarantee applies if N_r is active.*

Notice that if $\mathcal{B}_2 - M_{23}\mathcal{B}_3M_{32}$ is singular, a perturbation of $\mathcal{B}_2$ to, say, $\mathcal{B}_2 + \epsilon I$ for an arbitrary small ϵ will render the matrix nonsingular. In other words, (4.5.9) *fails to have a solution for $\dot{E}_2$ only for particular values of the inductors and capacitors.*

Of course, if (4.5.9) is solvable for $\dot{E}_2$, then there is no difficulty in constructing state-space equations for N. The input vector u agrees with E_1, the output vector y with R_1. Terms in $\dot{E}_1$ may be eliminated from the differential equation for the state vector by appropriate redefinition of the state vector, if necessary. Terms in $\dot{E}_1$ may occur in the equation for R_1, but of course these may not be eliminated.

If $\mathcal{B}_2 - M_{23}\mathcal{B}_3M_{32}$ is singular, it may or may not be possible to obtain state-space equations in which each entry of E_1, i.e., each excitation, can be selected independently. Examination of this point is requested in the problems.

Example 4.5.3 We return to the circuit of Fig. 4.5.4a. With quantities as defined in Fig. 4.5.4c, we found

$$\begin{bmatrix} V_o \\ I_3 \\ I_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -R_4^{-1} & R_1^{-1} + R_3^{-1} + R_4^{-1} + g_m & -R_3^{-1} & -1 \\ 0 & -R_3^{-1} - g_m & R_2^{-1} + R_3^{-1} & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_i \\ V_3 \\ V_4 \\ I_5 \end{bmatrix}$$

Using Fig. 4.5.4b, we see that

$$I_3 = -C_1\dot{V}_3 \qquad I_4 = -C_2\dot{V}_4 \qquad I_5 = -C_3\dot{V}_5$$

It is then easy to derive

$$V_5 = [1 \quad -1]\begin{bmatrix} V_3 \\ V_4 \end{bmatrix}$$

and then

$$\begin{bmatrix} -C_1\dot{V}_3 \\ -C_2\dot{V}_4 \end{bmatrix} = \begin{bmatrix} R_1^{-1} + R_3^{-1} + R_4^{-1} + g_m & -R_3^{-1} \\ -R_3^{-1} - g_m & R_2^{-1} + R_3^{-1} \end{bmatrix} \begin{bmatrix} V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} -R_4^{-1} \\ 0 \end{bmatrix} V_i - \begin{bmatrix} -1 \\ 1 \end{bmatrix} C_3 [1 \quad -1] \begin{bmatrix} \dot{V}_3 \\ \dot{V}_4 \end{bmatrix}$$

or

$$\begin{bmatrix} -(C_1 + C_3) & C_3 \\ C_3 & -(C_2 + C_3) \end{bmatrix} \begin{bmatrix} \dot{V}_3 \\ \dot{V}_4 \end{bmatrix} = \begin{bmatrix} R_1^{-1} + R_3^{-1} + R_4^{-1} + g_m & -R_3^{-1} \\ -R_3^{-1} - g_m & R_2^{-1} + R_3^{-1} \end{bmatrix} \begin{bmatrix} V_3 \\ V_4 \end{bmatrix} + \begin{bmatrix} -R_4^{-1} \\ 0 \end{bmatrix} V_i$$

The coefficient matrix multiplying $[\dot{V}_3 \quad \dot{V}_4]'$ is nonsingular, and so we can rewrite this equation in the standard form $\dot{x} = Fx + Gu$. The equation for V_o easily follows from the modified hybrid matrix as

$$V_o = [1 \quad -1] \begin{bmatrix} V_3 \\ V_4 \end{bmatrix}$$

The characterization in topological terms of the difficulties that can be encountered in setting up state-space equations for active networks is very difficult. More topologically based approaches to setting up the equations therefore tend to be ad hoc (more so, in fact, than the present procedure), relying on presenting sufficient conditions for the generation of equations, or on presenting algorithms that may or may not work (see, e.g., [8]). We prefer the procedure given here on the grounds that topological ideas do not play a key role in the formation of the equations—of course they may or may not be employed in generating the modified hybrid matrix of N_r.

There is one additional danger that the reader should be aware of in analyzing active networks. If the networks have some form of instability, they may not behave like linear networks because some elements—generally active ones—may be forced into nonlinear operation. This movement into the nonlinear regime is in fact the basis of operation of such circuits as the multivibrator. Many active networks do not exhibit this kind of instability of course, and the methods of this section are applicable to them.

Problem 4.5.1 Develop controlled source models for the transformer and the gyrator to conclude that the basic elements allowed in networks (active or passive) can all be assumed two terminal. (This is the approach adopted in many cases, e.g., [8], in applying topological methods to circuits containing gyrators and transformers.)

Problem 4.5.2 Figure 4.5.5 shows a circuit model of a transistor tuned amplifier based on the y-parameter model of the transistor. Develop state-space equations

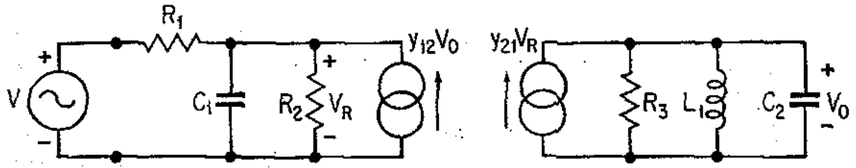


FIGURE 4.5.5. Network for Problem 4.5.2.

for the circuit. Assume first that y_{12} and y_{21} are real constants. What happens if y_{12} and y_{21} are complex and frequency dependent?

Problem 4.5.3 Develop procedures for forming state-space equations from (4.5.9) for the case when $\mathcal{B}_2 - M_{23}\mathcal{B}_3 M_{32}$ is singular.

Problem 4.5.4 Suppose that an active network possesses state-space equations derivable via the procedure of this section. Let $W(s)$ be the associated transfer-function matrix. Show that $\delta[W(s)]$ is a lower bound on the number of capacitors or inductors in the network.

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Part IV

THE STATE-SPACE DESCRIPTION OF PASSIVITY AND RECIPROCITY

The major task in this part is to state and prove the *positive real lemma*. This is a statement in state-space terms of the positive real constraint and is fundamental to the various synthesis procedures.

We break up this part into five segments:

1. *Statement and partial proof of the positive real lemma.*
2. Various proofs of the positive real lemma, completing the partial proof mentioned in 1.
3. Computation of certain matrices occurring in the positive real lemma.
4. The bounded real lemma.
5. The reciprocity constraint in state-space terms.

Part IV can be read in three degrees of depth. For the reader who is interested in how to synthesize networks, with scant regard for computations, 1, 4, and 5 above will suffice. This material is covered in Sections 5.1 and 5.2 and Sections 7.1, 7.2, 7.4, and 7.5. For information regarding computation, see Chapter 6 and Section 7.3. For the reader who desires the full story, all of Part IV should be read.

5

Positive Real Matrices and the Positive Real Lemma

5.1 INTRODUCTION

In this chapter our aim is to present an interpretation of the positive real constraint—usually viewed as a frequency-domain constraint—in terms of the matrices of a ***state-space realization*** of a prescribed ***positive real matrix***. In this introductory section we restate the positive real definition and note several minor consequences of it. Then we go on in the next section to state the positive real lemma, a set of constraints on the matrices of a state-space realization guaranteeing that the associated transfer-function matrix is positive real. We also prove that the stated conditions are sufficient to guarantee the positive real property.

The proof that the conditions on the matrices of a state-space realization stated in the positive real lemma are in fact necessary to guarantee the positive real property is quite difficult. In later sections we tackle this problem from different points of view. In fact, we present several proofs, each relying on different properties associated with a positive real matrix. One's familiarity with these properties is a function of background; therefore, the degree of acceptability of the different proofs is also a function of background. It is not necessary even to read, let alone understand, any of the proofs for later chapters (though of course an understanding of the positive real lemma statement is required). So we invite the reader to omit, skim, or study in detail as much or as little of the proofs as he wishes. For those unsure as to which proof may be of interest, we suggest that they be guided by the section titles.

As we know, an $m \times m$ matrix $Z(s)$ of real rational functions is positive real if

1. All elements of $Z(s)$ are analytic in $\mathrm{Re}[s] > 0$.
2. Any pure imaginary pole $j\omega_0$ of any element of $Z(s)$ is a simple pole, and the associated residue matrix of $Z(s)$ is nonnegative definite Hermitian.
3. For all real ω for which $j\omega$ is not a pole of any element of $Z(s)$, the matrix $Z'^*(j\omega) + Z(j\omega)$ is nonnegative definite Hermitian.

Conditions 2 and 3 may alternatively be replaced by

4. The matrix $Z'^*(s) + Z(s)$ is nonnegative definite Hermitian in $\mathrm{Re}[s] > 0$.

These conditions for positive realness are essentially analytic rather than algebraic; the positive real lemma, on the other hand, presents algebraic conditions on the matrices of a state-space realization of $Z(s)$. Roughly speaking, the positive real lemma enables the translation of network synthesis problems from problems of analysis to problems of algebra.

In the remainder of this section we shall clear two preliminary details out of the way, both of which will be helpful in the sequel. These details relate to minor decompositions of $Z(s)$ to isolate the effect of pure imaginary poles.

We can conceive a partial fraction expansion of each element of $Z(s)$ [and thus of $Z(s)$ itself] being made, with a subsequent grouping together of terms with poles on the $j\omega$ axis, i.e., poles of the form $j\omega_0$ for some real ω_0, and poles in the half-plane $\mathrm{Re}[s] < 0$. (Such a grouping together is characteristic of the Foster preamble of classical network synthesis procedures, which may be familiar to some readers.) This allows us to write $Z(s)$ in the form

$$Z(s) = sL + \sum_i \frac{K_i}{s - j\omega_i} + s^{-1}C + Z_0(s) \tag{5.1.1}$$

where L, C, and the K_i are nonnegative definite Hermitian, and the poles of elements of $Z_0(s)$ all lie in $\mathrm{Re}[s] < 0$. We have already argued that L must be real and symmetric, and it is easy to argue that C must be real and symmetric. The matrix K_i, on the other hand, will in general be complex. However, if $j\omega_i$ for $\omega_i \neq 0$ is a pole of an element of $Z(s)$, $-j\omega_i$ must also be a pole, and the *real* rational nature of $Z(s)$ implies that the associated residue matrix is K_i^*. This means that terms in the summation

$$\sum_i \frac{K_i}{s - j\omega_i}$$

occur in pairs, a pair being

$$\frac{K_i}{s - j\omega_i} + \frac{K_i^*}{s + j\omega_i} = \frac{A_i s + B_i}{s^2 + \omega_i^2}$$

where A_i and B_i are real matrices; in fact, $A_i = K_i + K_i^*$ and is nonnegative definite symmetric, while $B_i = j\omega_i(K_i - K_i^*)$ and is skew symmetric. So we can write

$$Z(s) = sL + \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} + s^{-1}C + Z_0(s) \tag{5.1.2}$$

if desired. We now wish the reader to carefully note the following points:

1. sL, $s^{-1}C$, and $(A_i s + B_i)(s^2 + \omega_i^2)^{-1}$ are LPR. The fact that they are PR is immediate from the definitions, while the lossless character follows, for example, by observing that

$$\frac{A_i s + B_i}{s^2 + \omega_i^2} + \frac{A_i'(-s) + B_i'}{(-s)^2 + \omega_i^2} = 0$$

2. $$sL + \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} + s^{-1}C = Z_L(s)$$

   is LPR. This is immediate from 1.
3. $Z_0(s)$ is PR! To see this observe that all poles of elements of $Z_0(s)$ lie in $\text{Re}\,[s] < 0$, while $Z_0'^*(j\omega) + Z_0(j\omega) = Z_L'^*(j\omega) + Z_L(j\omega) + Z_0'^*(j\omega) + Z_0(j\omega) = Z'^*(j\omega) + Z(j\omega) \geq 0$ on using the lossless character of $Z_L(s)$.

Therefore, we are able [by identifying the $j\omega$-axis poles of elements of $Z(s)$] to *decompose an arbitrary PR $Z(s)$ into the sum of a LPR matrix, and a PR matrix such that all poles of all elements lie in* $\text{Re}\,[s] < 0$.

Also, it is clear that $Z_1(s) = Z(s) - sL$ is PR, being the sum of $Z_0(s)$, which is PR, and

$$s^{-1}C + \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2}$$

which is LPR. Further, $Z_1(\infty) < \infty$. So we are able to *decompose an arbitrary PR $Z(s)$ into the sum of a LPR matrix sL, and a PR matrix $Z_1(s)$ with $Z_1(\infty) < \infty$.*

In a similar way, we can decompose an arbitrary PR $Z(s)$ with one or more elements possessing a pole at $s = j\omega_0$ into the sum of two PR matrices

$$\frac{A_0 s + B_0}{s^2 + \omega_0^2} + Z_2(s)$$

with the first matrix LPR and the second matrix simply PR, with no element possessing a pole at $j\omega_0$.

Problem 5.1.1 Show that $Z(s)$ is LPR if *and only if* $Z(s)$ has the form

$$Z(s) = sL + J + \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} + s^{-1}C$$

where $L = L' \geq 0$, $J + J' = 0$, $C = C' \geq 0$, A_i and B_i are real, and $A_i + B_i/j\omega_i$ is nonnegative definite Hermitian.

5.2 STATEMENT AND SUFFICIENCY PROOF OF THE POSITIVE REAL LEMMA

In this section we consider a positive real matrix $Z(s)$ of rational functions, and we assume from the start that $Z(\infty) < \infty$. As we explained in Section 5.1, in the absence of this condition we can simply recover from $Z(s)$ a further positive real matrix that does satisfy the restriction. The lemma to be stated will describe conditions on matrices appearing in a minimal state-space realization that are necessary and sufficient for $Z(s)$ to be positive real. We shall also state a specialization of the lemma applying to lossless positive real matrices.

The first statement of the lemma, for positive real functions rather than positive real matrices, was given by Yakubovic [1] in 1962, and an alternative proof was presented by Kalman [2] in 1963. Kalman conjectured a matrix version of the result in 1963 [3], and a proof, due to Anderson [4], appeared in 1967. Related results also appear in [5].

In the following, when we talk of a realization $\{F, G, H, J\}$ of $Z(s)$, we shall, as explained earlier, imply that if

$$\dot{x} = Fx + Gu \qquad y = H'x + Ju \tag{5.2.1}$$

then

$$Y(s) = Z(s)U(s) \tag{5.2.2}$$

where $\mathcal{L}[y(\cdot)] = Y(s)$, $\mathcal{L}[u(\cdot)] = U(s)$. If now an $m \times m$ $Z(s)$ is the impedance of an m-port network, we understand it to be the transfer-function matrix relating the m vector of port currents to the m vector of port voltages. In other words, *we identify the vector u of the state-space equations with the port current and the vector y with the port voltage of the network.*

The formal statement of the positive real lemma now follows.

Positive Real Lemma. Let $Z(\cdot)$ be an $m \times m$ matrix of real rational functions of a complex variable s, with $Z(\infty) < \infty$. Let $\{F, G, H, J\}$ be a minimal realization of $Z(s)$. Then $Z(s)$ is positive real if and only if there exist real matrices P, L, and W_0 with P positive definite symmetric, such that

$$\begin{aligned} PF + F'P &= -LL' \\ PG &= H - LW_0 \\ W_0'W_0 &= J + J' \end{aligned} \tag{5.2.3}$$

(The number of rows of W_0 and columns of L are unspecified, while the other dimensions of P, L, and W_0 are automatically fixed.)

In the remainder of this section we shall show that (5.2.3) are *sufficient* for the positive real property to hold, and we shall exhibit a special factorization, actually termed a *spectral factorization*, of $Z'(-s) + Z(s)$. Finally, we shall state and prove a positive real lemma associated with lossless positive real matrices.

Sufficiency Proof of the Positive Real Lemma

First, we must check analyticity of elements of $Z(s)$ in Re $[s] > 0$. Since $Z(s) = J + H'(sI - F)^{-1}G$, an element of $Z(s)$ will have a pole at $s = s_i$ only if s_i is an eigenvalue of F. The first of equations (5.2.3), together with the positive definiteness of P, guarantees that all eigenvalues of F have nonpositive real part, by the lemma of Lyapunov. Accordingly, all poles of elements of $Z(s)$ have nonpositive real parts.

Finally, we must check the nonnegative character of $Z'^*(s) + Z(s)$ in Re $[s] > 0$. Consider the following sequence of equalities—with reasoning following the last equality.

$$\begin{aligned} Z'^*(s) + Z(s) \\ &= J' + J + G'(s^*I - F')^{-1}H + H'(sI - F)^{-1}G \\ &= W_0'W_0 + G'[(s^*I - F')^{-1}P + P(sI - F)^{-1}]G \\ &\quad + G'(s^*I - F')^{-1}LW_0 + W_0'L'(sI - F)^{-1}G \\ &= W_0'W_0 + G'(s^*I - F')^{-1}[P(s + s^*) - PF - F'P](sI - F)^{-1}G \\ &\quad + G'(s^*I - F')^{-1}LW_0 + W_0'L'(sI - F)^{-1}G \\ &= W_0'W_0 + G'(s^*I - F')^{-1}LL'(sI - F)^{-1}G + G'(s^*I - F')^{-1}LW_0 \\ &\quad + W_0'L'(sI - F)^{-1}G + G'(s^*I - F')^{-1}P(sI - F)^{-1}G(s + s^*) \\ &= [W_0' + G'(s^*I - F')^{-1}L][W_0 + L'(sI - F)^{-1}G] \\ &\quad + G'(s^*I - F')^{-1}P(sI - F)^{-1}G(s + s^*) \end{aligned}$$

The second equality follows by using the second and third equation of (5.2.3), the third by manipulation, the fourth by using the first equation of (5.2.3) and rearrangement, and the final equation by manipulation.

Now since any matrix of the form A'^*A is nonnegative definite Hermitian, since $s + s^*$ is real and positive in Re $[s] > 0$, and since P is positive definite, it follows, as required, that

$$Z'^*(s) + Z(s) \geq 0 \qquad \nabla\nabla\nabla$$

To this point, sufficiency only is proved. As remarked earlier, in later sections proofs of necessity will be given. Also, since the quantities P, L, and W_0 prove important in studying synthesis problems, we shall be discussing techniques for their computation. (Notice that the positive real lemma statement talks about existence, not construction or computation. Computation is a separate question.)

Example 5.2.1 It is generally difficult to compute P, L, and W_0 from F, G, H, and J. Nevertheless, these quantities can be computed by inspection in very simple cases. For example, $z(s) = (s + 1)^{-1}$ is positive real, with realization $\{-1, 1, 1, 0\}$. Observe then that

$$P[-1] + [-1]P = -LL'$$
$$P[1] = [1] + LW_0$$
$$[0] + [0] = W_0'W_0$$

is satisfied by $P = [1]$, $L = [\sqrt{2}]$, and $W_0 = [0]$. Clearly, P is positive definite.

A Spectral Factorization Result

We now wish to note a factorization of $Z'(-s) + Z(s)$. We can carry through on this quantity essentially the same calculations that we carried through on $Z'^*(s) + Z(s)$ in the sufficiency proof of the positive real lemma. The end result is

$$\begin{aligned} Z'(-s) + Z(s) &= [W_0' + G'(-sI - F')^{-1}L][W_0 + L'(sI - F)^{-1}G] \\ &= W'(-s)W(s) \end{aligned} \tag{5.2.4}$$

where $W(s)$ is a real rational transfer-function matrix. Such a decomposition of $Z'(-s) + Z(s)$ has been termed in the network and control literature a *spectral factorization*, and is a generalization of a well-known scalar result: if $z(s)$ is a positive real function, then Re $z(j\omega) \geq 0$ for all ω (save those ω for which $j\omega$ is a pole), and Re $z(j\omega)$ may be factored (nonuniquely) as

$$\text{Re } z(j\omega) = |w(j\omega)|^2 = w(-j\omega)w(j\omega)$$

where $w(s)$ is a real rational function of s.

Essentially then, the positive real lemma says something about the ability to do a spectral factorization of $Z'(-s) + Z(s)$, where $Z(\cdot)$ may be a matrix. *Moreover, the spectral factor $W(s)$ can have a state-space realization with the same F and G matrices as $Z(s)$.* [*Note:* This is not to say that *all* spectral factors of $Z'(-s) + Z(s)$ have a state-space realization with the same F and G matrices as $Z(s)$—indeed this is certainly not true.] Though the existence of spectral factors of $Z'(-s) + Z(s)$ has been known to network theorists for some time, the extra property regarding the state-space realizations of some spectral factors is, by comparison, novel. We sum up these notions as follows.

Existence of a Spectral Factor. Let $Z(s)$ be a positive real matrix of rational functions with $Z(\infty) < \infty$. Let $\{F, G, H, J\}$ be a minimal realization of $Z(s)$, and assume the validity of the positive real lemma. In terms of the matrices L and W_0 mentioned therein, the transfer-function matrix

$$W(s) = W_0 + L'(sI - F)^{-1}G \tag{5.2.5}$$

is a spectral factor of $Z'(-s) + Z(s)$ in the sense that

$$Z'(-s) + Z(s) = W'(-s)W(s) \tag{5.2.4}$$

Example 5.2.2 For the positive real impedance $(s + 1)^{-1}$, we obtained in Example 5.2.1 a minimal realization $\{-1, 1, 1, 0\}$, together with matrices P, L, and W_0 satisfying the positive real lemma equations. We had $L = [\sqrt{2}]$ and $W_0 = [0]$. This means that

$$W(s) = \frac{\sqrt{2}}{s + 1}$$

and indeed

$$\frac{1}{-s + 1} + \frac{1}{s + 1} = \frac{2}{(-s + 1)(s + 1)} = \frac{\sqrt{2}}{-s + 1} \cdot \frac{\sqrt{2}}{s + 1}$$

as predicted.

Positive Real Lemma in the Lossless Case

We wish to indicate a minor adjustment of the positive real lemma, important for applications, which covers the case when $Z(s)$ is lossless. Operationally, the matrices L and W_0 become zero.

Lossless Positive Real Lemma. Let $Z(\cdot)$ be an $m \times m$ matrix of real rational functions of a complex variable s, with $Z(\infty) < \infty$. Let $\{F, G, H, J\}$ be a minimal realization of $Z(s)$. Then

$Z(s)$ is lossless positive real if and only if there exists a (real) positive definite symmetric P such that

$$\begin{aligned} PF + F'P &= 0 \\ PG &= H \\ J + J' &= 0 \end{aligned} \tag{5.2.6}$$

If we assume validity of the main positive real lemma, the proof of the lossless positive real lemma is straightforward.

Sufficiency Proof. Since (5.2.6) are the same as (5.2.3) with L and W_0 set equal to zero, it follows that $Z(s)$ is at least positive real. Applying the spectral factorization result, it follows that $Z(s) + Z'(-s) = 0$. This equation and the positive real condition guarantee $Z(s)$ is lossless positive real, as we see by applying the definition.

Necessity Proof. If $Z(s)$ is lossless positive real, it is positive real, and there are therefore real matrices P, L, and W_0 satisfying (5.2.3). From L and W_0 we can construct a $W(s)$ satisfying $Z'(-s) + Z(s) = W'(-s)W(s)$. Applying the lossless constraint, we see that $W'(-s)W(s) = 0$. By setting $s = j\omega$, we conclude that $W'^*(j\omega)W(j\omega) = 0$ and thus $W(j\omega) = 0$ for all real ω. Therefore, $W(s) = W_0 + L'(sI - F)^{-1}G = 0$ for all s, whence $W_0 = 0$ and, by complete controllability,* $L = 0$. $\nabla\nabla\nabla$

Example 5.2.3 The impedance function $z(s) = s(s^2 + 2)/(s^2 + 1)(s^2 + 3)$ is lossless positive real, and one can check that a realization is given by

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & -4 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \quad J = 0$$

A matrix P satisfying (5.2.6) is given by

$$P = \begin{bmatrix} 6 & 0 & 3 & 0 \\ 0 & 11 & 0 & 2 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

(Procedures for computing P will be given subsequently.) It is easy to

*If $L'(sI - F)^{-1}G = 0$ for all s, then $L' \exp(Ft)G = 0$ for all t. Complete controllability of $[F, G]$ implies that $L' = 0$ by one of the complete controllability properties.

establish, by evaluating the leading minors of P, that P is positive definite.

Problem 5.2.1 Prove via the positive real lemma that positive realness of $Z(s)$ implies positive realness of $Z'(s)$, [30].

Problem 5.2.2 Assuming the positive real lemma as stated, prove the following extension of it: let $Z(s)$ be an $m \times m$ matrix of real rational transfer functions of a complex variable s, with $Z(\infty) < \infty$. Let $\{F, G, H, J\}$ be a completely controllable realization of $Z(s)$, with $[F, H]$ not necessarily completely observable. Then $Z(s)$ is positive real *if and only if there exist* real matrices P, L, and W_0, with P *nonnegative* definite symmetric, such that $PF + F'P = -LL'$, $PG = H - LW_0$, $J + J' = W_0'W_0$. [*Hint:* Use the fact that if $[F, H]$ is not completely observable, then there exists a nonsingular T such that

$$TFT^{-1} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \qquad H'T^{-1} = [H_1' \quad 0]$$

with $[F_{11}, H_1]$ completely observable.]

Problem 5.2.3 With $Z(s)$ positive real and $\{F, G, H, J\}$ a minimal realization, establish the existence of transfer-function matrices $V(s)$ satisfying $Z'(-s) + Z(s) = V(s)V'(-s)$, and possessing realizations with the same F and H matrices as $Z(s)$. [*Hint:* Apply the positive real lemma to $Z'(s)$, known to be positive real by Problem 5.2.1.]

Problem 5.2.4 Let $Z(s)$ be PR with minimal realization $\{F, G, H, J\}$. Suppose that J is nonsingular, so that $Z^{-1}(s)$ has a realization, actually minimal, $\{F - GJ^{-1}H', GJ^{-1}, -H(J^{-1})', J^{-1}\}$. Use the positive real lemma to establish that $Z^{-1}(s)$ is PR.

Problem 5.2.5 Let $Z(s)$ be an $m \times m$ matrix with realization $\{F, G, H, J\}$, and suppose that there exist real matrices P, L, and W_0 satisfying the positive real lemma equations, except that P is not necessarily positive definite, though it is symmetric. Show that $Z'^*(j\omega) + Z(j\omega) \geq 0$ for all real ω with $j\omega$ not a pole of any element of $Z(s)$.

5.3 POSITIVE REAL LEMMA: NECESSITY PROOF BASED ON NETWORK THEORY RESULTS*

Much of the material in later chapters will be concerned with the synthesis problem, i.e., passing from a prescribed positive real impedance matrix $Z(s)$ of rational functions to a network of resistor, capacitor, inductor, ideal transformer, and gyrator elements synthesizing it. Nevertheless, we shall

*This section may be omitted at a first, and even a second, reading.

base a necessity proof of the positive real lemma (i.e., a proof of the existence of P, L, and W_0 satisfying certain equations) on the assumption that given a positive real impedance matrix $Z(s)$ of rational functions, there exist networks synthesizing $Z(s)$. In fact, we shall assume more, viz., *that there exist networks synthesizing $Z(s)$ using precisely $\delta[Z(s)]$ reactive elements.* Classical network theory guarantees the existence of such *minimal reactive element syntheses* [6]. The reader may feel that we are laying the foundations for a circular argument, in the sense that we are assuming a result is true in order to prove the positive real lemma, when we are planning to use the positive real lemma to establish the result. In this restricted context, it is true that a circular argument is being used. However, we note that

1. There are classical synthesis procedures that establish the synthesis result.
2. There are other procedures for proving the positive real lemma, presented in following sections, that do not use the synthesis result.

Accordingly, the reader can regard this section as giving fresh insights into properties and results that can be *independently* established.

In this section we give two proofs due to Layton [7] and Anderson [8]. Alternative proofs, not relying on synthesis results, will be found in later sections.

Proof Based on Reactance Extraction

Layton's proof proceeds as follows. Suppose that $Z(s)$ has $Z(\infty) < \infty$, and that N is a network synthesizing $Z(s)$ with $\delta[Z(s)]$ reactive elements. Suppose also that $Z(s)$ is $m \times m$, and $\delta[Z(s)] = n$.

In general, both inductor and capacitor elements may be used in N. But without altering the total number of such elements, we can conceive first that all reactive elements have values of 1 H or 1 F as the case may be (using transformer normalizations if necessary), and then that all 1-F capacitors are replaced by gyrators of unit transimpedance terminated in 1-H inductors, as shown in Fig. 5.3.1.

This means that N is composed of resistors, transformers, gyrators, and precisely n inductors.

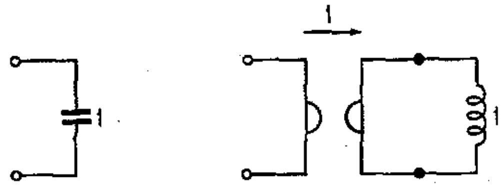

FIGURE 5.3.1. Elimination of Capacitors.

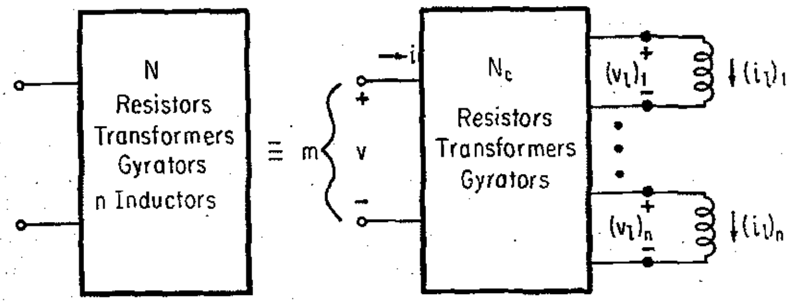

FIGURE 5.3.2. Redrawing of Network as Nondynamic Network Terminated in Inductors.

In Fig. 5.3.2 N is redrawn to emphasize the presence of the n inductors. Thus N is regarded as an $(m + n)$ port N_c—a coupling network of nondynamic elements, i.e., resistors, transformers, and gyrators—terminated in unit inductors at each of its last n ports. Let us now study N_c. We note the following preliminary result.

Lemma 5.3.1. With N_c as defined above, N_c possesses an impedance matrix

$$Z_c = \left[\begin{array}{c|c} z_{11} & z_{12} \\ \hline z_{21} & z_{22} \end{array}\right] \tag{5.3.1}$$

the partitioning being such that z_{11} is $m \times m$ and z_{22} is $n \times n$. Further, a minimal state-space realization for $Z(s)$ is provided by $\{-z_{22}, z_{21}, -z'_{12}, z_{11}\}$; i.e.,

$$Z(s) = z_{11} - z_{12}(sI + z_{22})^{-1}z_{21} \tag{5.3.2}$$

Proof. Let i_l be a vector whose entries are the instantaneous currents in the inductors of N (see Fig. 5.3.2). As established in Chapter 4, the fact that $Z(\infty) < \infty$ means that i_l works as a state vector for a state-space description of N. It is even a state vector of minimum possible dimension, viz., $\delta[Z(s)]$. Hence N has a description of the form

$$\frac{di_l}{dt} = F_N i_l + G_N i$$

$$v = H'_N i_l + J_N i$$

where i and v are the port current and voltage vectors for N. Let v_l be the vector of inductor voltages, with orientation chosen so that $v_l = di_l/dt$ (see Fig. 5.3.2). Thus

$$\begin{aligned} v_l &= F_N i_l + G_N i \\ v &= H'_N i_l + J_N i \end{aligned} \tag{5.3.3}$$

These equations describe the behavior of N_c when it is terminated in unit inductors, i.e., when $v_l = di_l/dt$. We claim that they also describe the performance of N_c when it is not terminated in unit inductors, and v_l is not constrained to be di_l/dt. For suppose that $i_l(0) = 0$ and that a current step is applied at the ports of N. Then on physical grounds the inductors act momentarily as open circuits, and so $v_l = G_N i(0+)$ and $v = J_N i(0+)$ momentarily, in the presence or absence of the inductors. Also, suppose that $i(t) \equiv 0$, $i_l(0) \neq 0$. The inductors now behave instantaneously like current sources $i_l(0)$ in parallel with an open circuit. So, *irrespective of the presence or absence of inductors*, i.e., irrespective of the mechanism producing $i_l(0)$, $v_l = F_N i_l(0)$ and $v = H'_N i_l(0)$ momentarily.

By superposition, it follows that (5.3.3) describes the performance of N_c at any fixed instant of time; then, because N_c is *constant, the equations describe* N_c over all time. Finally, by observing the orientations shown in Fig. 5.3.2, we see that N_c possesses an impedance matrix, with $z_{11} = J_N$, $z_{12} = -H'_N$, $z_{21} = G_N$, and $z_{22} = -F_N$. ∇ ∇ ∇

With this lemma in hand, the proof of the positive real lemma is straightforward. Suppose that $\{F, G, H, J\}$ is an arbitrary minimal realization of $Z(s)$. Then because $\{-z_{22}, z_{21}, -z'_{12}, z_{11}\}$ is another minimal realization, there exists a nonsingular matrix T such that $z_{22} = -TFT^{-1}$, $z_{21} = TG$, $z'_{12} = -(T^{-1})'H$, and $z_{11} = J$. The impedance matrix Z_c of N_c is thus

$$Z_c = \begin{bmatrix} J & -H'T^{-1} \\ TG & -TFT^{-1} \end{bmatrix}$$

Now N_c is a passive network, and a necessary and sufficient condition for this is that

$$Z_c + Z'_c \geq 0$$

or equivalently that

$$(I_m \dotplus T')(Z_c + Z'_c)(I_m \dotplus T) \geq 0 \tag{5.3.4}$$

where $\dotplus$ denotes the direct sum operation. Writing (5.3.4) out in terms of F, G, H, and J, and setting $P = T'T$, there results

$$\begin{bmatrix} J + J' & (-H + PG)' \\ -H + PG & -(PF + F'P) \end{bmatrix} \geq 0$$

(Note that because T is nonsingular, P will be positive definite.) Now any nonnegative definite matrix can be written in the form MM', with the number of columns of M greater than or equal to the rank of the matrix. Applying this result here, with the partition $M' = [-W_0 \mid L']$, we obtain

$$\begin{bmatrix} J + J' & (-H + PG)' \\ -H + PG & -(PF + F'P) \end{bmatrix} = \begin{bmatrix} W_0'W_0 & -(LW_0)' \\ -LW_0 & LL' \end{bmatrix}$$

or

$$PF + F'P = -LL'$$
$$PG = H - LW_0$$
$$J + J' = W_0'W_0$$

These are none other than the equations of the positive real lemma, and thus the proof is complete. ∇ ∇ ∇

We wish to stress several points. First, short of carrying out a synthesis of $Z(s)$, there is no technique suggested in the above proof for the computation of P, L, and W_0, given simply F, G, H, and J. Second, classical network theory exhibits an infinity of minimal reactive element syntheses of a prescribed $Z(s)$; the above remarks suggest, in a rough fashion, that to each such synthesis will correspond a P, in general differing from synthesis to synthesis. Therefore, we might expect an infinity of solutions P, L, W_0 of the positive real lemma equations. Third, the assumption of a minimal reactive synthesis above is critical; without it, the existence of T (and thus P) cannot be guaranteed.

Proof Based on Energy-Balance Arguments

Now we turn to a second proof that also assumes existence of a minimal reactive element synthesis.

As before, we assume the existence of a network N synthesizing $Z(s)$. Moreover, we suppose that N contains precisely $\delta[Z(s)]$ reactive elements (although we do not require that all reactive elements be inductive). Since $Z(\infty) < \infty$, there is a state-space description of N with state vector x_N whose entries are all inductor currents or capacitor voltages; assuming that all inductors are 1 H and capacitors 1 F (using transformer normalizations if necessary), we observe that the instantaneous energy stored in N is $\frac{1}{2}x_N'x_N$.

Now we also assume that there is prescribed a state-variable description $\dot{x} = Fx + Gu$, $y = H'x + Ju$, of minimal dimension. Therefore, there exists a nonsingular matrix T such that $Tx = x_N$, and then, with $P = T'T$, a positive definite symmetric matrix, we have that

$$\text{energy stored in } N = \tfrac{1}{2}x'Px \tag{5.3.5}$$

We shall also compute the power flowing into N and the rate of energy dissipation in N. These quantities and the derivative of (5.3.5) satisfy an energy-balance (more properly, a power-balance) equation from which we shall derive the positive real lemma equations. Thus we have, first,

$$\begin{aligned}\text{power flowing into } N &= (H'x + Ju)'u \\ &= x'Hu + \tfrac{1}{2}u'(J + J')u\end{aligned} \tag{5.3.6}$$

(Recall that u is the port current and $y = H'x + Ju$ is the port voltage of N.) Also, consider the currents in each resistor of N. At time t these must be linear functions of the state $x(t)$ and input current $u(t)$. With $i_R(t)$ denoting the vector of such currents, we have therefore

$$i_R(t) = Ax + Bu$$

for some matrices A and B. Let R be a diagonal matrix, the diagonal entries of which are the values of the network resistors. These values are real and nonnegative. Then

$$\text{rate of energy dissipation in } N = (Ax + Bu)'R(Ax + Bu) \tag{5.3.7}$$

Equating the power inflow with the sum of the rate of dissipation and rate of increase of stored energy yields

$$\frac{1}{2}u'(J + J')u + x'Hu = (Ax + Bu)'R(Ax + Bu) + \frac{d}{dt}\left(\frac{1}{2}x'Px\right)$$

We compute the derivative using $\dot{x} = Fx + Gu$ and obtain

$$\begin{aligned}&\tfrac{1}{2}u'(J + J')u + x'Hu \\ &\quad = u'B'RBu + x'(2A'RB + PG)u + x'(A'RA + \tfrac{1}{2}PF + \tfrac{1}{2}F'P)x\end{aligned}$$

Since this equation holds for all x and u, we have

$$J + J' = 2B'RB$$
$$H = PG + 2A'RB$$
$$0 = A'RA + \tfrac{1}{2}PF + \tfrac{1}{2}F'P$$

By setting $W_0 = \sqrt{2}R^{1/2}B$ and $L = \sqrt{2}A'R^{1/2}$, the equations $PF + F'P = -LL'$, $PG = H - LW_0$, and $J + J' = W_0'W_0$ are recovered. $\nabla\nabla\nabla$

Essentially the same comments can be made regarding the second proof of this section as were made regarding the first proof. There is little to choose between them from the point of view of the amount of insight they offer into the positive real lemma, although the second proof does offer the insight of energy-balance ideas, which is absent from the first proof. Both proofs assume that $Z(s)$ is a positive real *impedance* matrix; Problem 5.3.2 seeks a straightforward generalization to hybrid matrices.

Problem 5.3.1 *Define* a lossless network as one containing no resistors. Suppose that $Z(s)$ is positive real and the impedance of a lossless network. Suppose that $Z(s)$ is synthesizable using $\delta[Z(s)]$ reactive elements. Show that if $\{F, G, H, J\}$ is a minimal realization of $Z(s)$, there exists a real positive definite symmetric P such that

$$PF + F'P = 0$$
$$PG = H$$
$$J + J' = 0$$

(Use an argument based on the second proof of this section.) This proves that $Z(s)$ is a lossless positive real impedance in the sense of our earlier definition.

Problem 5.3.2 Suppose that $M(s)$ is an $n \times n$ hybrid matrix with minimal realization $\{F, G, H, J\}$. Prove that if $M(s)$ is synthesizable by a network with $\delta[M(s)]$ reactive elements, then there exist a positive definite symmetric P and matrices L and W_0, all real, such that $PF + F'P = -LL'$, $PG = H - LW_0$, and $J + J' = W_0'W_0$.

5.4 POSITIVE REAL LEMMA: NECESSITY PROOF BASED ON A VARIATIONAL PROBLEM*

In this section we present another necessity proof for the positive real lemma; i.e., starting with a rational positive real $Z(s)$ with minimal

*This section may be omitted at a first, and even a second, reading.

realization $\{F, G, H, J\}$, we prove the existence of a matrix $P = P' > 0$ and matrices L and W_0 for which

$$\begin{aligned} PF + F'P &= -LL' \\ PG &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{5.4.1}$$

Of course, $Z(\infty) < \infty$. *But to avoid many complications, we shall impose the restriction for most of this section that $J + J'$ be nonsingular.* The assumption that $J + J'$ is nonsingular serves to ensure that the variational problem we formulate has no singular solutions. When $J + J'$ is singular, we are forced to go through a sequence of manipulations that effectively allow us to replace $J + J'$ by a nonsingular matrix, so that the variational procedure may then be used.

To begin with, we shall pose and solve a variational problem with the constraint in force that $J + J'$ is nonsingular; the solvability of this problem is guaranteed precisely by the positive real property [or, more precisely, a time-domain interpretation of it involving the impulse response matrix $J\,\delta(t) + H'e^{Ft}G1(t)$, with $\delta(\cdot)$ the unit impulse, and $1(t)$ the unit step function]. The solution of the variational problem will involve a matrix that essentially gives us the P matrix of (5.4.1); the remaining matrices in (5.4.1) then prove easy to construct once P is known. Theorem 5.4.1 contains the main result used to generate P, L, and W_0.

In subsequent material of this section, we discuss the connection between Eq. (5.4.1) and the existence of a spectral factor $W(s) = W_0 + L'(sI - F)^{-1}G$ of $Z'(-s) + Z(s)$. We shall note frequency-domain properties of $W(\cdot)$ that follow from a deeper analysis of the variational problem. In particular, we shall consider conditions guaranteeing that rank $W(s)$ be constant in Re $[s] > 0$ and in Re $[s] \geq 0$. Finally, we shall remark on the removal of the nonsingularity constraint on $J + J'$.

Time-Domain Statement of the Positive Real Property

The positive real property of $Z(s)$ is equivalent to the following inequality:

$$\int_{t_0}^{t_1} u'(t)\left\{\int_{t_0}^{t} [J\,\delta(t-\tau) + H'e^{F(t-\tau)}G1(t-\tau)]u(\tau)\,d\tau\right\} dt \geq 0 \tag{5.4.2}$$

for all t_0, t_1, and $u(\cdot)$ for which the integrals exist. (It is assumed that $t_1 > t_0$.) We assume that the reader is capable of deriving this condition from the positive real condition on $Z(s)$. (It may be done, for example, by using a matrix version of Parseval's theorem.) The inequality has an obvious physical interpretation: Suppose that a system with impulse response $J\,\delta(t) + H'e^{Ft}G1(t)$ is initially in the zero state, and is excited with an input $u(\cdot)$. If $y(\cdot)$ is the corresponding output, $\int_{t_0}^{t_1} y'(t)u(t)\,dt$ is the energy flow into the system up to time t_1. This quantity may also be expressed as the integral on the left side of (5.4.2); its nonnegativity corresponds to passivity of the associated system.

In the following, our main application of the positive real property will be via (5.4.2). We also recall from the positive real definition that F can have no eigenvalues with positive real part, nor multiple eigenvalues with zero real part.

Throughout this section until further notice we shall use the symmetric matrix R, assumed positive definite, and defined by

$$R = J + J' \tag{5.4.3}$$

(Of course, the positive real property guarantees nonnegativeness of $J + J'$. Nonsingularity is a special assumption however.)

A Variational Problem. Consider the following minimization problem. Given the system

$$\dot{x} = Fx + Gu \qquad x(0) = x_0 \tag{5.4.4}$$

find $u(\cdot)$ so as to minimize

$$V(x_0, u(\cdot), t_1) = \int_0^{t_1} (u'Ru + 2x'Hu)\,dt \tag{5.4.5}$$

[Of course, $u(\cdot)$ is assumed constrained a priori to be such that the integrand in (5.4.5) is actually integrable.]

The positive real property is related to the variational problem in a simple way:

Lemma 5.4.1. The performance index (5.4.5) is bounded below for all x_0, $u(\cdot)$, t_1 independently of $u(\cdot)$ and t_1 if and only if $Z(s) = J + H'(sI - F)^{-1}G$ is positive real.

Proof First, suppose that $Z(\cdot)$ is not positive real. Then we can prove that V is not bounded below, as follows: Let $u_\alpha(\cdot)$ be a control defined on $[0, t_\alpha]$ taking (5.4.4) to the zero state at an arbitrary fixed time $t_\alpha \geq 0$. Existence of $u_\alpha(\cdot)$ follows from complete controllability. Let $u_\beta(\cdot)$ be a control defined on $[t_\alpha, t_\beta]$, when t_β and $u_\beta(\cdot)$ are chosen so that

$$\int_{t_\alpha}^{t_\beta} u'_\beta(t)\left\{\int_{t_\alpha}^{t} [J\,\delta(t-\tau) + H'e^{F(t-\tau)}G1(t-\tau)]u_\beta(\tau)\,d\tau\right\} dt < 0$$

Existence of t_β and $u_\beta(\cdot)$ follows by failure of the positive real property. Denoting the above integral by I, we observe that

$$2I = \int_{t_\alpha}^{t_\beta} (u'_\beta R u_\beta + 2x'Hu_\beta)\,dt$$

Now define $u_k(\cdot)$ as the concatenation of u_α and ku_β, where k is a constant, and set $t_1 = t_\beta$. Then

$$V(x_0, u_k(\cdot), t_1) = \int_0^{t_\alpha} (u'_\alpha R u_\alpha + 2x'Hu_\alpha)dt + 2k^2 I$$

The first term on the right side is independent of k, while the second can be made as negative as desired through choice of appropriately large k. Hence $V(x_0, u_k(\cdot), t_1)$ is not bounded below independently of k, and $V(x_0, u(\cdot), t_1)$ cannot be bounded below independently of $u(\cdot)$ and t_1.

For the converse, suppose that $Z(s)$ is positive real; we shall prove the boundedness property of the performance index. Let $t_\gamma < 0$ and $u_\gamma(\cdot)$ defined on $[t_\gamma, 0]$ be such that $u_\gamma(\cdot)$ takes the zero state at time t_γ to the state x_0 at time zero. Then if $u(\cdot)$ is defined on $[t_\gamma, t_1]$, is equal to u_γ on $[t_\gamma, 0]$, and is otherwise arbitrary, we have

$$\begin{aligned} V(x_0, u(\cdot), t_1) &= \int_{t_\gamma}^{t_1} (u'Ru + 2x'Hu)\,dt - \int_{t_\gamma}^{0} (u'_\gamma R u_\gamma + 2x'Hu_\gamma)\,dt \\ &\geq -\int_{t_\gamma}^{0} (u'_\gamma R u_\gamma + 2x'Hu_\gamma)\,dt \end{aligned}$$

since, by the constraint (5.4.2), the first integral on the right of the equality is nonnegative for all $u(\cdot)$ and t_1. The integral on the right of the inequality is independent of $u(\cdot)$ on $[0, t_1]$ and of t_1, but depends on x_0 since $u_\gamma(\cdot)$ depends on x_0. That is, $V(x_0, u(\cdot), t_1) \geq k(x_0)$ for some scalar function k taking finite values. ∇ ∇ ∇

Although the lemma guarantees a lower bound on any performance index value, including therefore the value of the optimal performance index, we have not obtained any upper bound. But such is easy to obtain. Reference to (5.4.5) shows that $V(x_0, u(t) \equiv 0, t_1) = 0$ for all x_0 and t_1, and so *the optimal performance index is bounded above by zero for all* x_0 *and* t_1.

Calculation of the optimal performance index proceeds much as in the standard quadratic regulator problem (see, e.g., [9, 10]). Since a review of this material could take us too far afield, we merely state here the consequences of carrying through a standard argument.

The optimal performance index associated with (5.4.5), i.e., $\min_{u(\cdot)} V(x_0, u(\cdot), t_1) = V^0(x_0, t_1)$, is given by

$$V^0(x_0, t_1) = x_0'\Pi(0, t_1)x_0 \tag{5.4.6}$$

where $\Pi(\cdot, t_1)$ is a symmetric matrix defined as the solution of the Riccati equation

$$\begin{aligned} -\dot{\Pi} &= \Pi(F - GR^{-1}H') + (F' - HR^{-1}G')\Pi \\ &\quad - \Pi GR^{-1}G'\Pi - HR^{-1}H' \\ \Pi(t_1, t_1) &= 0 \end{aligned} \tag{5.4.7}$$

The associated optimal control, call it $u^0(\cdot)$, is given by

$$u^0(t) = -R^{-1}[G'\Pi(t, t_1) + H']x(t) \tag{5.4.8}$$

Since (5.4.7) is a nonlinear equation, the question arises as to whether it has a solution outside a neighborhood of t_1. The answer is yes. Again the reasoning is similar to the standard quadratic regulator problem. Because we know that

$$k(x_0) \leq V^0(x_0, t_1) = x_0'\Pi(0, t_1)x_0 \leq 0$$

for some scalar function $k(\cdot)$ of x_0, we can ensure that as t_1 increases away from the origin, $\Pi(0, t_1)$ cannot have any element become unbounded; i.e., $\Pi(0, t_1)$ exists for all $t_1 \geq 0$. Since the constant coefficient matrices in (5.4.7) guarantee $\Pi(t, t_1) = \Pi(0, t_1 - t)$, it follows that if t_1 is fixed and (5.4.7) solved backward in time, $\Pi(t, t_1)$ exists for all $t \leq t_1$, since $\Pi(0, t_1 - t)$ exists for all $t_1 - t \geq 0$.

Having now established the existence of $\Pi(t, t_1)$, we wish to establish the limiting property contained in the following lemma:

Lemma 5.4.2 [*Limiting Property of* $\Pi(t, t_1)$]. Suppose that $Z(\cdot)$ is positive real, so that the matrix $\Pi(t, t_1)$ defined by (5.4.7) exists for all $t \leq t_1$. Then

$$\lim_{t_1 \to \infty} \Pi(t, t_1) = \bar{\Pi}$$

exists and is independent of t; moreover, $\bar{\Pi}$ satisfies a limiting version of the differential equation part of (5.4.7), viz.,

$$\begin{aligned} &\bar{\Pi}(F - GR^{-1}H') + (F' - HR^{-1}G')\bar{\Pi} \\ &\qquad - \bar{\Pi}GR^{-1}G'\bar{\Pi} - HR^{-1}H' = 0 \end{aligned} \tag{5.4.9}$$

Proof. We show first that $\Pi(0, t_1)$ is a monotonically decreasing matrix function of t_1. Let u_2^0 be a control optimum for initial state x_0, final time t_2, and let $u(\cdot)$ equal u_2^0 on $[0, t_2]$, and equal zero after t_2. Then for arbitrary $t_1 > t_2$,

$$\begin{aligned} x_0'\Pi(0, t_1)x_0 &= V^0(x_0, t_1) \\ &\leq V(x_0, u(\cdot), t_1) \\ &= V^0(x_0, t_2) \end{aligned}$$

or

$$x_0'\Pi(0, t_1)x_0 \leq x_0'\Pi(0, t_2)x_0$$

The existence of the lower bound $k(x_0)$ on $x_0'\Pi(0, t_1)x_0$, independent of t_1, and the just proved monotonicity then show after a slightly nontrivial argument that $\lim_{t_1 \to \infty} \Pi(0, t_1)$ exists. Let this limit be $\bar{\Pi}$. Since $\Pi(t, t_1) = \Pi(0, t_1 - t)$, as noted earlier, $\lim_{t_1 \to \infty} \Pi(t, t_1) = \bar{\Pi}$ for arbitrary fixed t. This proves the first part of the lemma.

Introduce the notation $\Pi(t, t_1, A)$ to denote the solution of (5.4.7) with boundary condition $\Pi(t_1, t_1) = A$, for any $A = A'$. A standard differential equation property yields that

$$\Pi(t, t_1, \Pi(t_1, t_2)) = \Pi(t, t_2)$$

Also, solutions of (5.4.7) are continuous in the boundary condition, at least in the vicinity of the end point. Thus for t close to t_1,

$$\begin{aligned} \lim_{t_2 \to \infty} \Pi(t, t_1, \Pi(t_1, t_2)) &= \Pi(t, t_1, \lim_{t_2 \to \infty} \Pi(t_1, t_2)) \\ &= \Pi(t, t_1, \bar{\Pi}) \end{aligned}$$

But also

$$\begin{aligned} \lim_{t_2 \to \infty} \Pi(t, t_1, \Pi(t_1, t_2)) &= \lim_{t_2 \to \infty} \Pi(t, t_2) \\ &= \bar{\Pi} \end{aligned}$$

So the boundary condition $\Pi(t_1, t_1) = \bar{\Pi}$ leads, in the vicinity of t_1, to a constant solution $\bar{\Pi}$. Then $\bar{\Pi}$ must be a global solution; i.e., (5.4.9) holds. $\nabla \nabla \nabla$

The computation of $\bar{\Pi}$ via the formula contained in the lemma [i.e., compute $\Pi(t, t_1)$ for fixed t and a sequence of increasing values of t_1 by repeated solutions of the Riccati equation (5.4.7)] would clearly be cumbersome. But since $\Pi(t, t_1) = \Pi(0, t_1 - t)$, it follows that

$$\lim_{t_1 \to \infty} \Pi(t, t_1) = \lim_{t \to -\infty} \Pi(t, t_1)$$

and so in practice the formula

$$\bar{\Pi} = \lim_{t \to -\infty} \Pi(t, t_1) \tag{5.4.10}$$

would be more appropriate for computation of $\bar{\Pi}$.

Example 5.4.1 Consider the positive real function $z(s) = \frac{1}{2} + (s + 1)^{-1}$, for which a minimal realization is provided by $(-1, 1, 1, \frac{1}{2})$. The Riccati equation (5.4.7) becomes, with $R = 1$,

$$-\dot{\Pi} = -4\Pi - \Pi^2 - 1 \qquad \Pi(t_1, t_1) = 0$$

of which a solution may be found to be

$$\Pi(t, t_1) = \frac{(2 - \sqrt{3})[e^{2\sqrt{3}(t-t_1)} - 1]}{1 - [(2 - \sqrt{3})/(2 + \sqrt{3})]e^{2\sqrt{3}(t-t_1)}}$$

Observe that $\Pi(t, t_1)$ exists for all $t \leq t_1$, because the denominator never becomes zero. Also, we see that $\Pi(t, t_1) = \Pi(0, t_1 - t)$, and

$$\lim_{t_1 \to \infty} \Pi(t, t_1) = \lim_{t \to -\infty} \Pi(t, t_1) = \sqrt{3} - 2$$

Finally, notice that $\bar{\Pi} = \sqrt{3} - 2$ satisfies

$$-4\bar{\Pi} - \bar{\Pi}^2 - 1 = 0$$

Before spelling out the necessity proof of the positive real lemma, we need one last fact concerning $\bar{\Pi}$.

Lemma 5.4.3. If $\bar{\Pi}$ is defined as described in Lemma 5.4.2, then $\bar{\Pi}$ is negative definite.

Proof Certainly, $\bar{\Pi}$ is nonpositive definite. For as remarked before, $x_0'\Pi(0, t_1)x_0 \leq 0$ for all x_0, and we found $\bar{\Pi} \leq \Pi(0, t_1)$ in proving Lemma 5.4.2. Suppose that $\bar{\Pi}$ is singular. We argue by contradiction. Let x_0 be a nonzero vector such that $x_0'\bar{\Pi}x_0 = 0$. Then $x_0'\Pi(0, t_1)x_0 = 0$ for any t_1. Now optimal-control theory tells us that the minimization of $V(x_0, u(\cdot), t_1)$ is achieved with a *unique* optimal control [9, 10]. The minimum value of the index is 0, and inspection of the integral used in computing the index shows that $u(t) \equiv 0$ leads to a value of the index of zero. Hence $u(t) \equiv 0$ must be the optimal control.

By the principle of optimality, $u(t) \equiv 0$ is optimal if we restrict attention to a problem over the time interval $[t_\alpha, t_1]$ for any $t_1 > t_\alpha > 0$, provided we take as $x(t_\alpha)$ the state at time t_α that arises when the optimal control for the original optimal-control problem is applied over $[0, t_\alpha]$; i.e., $x(t_\alpha) = \exp(Ft_\alpha)x_0$. Therefore, by stationarity, $u(t) \equiv 0$ is the optimal control if the end point is $t_1 - t_\alpha$ and the initial state $\exp(Ft_\alpha)x_0$. The optimal performance index is also zero; i.e.,

$$\begin{aligned} x_0' \exp(F't_\alpha)\Pi(0, t_1 - t_\alpha) \exp(Ft_\alpha)x_0 \\ &= x_0' \exp(F't_\alpha)\Pi(t_\alpha, t_1) \exp(Ft_\alpha)x_0 \\ &= 0 \end{aligned}$$

This implies that $\Pi(t_\alpha, t_1) \exp(Ft_\alpha)x_0 = 0$ in view of the nonpositive definite nature of $\Pi(t_\alpha, t_1)$.

Now consider the original problem again. As we know, the optimal control is given by

$$u^0(t) = -R^{-1}[G'\Pi(t, t_1) + H']x(t) \tag{5.4.8}$$

and is identically zero. Setting $x(t) = \exp(Ft)x_0$ and using the fact that $\Pi(t_\alpha, t_1) \exp(Ft_\alpha)x_0 = 0$ for all $0 \leq t_\alpha \leq t_1$, it follows that $H' \exp(Ft)x_0$ is identically zero. This contradicts the minimality of $\{F, G, H, J\}$, which demands complete observability of $[F, H]$. $\nabla\nabla\nabla$

Let us sum up what we have proved so far in a single theorem.

Theorem 5.4.1. (*Existence, Construction, and Properties of* $\bar{\Pi}$) Let $Z(s)$ be a positive real matrix of rational functions of s, with $Z(\infty) < \infty$. Suppose that $\{F, G, H, J\}$ is a minimal realization of $Z(\cdot)$, with $J + J' = R$ nonsingular. Then there exists a negative definite matrix $\bar{\Pi}$ satisfying the equation

$$\begin{aligned} \bar{\Pi}(F - GR^{-1}H') + (F' - HR^{-1}G')\bar{\Pi} \\ - \bar{\Pi}GR^{-1}G'\bar{\Pi} - HR^{-1}H' = 0 \end{aligned} \tag{5.4.9}$$

Moreover, $\bar{\Pi} = \lim_{t_1 \to \infty} \Pi(t, t_1) = \lim_{t \to -\infty} \Pi(t, t_1)$, where $\Pi(\cdot, t_1)$ is the solution of the Riccati equation

$$\begin{aligned} -\dot{\Pi} = \Pi(F - GR^{-1}H') + (F' - HR^{-1}G')\Pi \\ - \Pi GR^{-1}G'\Pi - HR^{-1}H' \end{aligned} \tag{5.4.7}$$

with boundary condition $\Pi(t_1, t_1) = 0$.

Necessity Proof of the Positive Real Lemma

The summarizing theorem contains all the material we need to establish the positive real lemma equations (5.4.1), under, we recall, the mildly restrictive condition of nonsingularity on $J + J' = R$.

We define

$$P = -\bar{\Pi} \qquad W_0 = R^{1/2} \qquad L = (\bar{\Pi}G + H)R^{-1/2} \tag{5.4.11}$$

Observe that $P = P' > 0$ follows by the negative definite property of $\bar{\Pi}$. The first of equations (5.4.1) follows by rearranging (5.4.9) as

$$\bar{\Pi}F + F'\bar{\Pi} = (\bar{\Pi}G + H)R^{-1}(\bar{\Pi}G + H)'$$

and making the necessary substitutions. The second and third equations follow immediately from the definitions (5.4.11). We have therefore constructed a set of matrices P, L, and W_0 that satisfy the positive real lemma equations. $\nabla\nabla\nabla$

Example 5.4.2 For $z(s) = \frac{1}{2} + (s + 1)^{-1}$ in Example 5.4.1, we obtained a minimal realization $\{-1, 1, 1, \frac{1}{2}\}$ and found the associated $\bar{\Pi}$ to be $\sqrt{3} - 2$. According to (5.4.11), we should then take

$$P = 2 - \sqrt{3} \qquad W_0 = 1 \qquad L = \sqrt{3} - 1$$

Observe then that

$$PF + F'P = -4 + 2\sqrt{3} = -(\sqrt{3} - 1)^2 = -LL'$$
$$PG = 2 - \sqrt{3} = 1 - (\sqrt{3} - 1) = H - LW_0$$

and

$$W_0'W_0 = 1 = J + J'$$

In providing the above necessity proof of the positive real lemma, we have also provided a constructive procedure leading to one set of matrices P, L, and W_0 satisfying the associated equations. As it turns out though, there are an infinity of solutions of the equations. Later we shall study techniques for their computation.

One might well ask then whether the P, L, and W_0 we have constructed have any distinguishing properties other than the obvious property of constructability according to the technique outlined above. The answer is yes, and the property is best described with the aid of the spectral factor $R^{1/2} + L'(sI - F)^{-1}G$ of $Z'(-s) + Z(s)$.

Spectral-Factor Properties

It will probably be well known to control and network theorists that in general the operation of spectral factorization leads to nonunique spectral factors; i.e., given a positive real $Z(s)$, there will not be a unique $W(s)$ such that

$$Z'(-s) + Z(s) = W'(-s)W(s) \tag{5.4.12}$$

Indeed, as we shall see later, even if we constrain $W(s)$ to have a minimal realization quadruple of the form $\{F, G, L, W_0\}$ i.e., one with the same F and G matrices as $Z(s)$, $W(s)$ is still not uniquely specified. Though the poles of elements of $W(s)$ are uniquely specified, the poles of elements of $W^{-1}(s)$, assuming for the moment nonsingularity of $W(s)$, turn out to be nonuniquely specified.

In both control and network theory there is sometimes interest in constraining the poles of elements of $W^{-1}(s)$; typically it is demanded that entries of $W^{-1}(s)$ be analytic in one of the regions $\text{Re}\,[s] > 0$, $\text{Re}\,[s] \geq 0$, $\text{Re}\,[s] < 0$, or $\text{Re}\,[s] \leq 0$. In the case in which $W(s)$ is singular, the constraint becomes one of demanding constant rank of $W(s)$ in these regions, excluding points where an entry of $W(s)$ has a pole.

Below, we give two theorems, one guaranteeing analyticity in $\text{Re}\,[s] > 0$ of the elements of the inverse of that particular spectral factor $W(s)$ formed via the procedure spelled out earlier in this section. The second theorem, by imposing extra constraints on $Z(s)$, yields a condition guaranteeing analyticity in $\text{Re}\,[s] \geq 0$.

Theorem 5.4.2. Let $Z(s)$ be a positive real matrix of rational functions of s, with $Z(\infty) < \infty$. Let $\{F, G, H, J\}$ be a minimal realization with $J + J' = R$ nonsingular. Let $\bar{\Pi}$ be the matrix whose construction is described in the statement of Theorem 5.4.1. Then a spectral factor $W(s)$ of $Z'(-s) + Z(s)$ is defined by

$$W(s) = R^{1/2} + R^{-1/2}(\bar{\Pi}G + H)'(sI - F)^{-1}G \tag{5.4.13}$$

Moreover, $W^{-1}(s)$ exists in $\text{Re}\,[s] > 0$; i.e., all entries of $W^{-1}(s)$ are analytic in this region.

The proof will follow the following lemma.

Lemma 5.4.4. With $W(s)$ defined as in (5.4.13), the following equation holds:

$$\det\,[sI - F + GR^{-1}(\bar{\Pi}G + H)'] = \det R^{-1/2} \det\,(sI - F) \det W(s) \tag{5.4.14}$$

Proof*

$$\begin{aligned}
&\det[sI - F + GR^{-1}(\bar{\Pi}G + H)'] \\
&= \det(sI - F)\det[I + (sI - F)^{-1}GR^{-1}(\bar{\Pi}G + H)'] \\
&= \det(sI - F)\det[I + R^{-1}(\bar{\Pi}G + H)'(sI - F)^{-1}G] \\
&= \det(sI - F)\det R^{-1/2}\det[R^{1/2} + R^{-1/2}(\bar{\Pi}G + H)'(sI - F)^{-1}G] \\
&= \det R^{-1/2}\det(sI - F)\det W(s) \qquad \nabla\nabla\nabla
\end{aligned}$$

This lemma will be used in the proofs of both Theorems 5.4.2 and 5.4.3.

Proof of Theorem 5.4.2. First note that the argument justifying that $W(s)$ is a spectral factor of $Z'(-s) + Z(s)$ depends on

1. The definition of matrices P, L, and W_0 satisfying the positive real lemma equations in terms of F, G, H, R, and $\bar{\Pi}$, as contained in the previous subsection.
2. The observation made in an earlier section that if P, L, and W_0 are matrices satisfying the positive real lemma equations, then $W_0 + L'(sI - F)^{-1}G$ is a spectral factor of $Z'(-s) + Z(s)$.

In view of Lemma 5.4.4 and the fact that F can have no eigenvalues in $\text{Re}[s] > 0$, it follows that we have to prove that all eigenvalues of $F - GR^{-1}(\bar{\Pi}G + H)'$ lie in $\text{Re}[s] \leq 0$ in order to establish that $W^{-1}(s)$ exists throughout $\text{Re}[s] > 0$. This we shall now do. For convenience, set

$$\bar{F} = F - GR^{-1}(\bar{\Pi}G + H)' \tag{5.4.15}$$

From (5.4.7) and (5.4.9) it follows, after a little manipulation, and with the definition $V(t) = \Pi(t, t_1) - \bar{\Pi}$, that

$$\begin{aligned}
\dot{V} &= -V\bar{F} - \bar{F}'V + VGR^{-1}G'V \\
V(t_1) &= -\bar{\Pi}
\end{aligned}$$

The matrix $V(t)$ is defined for all $t \leq t_1$, and by definition of $\bar{\Pi}$ as $\lim_{t \to -\infty} \Pi(t, t_1)$, we must have $\lim_{t \to -\infty} V(t) = 0$. By the monotonicity property of $\Pi(0, t_1)$, it is easily argued that $V(t)$ is nonnegative definite for all t. Setting $X(t) = V(t_1 - t)$, it follows that $\lim_{t \to \infty} X(t) = 0$, and that

*This proof uses the identity $\det[I + AB] = \det[I + BA]$.

$$\dot{X} = X\bar{F} + \bar{F}'X - XGR^{-1}G'X$$
$$X(0) = -\bar{\Pi}$$

with $X(t)$ nonnegative definite for all t.

Since $\bar{\Pi}$ is nonsingular by Theorem 5.4.1, $X^{-1}(t)$ exists, at least near $t = 0$. Setting $Y(t) = X^{-1}(t)$, it is easy to check that

$$\dot{Y} = -\bar{F}Y - Y\bar{F}' + GR^{-1}G'$$
$$Y(0) = (-\bar{\Pi})^{-1} \tag{5.4.16}$$

Moreover, since this equation is linear, it follows that $Y(t) = X^{-1}(t)$ exists for all $t \geq 0$. The matrix $Y(t)$ must be positive definite, because $X(t)$ is positive definite. Also, for any real vector m, it is known that $[m'X^{-1}(t)m][m'X(t)m] \geq (m'm)^2$, and since $\lim_{t\to+\infty} X(t) = 0$, $m'Y(t)m$ must diverge for any nonzero m.

Using these properties of $Y(t)$, we can now easily establish that $\bar{F}$ has no eigenvalue with positive real part. Suppose to the contrary, and for the moment suppose that $\bar{F}$ has a real positive eigenvalue λ. Let m be the associated eigenvector of $\bar{F}'$. Multiply (5.4.16) on the left by m' and on the right by m; set $y(t) = m'Y(t)m$ and $q = m'GR^{-1}G'm$. Then

$$\dot{y} = -2\lambda y + q$$

with $y(0) = -m'\bar{\Pi}^{-1}m \neq 0$. The solution $y(t)$ of this equation is obviously bounded for all t, which is a contradiction. If $\bar{F}$ has no real positive eigenvalue, but a complex eigenvalue with positive real part, we can proceed similarly to show there exists a complex nonzero $m = m_1 + jm_2$, m_1 and m_2 real, such that $m_1'Y(t)m_1$ and $m_2'Y(t)m_2$ are bounded. Again this is a contradiction. $\nabla\nabla\nabla$

Let us now attempt to improve on this result, to the extent of guaranteeing existence of $W^{-1}(s)$ in Re $[s] \geq 0$, rather than Re $[s] > 0$. The task is a simple one, with Theorem 5.4.2 and Lemma 5.4.4 in hand. Recall that

$$Z(s) + Z'(-s) = W'(-s)W(s) \tag{5.4.12}$$

and so, setting $s = j\omega$ and noting that $W(s)$ is *real* rational,

$$Z(j\omega) + Z'(-j\omega) = W'^*(j\omega)W(j\omega) \tag{5.4.17}$$

where the superscript asterisk denotes as usual complex conjugation.

If $j\omega$ is not a pole of any element of $Z(\cdot)$, $W(j\omega)$ is nonsingular if and only if $Z(j\omega) + Z'(-j\omega)$ is positive definite, as distinct from merely nonnegative definite.

If $j\omega$ is a pole of some element of $Z(\cdot)$ for $\omega = \omega_0$, say, i.e., $j\omega_0$ is an eigenvalue of F, we would imagine from the definition $W(s) = W_0 + L'(sI - F)^{-1}G$ that $W(s)$ would have some element with a pole at $j\omega_0$, and therefore care would need to be exercised in examining $W^{-1}(s)$. This is slightly misleading: As we now show, there is some sort of cancellation between the numerator and the denominator of each element of $W(s)$, which means that $W(s)$ does not have an element with a pole at $s = j\omega_0$.

As discussed in Section 5.1, we can write

$$Z(s) = Z_1(s) + \frac{As + B}{s^2 + \omega_0^2}$$

where no element of $Z_1(s)$ has a pole at $j\omega_0$, $Z_1(s)$ is positive real, A is real symmetric, and B is real and skew symmetric. Immediately,

$$Z(s) + Z'(-s) = Z_1(s) + Z_1'(-s) \tag{5.4.18}$$

Therefore, as $s \to j\omega_0$, though elements of each summand on the left side behave unboundedly, the sum does not—every element of $Z_1(s)$ and $Z_1'(-s)$ being analytic at $s = j\omega_0$. Therefore [see (5.4.17)], no element of $W(s)$ can have a pole at $s = j\omega_0$, and $W^{-1}(j\omega)$ exists again precisely when $Z(j\omega) + Z'(-j\omega)$, *interpreted as in* (5.4.18), is nonsingular.

We can combine Theorem 5.4.2 with the above remarks to conclude

Theorem 5.4.3. Suppose that the same hypotheses apply as in the case of Theorem 5.4.2, save that also $Z(j\omega) + Z'(-j\omega)$ is positive definite for all ω, including those ω for which $j\omega$ is a pole of an element of $Z(s)$. Then the spectral factor $W(s)$ defined in Theorem 5.4.2 is such that $W^{-1}(s)$ exists in $\text{Re}[s] \geq 0$; i.e., all entries of $W^{-1}(s)$ are analytic in this region.

Example 5.4.3 We have earlier found that for the impedance $\frac{1}{2} + (s + 1)^{-1}$, a spectral factor resulting from computing $\bar{\Pi}$ is

$$W(s) = 1 + (\sqrt{3} - 1)(s + 1)^{-1}$$
$$= \frac{s + \sqrt{3}}{s + 1}$$

We observe that $W^{-1}(s)$ is analytic in $\text{Re}[s] > 0$, as required, and in fact in $\text{Re}[s] \geq 0$. The fact that $W^{-1}(s)$ has no pole on $\text{Re}[s] = 0$ results from the positivity of $z(j\omega) + z(-j\omega)$, verifiable as follows:

$$z(j\omega) + z(-j\omega) = 1 + \frac{1}{j\omega + 1} + \frac{1}{-j\omega + 1} = 1 + \frac{2}{\omega^2 + 1} \geq 1$$

Extension to Case of Singular $J + J'$

In the foregoing we have given a constructive proof for the existence of solutions to the positive real lemma equations, under the assumption that $J + J'$ is nonsingular. We now wish to make several remarks applying to the case when $J + J'$ is singular. These remarks will be made in response to the following two questions:

1. Given a problem of synthesizing a positive real $Z(s)$ for which $Z(\infty) + Z'(-\infty)$ or $J + J'$ is singular, can simple preliminary synthesis steps be carried out to reduce the problem to one of synthesizing another positive real impedance, $\hat{Z}(s)$ say, for which $\hat{J} + \hat{J}'$ is nonsingular?
2. Forgetting the synthesis problem for the moment, can we set up a variational problem for a matrix $Z(s)$ with $J + J'$ singular along the lines of the case where $J + J'$ is nonsingular, compute a P matrix via this procedure, and obtain a spectral factor $W(s)$ with essentially the same properties as above?

The answer to both these questions is yes. We shall elsewhere in this book give a reasoned answer to question 1; the preliminary synthesis steps require techniques such as series inductor and gyrator extraction, cascade transformer extraction, and shunt capacitor extraction—all operations that are standard in the (multiport generalization of the) Foster preamble.

We shall not present a reasoned answer to question 2 here. The reader may consult [11]. Essentially, what happens is that a second positive real impedance $Z_1(s)$ is formed from $Z(s)$, which possesses the same set of P matrix solutions of the positive real lemma equations. The matrix $Z_1(s)$ has a minimal realization with the same F and G as $Z(s)$, but different H and J. Also, $Z_1(\infty) + Z_1'(-\infty)$ is nonsingular.

The Riccati equation technique can be used on $Z_1(s)$, and this yields a P matrix satisfying the positive real lemma equations for $Z(s)$. This matrix P enables a spectral factor $W(s)$ to be defined, satisfying $Z(s) + Z'(-s) = W'(-s)W(s)$, and with $W(s)$ of constant rank in $\text{Re}\,[s] > 0$.

The construction of $Z_1(s)$ from $Z(s)$ proceeds by a sequence of operations on $Z(s) + Z'(-s)$, which multiply this matrix on the left and right by constant matrices, and diagonal matrices whose nonzero entries are powers of s. The algorithm is straightforward to perform, but can only be stated with a great deal of algebraic detail.

A third approach to the singular problem appears in Problem 5.4.3. This approach is of theoretical interest, but appears to lack any computational utility.

Problem 5.4.1 The impedance $z(s) = 1 - 1/(s + 1)$ is positive real and possesses a minimal realization $\{-1, 1, -1, 1\}$. Apply the technique outlined in the statements of Theorems 5.4.1 and 5.4.2 to find a solution of the positive real lemma equations and a spectral factor with "stable" inverse, i.e., an inverse analytic in $\text{Re}[s] > 0$. Is the inverse analytic in $\text{Re}[s] \geq 0$?

Problem 5.4.2 Suppose that a positive real matrix $Z(s)$ has a minimal realization $\{F, G, H, J\}$ with $J + J'$ nonsingular. Apply the techniques of this section to $Z'(s)$ to find a $V(s)$ satisfying $Z'(-s) + Z(s) = V(s)V'(-s)$, with $V(s) = J_v + H'(sI - F)^{-1}L_v$ and $V^{-1}(s)$ existing in $\text{Re}[s] > 0$.

Problem 5.4.3 Suppose that a positive real matrix $Z(s)$ has a minimal realization $\{F, G, H, J\}$ with $J + J'$ singular. Let ϵ be a positive constant, and let $Z_\epsilon(s) = Z(s) + \epsilon I$. Let $\bar{\Pi}_\epsilon$ be the matrix associated with $Z_\epsilon(s)$ in accordance with Theorem 5.4.1, and let P_ϵ, L_ϵ, and $W_{0\epsilon}$ be solutions of the positive real lemma equations for $Z_\epsilon(s)$ obtained from $\bar{\Pi}_\epsilon$ in accordance with Eq. (5.4.11). Show that P_ϵ varies monotonically with ϵ and that $\lim_{\epsilon \to 0} P_\epsilon$ exists. Show also that

$$\lim_{\epsilon \to 0} \begin{bmatrix} L_\epsilon L'_\epsilon & L_\epsilon W'_{0\epsilon} \\ W'_{0\epsilon} L'_\epsilon & W'_{0\epsilon} W_{0\epsilon} \end{bmatrix}$$

exists; deduce that there exist real matrices P, L and W_0 which satisfy the positive real lemma equations for $Z(s)$, with P positive definite symmetric. Discuss the analyticity of the inverse of the spectral factor associated with $Z(s)$.

Problem 5.4.4 *Suppose that $Z(s)$ has a minimal realization $\{F, G, H, J\}$ with $R = J + J'$* nonsingular. Let $\{Q, M, R^{1/2}\}$ be a solution of the positive real lemma equations for $Z'(s)$, obtained by the method of this section. Show [30] that $\{Q^{-1}, Q^{-1}M, R^{1/2}\}$ is a solution of the positive real lemma equations for $Z(s)$, with the property that all eigenvalues of $F - GR^{-1}(H' - G'Q^{-1})$ lie in $\text{Re}[s] > 0$, i.e. that the associated spectral factor $W(s)$ has $W^{-1}(s)$ analytic in $\text{Re}[s] < 0$. (Using arguments like those of Problem 5.4.3, this result may be extended to the case of singular R).

5.5 POSITIVE REAL LEMMA: NECESSITY PROOF BASED ON SPECTRAL FACTORIZATION*

In this section we prove the positive real lemma by assuming the existence of spectral factors defined in transfer-function matrix form. In the case of a scalar positive real $Z(s)$, it is a simple task both to exhibit spectral factors and to use their existence to establish the positive real lemma. In the case of a matrix $Z(s)$, existence of spectral factors is not straightforward to establish, and we shall refer the reader to the literature for the relevant proofs, rather than present them here; we shall prove the positive real lemma by taking existence as a starting point.

*This section may be omitted at a first, and even a second, reading.

Although the matrix case $Z(s)$ obviously subsumes the scalar case, and thus a separate proof for the latter is somewhat superfluous, we shall nevertheless present such a proof on the grounds of its simplicity and motivational content. For the same reason, we shall present a simple proof for the matrix case that restricts the F matrix in a minimal realization of $Z(s)$. This will be in addition to a general proof for the matrix case.

Finally, we comment that it is helpful to restrict initially the poles of elements of $Z(s)$ to lie in the half-plane $\text{Re}\,[s] < 0$. This restriction will be removed at the end of the section.

The proof to be presented for scalar impedances appeared in [2], the proof for a restricted class of matrix $Z(s)$ in [12], and the full proof, applicable for any $Z(s)$, in [4].

Proof for Scalar Impedances

We assume for the moment that $Z(s)$ has a minimal realization $\{F, g, h, j\}$ (lowercase letters denoting vectors or scalars). Moreover, we assume that all eigenvalues of F lie in $\text{Re}\,[s] < 0$. As noted above, this restriction will be lifted at the end of the section. Let $p(s)$ and $q(s)$ be polynomials such that

$$\frac{q(s)}{p(s)} = j + h'(sI - F)^{-1}g \tag{5.5.1}$$

and

$$p(s) = \det(sI - F) \tag{5.5.2}$$

As we know, minimality of the realization $\{F, g, h, j\}$ is equivalent to the polynomials $p(s)$ and $q(s)$ being relatively prime. Now*

$$\text{Re}\, z(j\omega) = \frac{q(j\omega)p(-j\omega) + p(j\omega)q(-j\omega)}{|p(j\omega)|^2}$$

It follows from the positive real property that the polynomial $\lambda(\omega) = q(j\omega)p(-j\omega) + p(j\omega)q(-j\omega)$, which has real coefficients and only even powers of ω, is never negative.

Hence $\lambda(\omega)$, regarded as a polynomial in ω, must have zeros that are either complex conjugate or real and of even multiplicity. It follows that the polynomial $\mu(s) = q(s)p(-s) + p(s)q(-s)$ has zeros that are either purely imaginary and of even multiplicity or are such that for each zero s_0 in $\text{Re}\,[s] > 0$ there is a zero $-s_0$ in $\text{Re}\,[s] < 0$. Form the polynomial $r(s) = \prod (s - s_0)$, where the product is over (1) all zeros s_0 in $\text{Re}\,[s] > 0$, and (2) all zeros on $\text{Re}\,[s] = 0$ to half the multiplicity with which they occur in $\mu(s)$. Then $r(s)$

*When j appears as j_ω, it denotes $\sqrt{-1}$, and when it appears by itself, it denotes $Z(\infty)$.

will be a real polynomial and $\mu(s) = \alpha \hat{r}(s)\hat{r}(-s)$ for some constant α. Since $\lambda(\omega) = \alpha |\hat{r}(j\omega)|^2$, it follows that $\alpha > 0$; set $r(s) = \sqrt{\alpha}\,\hat{r}(s)$ to conclude that

$$q(s)p(-s) + p(s)q(-s) = r(-s)r(s) \tag{5.5.3}$$

and therefore

$$\frac{q(s)}{p(s)} + \frac{q(-s)}{p(-s)} = \frac{r(-s)}{p(-s)}\frac{r(s)}{p(s)} \tag{5.5.4}$$

Equation (5.5.4) yields a spectral factorization of $Z(s) + Z(-s)$. From it we can establish the positive real lemma equations.

Because $[F, g]$ is completely controllable, it follows that for some nonsingular T

$$TFT^{-1} = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & 1 \\ -a_1 & -a_2 & & & -a_n \end{bmatrix}$$

$$Tg = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \qquad (T^{-1})'h = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}$$

where the a_i and b_i are such that

$$\frac{q(s)}{p(s)} = j + \frac{b_n s^{n-1} + b_{n-1}s^{n-2} + \cdots + b_1}{s^n + a_n s^{n-1} + \cdots + a_1} \tag{5.5.5}$$

From (5.5.4) and (5.5.5) we see that $\lim_{s\to\infty} r(s)/p(s) = \sqrt{2j}$, and thus

$$\frac{r(s)}{p(s)} = \sqrt{2j} + \frac{\hat{l}_n s^{n-1} + \hat{l}_{n-1}s^{n-2} + \cdots + \hat{l}_1}{s^n + a_n s^{n-1} + \cdots + a_1}$$

for some set of coefficients $\hat{l}_i$. Now set $\hat{l} = [\hat{l}_1 \quad \hat{l}_2 \quad \cdots \quad \hat{l}_n]'$, to obtain

$$\frac{r(s)}{p(s)} = \sqrt{2j} + \hat{l}'[sI - TFT^{-1}]^{-1}(Tg)$$

Therefore, with $l = T'\hat{l}$, it follows that $\{F, g, l, \sqrt{2j}\}$ is a realization for $r(s)/p(s)$.

This realization is certainly completely controllable. It is also completely observable. For if not, there exists a polynomial dividing both $r(s)$ and $p(s)$ of degree at least 1 and with all zeros possessing negative real part. From (5.5.3) we see that this polynomial must divide $q(s)p(-s)$, and thus $q(s)$, since all zeros of $p(-s)$ have positive real part. Consequently, $q(s)$ and $p(s)$ have a common factor; this contradicts the earlier stated requirement that $p(s)$ and $q(s)$ be relatively prime. Hence, by contradiction, $[F, l]$ is completely observable.

Now consider (5.5.4). This can be written as

$$\begin{aligned}2j &+ h'(sI - F)^{-1}g + g'(-sI - F')^{-1}h \\ &= [\sqrt{2j} + g'(-sI - F')^{-1}l][\sqrt{2j} + l'(sI - F)^{-1}g] \\ &= 2j + \sqrt{2j}l'(sI - F)^{-1}g + \sqrt{2j}g'(-sI - F')^{-1}l \\ &\quad + g'(-sI - F')^{-1}ll'(sI - F)^{-1}g \end{aligned} \tag{5.5.6}$$

Define P as the unique positive definite symmetric solution of

$$PF + F'P = -ll' \tag{5.5.7}$$

The fact that P exists and has the stated properties follows from the lemma of Lyapunov. This definition of P allows us to rewrite the final term on the right side of (5.5.6) as follows:

$$\begin{aligned}g'(-sI &- F')^{-1}ll'(sI - F)^{-1}g \\ &= g'(-sI - F')^{-1}[P(sI - F) + (-sI - F')P](sI - F)^{-1}g \\ &= g'(-sI - F')^{-1}Pg + g'P(sI - F)^{-1}g\end{aligned}$$

Equation (5.5.6) then yields

$$(h' - g'P - \sqrt{2j}l')(sI - F)^{-1}g + g'(-sI - F')^{-1}(h - Pg - \sqrt{2j}l) = 0$$

Because all eigenvalues of F are restricted to $\text{Re}\,[s] < 0$, it follows that the two terms on the left side of this equation are separately zero. (What is the basis of this reasoning?) Then, because $[F, g]$ is completely controllable, it follows that

$$Pg = h - \sqrt{2j}l \tag{5.5.8}$$

Equations (5.5.7), (5.5.8), and the identification $W_0 = \sqrt{2j}$ (so that $2j = W_0'W_0$) then constitute the positive real lemma equations. $\nabla\nabla\nabla$

Example 5.5.1 Let $z(s) = \frac{1}{2} + (s + 1)^{-1}$, so that a minimal realization is $[-1, 1, 1, \frac{1}{2}]$. Evidently,

$$z(s) + z(-s) = 1 + \frac{1}{s+1} + \frac{1}{-s+1} = \frac{-s^2+3}{-s^2+1}$$
$$= \frac{(-s+\sqrt{3})(s+\sqrt{3})}{(-s+1)(s+1)}$$

so that a spectral factor is given by $w(s) = (s + \sqrt{3})(s + 1)^{-1}$. A minimal realization for $w(s)$ is $\{-1, 1, \sqrt{3} - 1, 1\}$, which has the same F matrix and g vector as the minimal realization for $z(s)$. The equation $PF + F'P = -ll'$ yields simply $P = 2 - \sqrt{3}$.

Alternatively, we might have taken the spectral factor $w(s) = (-s + \sqrt{3})(s + 1)^{-1}$. This spectral factor has minimal realization $\{-1, 1, \sqrt{3} + 1, -1\}$. It still has the same F matrix and g vector as the minimal realization for $z(s)$. The equation $PF + F'P = -ll'$ yields now $P = 2 + \sqrt{3}$. So we see that the positive real lemma equations do not yield, in general anyway, a unique solution.

Before proceeding to the matrix case, we wish to note three points concerning the proof above:

1. We have exhibited only one solution to the positive real lemma equations. Subsequently, we shall exhibit many.
2. We have exhibited a constructive procedure for obtaining a solution to the positive real lemma equations. A necessary part of the computation is the operation of spectral factorization of a transfer function, performed by reducing the spectral factorization problem to one involving only polynomials.
3. We have yet to permit the prescribed $Z(s)$ to have elements with poles that are pure imaginary, so the above does not constitute a complete proof of the positive real lemma for a scalar $Z(s)$. As earlier noted, we shall subsequently see how to incorporate $j\omega$-axis, i.e., pure imaginary, poles.

Matrix Spectral Factorization

The problem of matrix spectral factorization has been considered in a number of places, e.g., [13–17]. Here we shall quote a result as obtained by Youla [14].

Youla's Spectral Factorization Statement. Let $Z(s)$ be an $m \times m$ positive real rational matrix, with no element of $Z(s)$ possessing a pure imaginary pole. Then there exists an $r \times m$ matrix $W(\cdot)$ of real rational functions of s satisfying

$$Z(s) + Z'(-s) = W'(-s)W(s) \tag{5.5.9}$$

where r is the normal rank of $Z(s) + Z'(-s)$, i.e., the rank almost everywhere.* Further, $W(s)$ has no element with a pole in Re $[s] \geq 0$, $W(s)$ has constant, as opposed to merely normal, rank in Re $[s] > 0$, and $W(s)$ is unique to within left multiplication by an arbitrary real constant orthogonal matrix.

We shall not prove this result here. From it we shall, however, prove the existence, via a constructive procedure, of matrices P, L, and W_0 satisfying the positive real lemma equations. This constructive procedure uses as its starting point the matrix $W(s)$ above; in references [14–16], techniques for the construction of $W(s)$ may be found. Elsewhere in the book, construction of P, L, and W_0 will be required. However, other than in this section, the procedures of [14–16] will almost never be needed to carry out this construction, and this is why we omit presentation of these procedures.

Example 5.5.2 Suppose that $Z(s)$ is the 2×2 matrix

$$Z(s) = \begin{bmatrix} 2 & 4\dfrac{s-1}{s+1} \\ 0 & 2 \end{bmatrix}$$

The positive real nature of $Z(s)$ is not hard to check. Also,

$$Z'(-s) + Z(s) = 4\begin{bmatrix} 1 & \dfrac{s-1}{s+1} \\ \dfrac{s+1}{s-1} & 1 \end{bmatrix} = \begin{bmatrix} 2 \\ \dfrac{2(s+1)}{s-1} \end{bmatrix} \begin{bmatrix} 2 & 2\dfrac{s-1}{s+1} \end{bmatrix}$$

Thus we take as $W(s)$ the matrix $[2 \quad 2(s-1)/(s+1)]$. Notice that $W(s)$ has the correct size, viz., one row and two columns, 1 being normal rank of $Z'(-s) + Z(s)$. Further, $W(s)$ has no element with a pole in Re $[s] > 0$, and rank $W(s) = 1$ throughout Re $[s] > 0$. The only 1×1 real constant orthogonal matrices are $+1$ and -1, and so $W(s)$ is unique up to a ± 1 multiplier.

Before using Youla's result to prove the positive real lemma, we shall isolate an important property of $W(s)$.

Important Property of Spectral Factor Minimal Realizations

In this subsection we wish to establish the following result.

*For example, the 1×1 matrix $[(s-1)/(s+1)]$ has normal rank 1 throughout Re $[s] > 0$, but not constant rank throughout Re $[s] > 0$; that is, it has rank 1 almost everywhere, but not at $s = 1$.

Lemma 5.5.1. Let $Z(s)$ be an $m \times m$ positive real matrix with no element possessing a pure imaginary pole, and let $W(s)$ be the spectral factor of $Z(s) + Z'(-s)$ satisfying (5.5.9) and fulfilling the various conditions stated in association with that equation. Suppose that $Z(s)$ has a minimal realization $\{F, G, H, J\}$. Then $W(s)$ has a minimal realization $\{F, G, L, W_0\}$.

We comment that this lemma is almost, but *not quite*, immediate when $Z(s)$ is scalar, because of the existence of a *companion matrix* form for F with $g' = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$. In our proof of the positive real lemma for scalar $Z(s)$, we constructed a minimal realization for $W(s)$ satisfying the constraint of the above lemma, with the aid of the special forms for F and g. Now we present two proofs for the lemma, one simple, but restricting F somewhat.

Simple proof for restricted F [12]. We shall suppose that all eigenvalues of F are distinct. Then

$$(sI - F)^{-1} = \sum_{k=1}^{n} \frac{F_k}{s - s_k}$$

where the s_k are eigenvalues of F and all have negative real part. It is immediate from (5.5.9) that the entries of $W(s)$ have only simple poles, and so, for some constant matrices W_k, $k = 0, 1, \ldots, n$,

$$W(s) = W_0 + \sum_{k=1}^{n} \frac{W_k}{s - s_k}$$

Equation (5.5.9) now implies that

$$J + J' = W_0' W_0 \tag{5.5.10}$$

and

$$H' F_k G = W'(-s_k) W_k$$

We must now identify L. The constraints on $W(s)$ listed in association with (5.5.9) guarantee that $W'(-s_k)$ has rank r, and thus, being an $n \times r$ matrix, possesses a left inverse, which we shall write simply as $[W'(-s_k)]^{-1}$. Define

$$H_k' = [W'(-s_k)]^{-1} H'$$

so that

$$W_k = H_k' F_k G \qquad k = 1, 2, \ldots, n$$

Now suppose T is such that $TFT^{-1} = \text{diag}\,[s_1, s_2, \ldots, s_n]$. Then

$F_k = T^{-1}E_kT$, where E_k is a diagonal matrix whose only nonzero entry is unity in the (k, k) entry. Define

$$L' = \sum_{k=1}^{n} H'_k T^{-1} E_k T$$

It follows, on observing that $E_iE_j = 0_n$, $i \neq j$, and $E_i^2 = E_i$, that

$$L'F_k = \sum_{j=1}^{n} H'_j T^{-1} E_j T T^{-1} E_k T = H'_k T^{-1} E_k T = H'_k F_k$$

and so

$$W_k = L'F_kG$$

Finally,

$$W(s) = W_0 + \sum_{k=1}^{n} \frac{L'F_kG}{s - s_k}$$
$$= W_0 + L'(sI - F)^{-1}G \qquad \nabla\nabla\nabla$$

Actually, we should show that $[F, L]$ is completely observable to complete the proof. Though this is not too difficult, we shall omit the proof here since this point is essentially covered in the fuller treatment of the next subsection.

Proof for arbitrary F. The proof will fall into three segments:

1. We shall show that $\delta[W'(-s)W(s)] = 2\delta[W(s)]$.
2. We shall show that $\delta[Z(s)] = \delta[W(s)]$ and $\delta[Z(s) + Z'(-s)] = \delta[W'(-s)W(s)] = 2\delta[Z(s)]$.
3. We shall conclude the desired result.

First, then, part (1). Note that one of the properties of the degree operator is that $\delta[AB] \leq \delta[A] + \delta[B]$. Thus we need to conclude equality. We rely on the following characterization of degree, noted earlier: for a real rational $X(s)$

$$\delta[X(s)] = \sum_{s_k} \delta[X(s); s_k] \tag{5.5.11}$$

where the s_k are poles of $X(s)$, and where $\delta[X(s); s_k]$ is the maximum multiplicity that s_k possesses as a pole of any minor of $X(s)$.

Now let s_k be a pole of $W(s)$, so that Re $[s_k] < 0$, and consider $\delta[W'(-s)W(s); s_k]$. We claim that there is a minor of $W'(-s)W(s)$ possessing s_k as a pole with multiplicity $\delta[W(s); s_k]$; i.e., we claim that $\delta[W'(-s)W(s); s_k] \geq \delta[W(s); s_k]$. The argument is as follows. Because $W(s)$ has rank r throughout Re $[s] > 0$, $W'(-s)$ has rank

r throughout $\mathrm{Re}\,[s] < 0$ and must possess a left inverse, which we shall denote by $[W'(-s)]^{-1}$. Now

$$[W'(-s)]^{-1}W'(-s)W(s) = W(s)$$

and so

$$\delta[[W'(-s)]^{-1}W'(-s)W(s); s_k] = \delta[W(s); s_k]$$

Therefore, there exists a minor of $[W'(-s)]^{-1}W'(-s)W(s)$ that has s_k as a pole of multiplicity $\delta[W(s); s_k]$. By the Binet–Cauchy theorem [18], such a minor is a sum of products of minors of $[W'(-s)]^{-1}$ and $W'(-s)W(s)$. Since $\mathrm{Re}\,[s_k] < 0$, s_k cannot be a pole of any entry of $[W'(-s)]^{-1}$, and it follows that there is at least one minor of $W'(-s)W(s)$ with a pole at s_k of multiplicity at least $\delta[W(s); s_k]$; i.e.,

$$\delta[W'(-s)W(s); s_k] \geq \delta[W(s); s_k]$$

If s_k is a pole of an element of $W(s)$, $-s_k$ is a pole of an element of $W'(-s)$. A similar argument to the above shows that

$$\delta[W'(-s)W(s); -s_k] \geq \delta[W'(-s); -s_k]$$

Summing these relations over all s_k and using (5.5.11) applied to $W'(-s)W(s)$, $W'(-s)$ and $W(s)$ yields

$$\delta[W'(-s)W(s)] \geq \delta[W(s)] + \delta[W'(-s)]$$

But as earlier noted, the inequality must apply the other way. Hence

$$\begin{aligned}\delta[W'(-s)W(s)] &= \delta[W(s)] + \delta[W'(-s)]\\ &= 2\delta[W(s)]\end{aligned} \tag{5.5.12}$$

since clearly $\delta[W(s)] = \delta[W'(-s)]$.

Part (2) is straightforward. Because no element of $Z(s)$ can have a pole in common with an element of $Z'(-s)$, application of (5.5.11) yields

$$\begin{aligned}\delta[Z(s) + Z'(-s)] &= \delta[Z(s)] + \delta[Z'(-s)]\\ &= 2\delta[Z(s)]\end{aligned} \tag{5.5.13}$$

Then (5.5.12), (5.5.13), and the fundamental equality $Z(s) + Z'(-s) = W'(-s)W(s)$ yield simply that $\delta[Z(s)] = \delta[W(s)]$ and $\delta[Z(s) + Z'(-s)] = \delta[W'(-s)W(s)] = 2\delta[Z(s)]$.

Now let $\{F_w, G_w, H_w, W_0\}$ be a minimal realization for $W(s)$. It is straightforward to verify by direct computation that a realization for $W'(-s)W(s)$ is provided by

$$F_1 = \begin{bmatrix} F_w & 0 \\ -H_w H'_w & -F'_w \end{bmatrix} \quad G_1 = \begin{bmatrix} G_w \\ -H_w W_0 \end{bmatrix}$$
$$H_1 = \begin{bmatrix} H_w W_0 \\ G_w \end{bmatrix} \quad J_1 = W'_0 W_0 \tag{5.5.14}$$

(Verification is requested in the problems.)

This realization has dimension $2\delta[W(s)]$ and is minimal by (5.5.12). With $\{F, G, H, J\}$ a minimal realization of $Z(s)$, a realization for $Z'(-s) + Z(s)$ is easily seen to be provided by

$$F_2 = \begin{bmatrix} F & 0 \\ 0 & -F' \end{bmatrix} \quad G_2 = \begin{bmatrix} G \\ H \end{bmatrix}$$
$$H_2 = \begin{bmatrix} H \\ -G \end{bmatrix} \quad J_2 = J + J' \tag{5.5.15}$$

By (5.5.13), this is minimal. Notice that (5.5.14) and (5.5.15) constitute minimal realizations of the same transfer-function matrix. From (5.5.14) let us construct a further minimal realization. Define P_w, existence being guaranteed by the lemma of Lyapunov, as the unique positive definite symmetric solution of

$$P_w F_w + F'_w P_w = -H_w H'_w \tag{5.5.16}$$

Set

$$T = \begin{bmatrix} I & 0 \\ -P_w & I \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} I & 0 \\ P_w & I \end{bmatrix}$$

and define a minimal realization of $W'(-s)W(s)$ by $F_3 = TF_1T^{-1}$, etc. Thus

$$F_3 = \begin{bmatrix} F_w & 0 \\ 0 & -F'_w \end{bmatrix} \quad G_3 = \begin{bmatrix} G_w \\ -P_w G_w - H_w W_0 \end{bmatrix}$$
$$H_3 = \begin{bmatrix} P_w G_w + H_w W_0 \\ G_w \end{bmatrix} \quad J_3 = W'_0 W_0 \tag{5.5.17}$$

Now (5.5.15) and (5.5.17) are minimal realizations of the same transfer-function matrix, and we have

$$J + J' + H'(sI - F)^{-1}G - G'(sI + F')^{-1}H$$
$$= W_0'W_0 + (P_wG_w + H_wW_0)'(sI - F_w)^{-1}G_w$$
$$- G_w'(sI + F_w')^{-1}(P_wG_w + H_wW_0)$$

Now F_w and F both have eigenvalues restricted to $\operatorname{Re}[s] < 0$ and therefore

$$H'(sI - F)^{-1}G = (P_wG_w + H_wW_0)'(sI - F_w)^{-1}G_w \tag{5.5.18}$$

The decomposition on the left side of (5.5.18) is known to be minimal, and that on the right side must be minimal, having the same dimension; so, for some T_w,

$$F = T_wF_wT_w^{-1} \qquad G = T_wG_w \tag{5.5.19}$$

Consequently, $W(s)$, which possesses a minimal realization $\{F_w, G_w, H_w, W_0\}$, possesses also a minimal realization of the form $\{F, G, L, W_0\}$ with $L = (T_w')^{-1}H_w$. ∇ ∇ ∇

Positive Real Lemma Proof—Restricted Pole Positions

With the lemma above in hand, the positive real lemma proof is very straightforward. With $Z(s)$ positive real and possessing a minimal realization $\{F, G, H, J\}$ and with $W(s)$ an associated spectral factor with minimal realization $\{F, G, L, W_0\}$, define P as the positive definite symmetric solution of

$$PF + F'P = -LL' \tag{5.5.20}$$

Then

$$W'(-s)W(s)$$
$$= W_0'W_0 + W_0'L'(sI - F)^{-1}G + G'(-sI - F')^{-1}LW_0$$
$$+ G'(-sI - F')^{-1}[P(sI - F) + (-sI - F')P](sI - F)^{-1}G$$
$$= W_0'W_0 + (W_0'L' + G'P)(sI - F)^{-1}G$$
$$+ G'(-sI - F')^{-1}(PG + LW_0)$$

Equating this with $Z(s) + Z'(-s)$, it follows that

$$J + J' = W_0'W_0 \tag{5.5.21}$$

and

$$H'(sI - F)^{-1}G = (W_0'L' + G'P)(sI - F)^{-1}G$$

By complete controllability,

$$PG = -LW_0 + H \tag{5.5.22}$$

Equations (5.5.20) through (5.5.22) constitute the positive real lemma equations. $\nabla\nabla\nabla$

We see that the constructive proof falls into three parts:

1. Computation of $W(s)$.
2. Computation of a minimal realization of $W(s)$ of the form $\{F, G, L, W_0\}$.
3. Calculation of P.

Step 1 is carried out in references [14] and [15]. For step 2 we can form any minimal realization of $W(s)$ and then proceed as in the previous section to obtain a minimal realization with the correct F and G. Actually, all that is required is the matrix T_w of Eq. (5.5.19). This is straightforward to compute directly from F, G, F_w, and G_w, because (5.5.19) implies

$$[G \quad FG \quad \cdots \quad F^{n-1}G] = T_w[G_w \quad F_wG_w \quad \cdots \quad F_w^{n-1}G_w]$$

as direct calculation will show. The matrix $[G_w \quad F_wG_w \quad \cdots \quad F_w^{n-1}G_w]$ possesses a right inverse because $[F_w, G_w]$ is completely controllable. Thus T_w follows.

Step 3 requires the solving of (5.5.20) [note that (5.5.21) and (5.5.22) will be automatically satisfied]. Solution procedures for (5.5.20) are discussed in, e.g., [18] and require the solving of linear simultaneous equations.

Example 5.5.3 Associated with the positive real matrix

$$Z(s) = \begin{bmatrix} 2 & 4\dfrac{s-1}{s+1} \\ 6 & 2 \end{bmatrix}$$

we found the spectral factor

$$W(s) = \begin{bmatrix} 2 & 2\dfrac{s-1}{s+1} \end{bmatrix}$$

Now a minimal realization for $Z(s)$ is readily found to be

$$F = [-1] \qquad G = [0 \quad 1] \qquad H = [-8 \quad 0] \qquad J = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$$

A minimal realization for $W(s)$ is

$$F_w = [-1] \qquad G_w = [0 \quad -4] \qquad L_w = [1] \qquad W_0 = [2 \quad 2]$$

We are assured that there exists a realization with the same F and G

matrices as the realization for $Z(s)$. Therefore, we seek T_w such that

$$[G \quad FG] = T_w[G_w \quad F_wG_w]$$

or

$$[0 \quad 1 \quad 0 \quad -1] = T_w[0 \quad -4 \quad 0 \quad 4]$$

whence $T_w = -\frac{1}{4}$. It follows that $L = (T_w')^{-1}L_w = -4$ so that $W(s)$ has a minimal realization $\{-1, [0 \quad 1], [-4], [2 \quad 2]\}$. The matrix P is given from (5.5.20) as $P = 8$ and, as may easily be checked, satisfies $PG = H - LW_0$. Also, $W_0'W_0 = J + J'$ follows simply.

Positive Real Lemma Proof—Unrestricted Pole Positions

Previously in this section we have presented a *constructive proof* for the positive real lemma, based on spectral factorization, for the case when no entry of a prescribed positive real $Z(s)$ possesses a purely imaginary pole. Our goal in this section is to remove the restriction. A broad outline of our approach will be as follows.

1. We shall establish that if the positive real lemma is proved for a particular state-space coordinate basis, it is true for any coordinate basis.
2. We shall take a special coordinate basis that separates the problem of proving the positive real lemma into the problem of proving it for two classes of positive real matrices, one class being that for which we have already proved it in this section, the other being the class of lossless positive real matrices.
3. We shall prove the positive real lemma for lossless positive real matrices.

Taking 1, 2, and 3 together, we achieve a full proof.

1. *Effect of Different Coordinate Basis:* Let $\{F_1, G_1, H_1, J\}$ and $\{F_2, G_2, H_2, J\}$ be two minimal realizations of a positive real matrix $Z(s)$. Suppose that we know the existence of a positive definite P_1 and matrices L_1 and W_0 for which $P_1F_1 + F_1'P_1 = -L_1L_1'$, $P_1G_1 = H_1 - L_1W_0$, $J + J' = W_0'W_0$. There exists T such that $TF_2T^{-1} = F_1$, $TG_2 = G_1$, and $(T^{-1})'H_2 = H_1$. Straightforward calculation will show that if $P_2 = T'P_1T$ and $L_2 = T'L_1$, then $P_2F_2 + F_2'P_2 = -L_2L_2'$, $P_2G_2 = H_2 - L_2W_0$, and again $J + J' = W_0'W_0$.

Therefore, if the positive real lemma can be proved in one particular coordinate basis, i.e., if P, L, and W_0 satisfying the positive real lemma equations can be found for one particular minimal realization of $Z(s)$, it is trivial to observe that the lemma is true in any coordinate basis, or that P, L, and W_0 satisfying the positive real lemma equations can be found for any minimal realization of $Z(s)$.

2. *Separation of the Problem.* We shall use the decomposition property for real rational $Z(s)$, noted earlier in this chapter. Suppose that $Z(s)$ is positive real, with $Z(\infty) < \infty$. Then we may write

$$Z(s) = Z_0(s) + \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} \tag{5.5.23}$$

with, actually,

$$\delta[Z(s)] = \delta[Z_0(s)] + \sum_i \delta\left[\frac{A_i s + B_i}{s^2 + \omega_i^2}\right] \tag{5.5.24}$$

The matrix $Z_0(s)$ is positive real, with no element possessing a pure imaginary pole. Each A_i is nonnegative definite symmetric and B_i skew symmetric, with $(A_i s + B_i)(s^2 + \omega_i^2)^{-1}$ lossless and, of course, positive real. Also, A_i and B_i are related to the residue matrix at ω_i. Equation (5.5.24) results from the fact that the elements of the different summands in (5.5.23) have no common poles.

Suppose that we find a minimal realization $\{F_0, G_0, H_0, J_0\}$ for $Z_0(s)$ and minimal realizations $\{F_i, G_i, H_i\}$ for each $Z_i(s) = (A_i s + B_i)(s^2 + \omega_i^2)^{-1}$. Suppose also that we find solutions P_0, L_0, and W_0 of the positive real lemma equations in the case of $Z_0(s)$, and solutions P_i (the L_i and W_{0i} being as we shall see, always zero) in the case of $Z_i(s)$. Then it is not hard to check that a realization for $Z(s)$, minimal by (5.5.24), is provided by

$$F = \begin{bmatrix} F_0 & 0 & 0 & \cdots \\ 0 & F_1 & 0 & \cdots \\ 0 & 0 & F_2 & \\ \cdot & \cdot & \cdot & \cdot\,\cdot \\ \cdot & \cdot & \cdot & \cdot\,\cdot \\ \cdot & \cdot & \cdot & \cdot\,\cdot \end{bmatrix} \qquad G = \begin{bmatrix} G_0 \\ G_1 \\ G_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad H = \begin{bmatrix} H_0 \\ H_1 \\ H_2 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad J = J_0 \tag{5.5.25}$$

and a solution of the positive real lemma equations is provided by W_0 and

$$P = \begin{bmatrix} P_0 & 0 & 0 & \cdots \\ 0 & P_1 & 0 & \cdots \\ 0 & 0 & P_2 & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \end{bmatrix} \qquad L = \begin{bmatrix} L_0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \tag{5.5.26}$$

Because the P_i are positive definite symmetric, so is P.

Consequently, choice of the special coordinate basis leading to (5.5.25) reduces the problem of proving the positive real lemma to a solved problem [that involving $Z_0(s)$ alone] and the so far unsolved problem of proving the positive real lemma for a lossless $Z_i(s)$.

3. *Proof of Positive Real Lemma for Lossless* $Z_i(s)$. Let us drop the subscript i and study

$$Z(s) = \frac{As + B}{s^2 + \omega_0^2} \tag{5.5.27}$$

Rewrite (5.5.27) as

$$Z(s) = \frac{K_0}{s - j\omega_0} + \frac{K_0^*}{s + j\omega_0}$$

where K_0 is nonnegative definite Hermitian. Then if $k = \text{rank } K_0$, $Z(s)$ has degree $2k$ and there exist k complex vectors x_i such that

$$Z(s) = \sum_{i=1}^{k} \left[\frac{x_i x_i'^*}{s - j\omega_0} + \frac{x_i^* x_i'}{s + j\omega_0} \right]$$

By using the same decomposition technique as discussed in 2 above, we can replace the problem of finding a solution of the positive real lemma equations for the above $Z(s)$ by one involving the degree 2 positive real matrix

$$Z(s) = \frac{xx'^*}{s - j\omega_0} + \frac{x^* x'}{s + j\omega_0}$$

Setting $y_1 = (x + x^*)/\sqrt{2}$, $y_2 = (x - x^*)/\sqrt{2}$, it is easy to verify that

$$Z(s) = [y_1 \quad y_2] \frac{1}{s^2 + \omega_0^2} \begin{bmatrix} s & \omega_0 \\ -\omega_0 & s \end{bmatrix} \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$$

Therefore, $Z(s)$ has a minimal realization

$$F = \begin{bmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \qquad H = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} \qquad J = 0$$

For this special realization, the construction of a solution of the positive real lemma equations is easy. We take simply

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad L = W_0 = 0$$

Verification that the positive real lemma equations are satisfied is then immediate.

When we build up realizations of degree 2 $Z(s)$ and their associated P to obtain a realization for $Z(s)$ of (5.5.27), it follows still that P is positive definite, and that the L and W_0 of the positive real lemma equations for the $Z(s)$

of (5.5.27) are both zero. A further building up leads to zero L and W_0 for the lossless matrix

$$Z(s) = J + \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} \tag{5.5.28}$$

where the A_i and B_i satisfy the usual constraints, and J is skew. Since (5.5.28) constitutes the most general form of lossless positive real matrix for which $Z(\infty) < \infty$, it follows that we have given a necessity proof for the lossless positive real lemma that does not involve use of the main positive real lemma in the proof. ∇ ∇ ∇

Problem 5.5.1 Let $Z(s)$ be positive real, $Z(\infty) < \infty$, and let $V(s)$ be such that

$$Z(s) + Z'(-s) = V(s)V'(-s)$$

Prove, by applying Youla's result to $Z'(s)$, that there exists a $V(s)$, with all elements analytic in $\text{Re}\,[s] \geq 0$, with a left inverse existing in $\text{Re}\,[s] \geq 0$, with rank $[Z(s) + Z'(-s)]$ columns, and unique up to right multiplication by an arbitrary real constant orthogonal matrix. Then use the theorems of the section to conclude that $\delta[V(s)] = \delta[Z(s)]$ and that V has a minimal realization with the same F and H matrix as $Z(s)$.

Problem 5.5.2 Let $Z(s) = 1 + 1/(s + 1)$. Find a minimal realization and a solution of the positive real lemma equations using spectral factorization. Can you find a second set of solutions to the positive real lemma equations by using a different spectral factor?

Problem 5.5.3 Let P be a solution of the positive real lemma equations. Show that if elements of $Z(s)$ have poles strictly in $\text{Re}\,[s] < 0$, $x'Px$ is a Lyapunov function establishing asymptotic stability, while if $Z(s)$ is lossless, $x'Px$ is a Lyapunov function establishing stability but not asymptotic stability.

Problem 5.5.4 Let $\{F_w, G_w, H_w, W_0\}$ be a minimal realization of $W(s)$. Prove that the matrices F_1, G_1, H_1, and J_1 of Eq. (5.5.14) constitute a realization of $W'(-s)W(s)$.

5.6 THE POSITIVE REAL LEMMA: OTHER PROOFS AND APPLICATIONS

In this section we shall briefly note the existence of other proofs of the positive real lemma, and then draw attention to some applications.

Other Proofs of the Positive Real Lemma

The proof of Yakubovic [1] was actually the first proof ever to be given of the lemma. Applying only to a scalar function, it proceeds by induction on the degree of the positive real function $Z(s)$, and a key step in the argument requires the determination of the value of ω minimizing $\mathrm{Re}[Z(j\omega)]$. The case of a scalar $Z(s)$ of degree n is reduced to the case of a scalar $Z(s)$ of degree $n-1$ by a procedure reminiscent of the Brune synthesis procedure (see, e.g., [19]). To the extent that a multiport Brune synthesis procedure is available (see, e.g., [6]), one would imagine that the Yakubovic proof could be generalized to positive real matrices $Z(s)$.

Another proof, again applying only to scalar positive real $Z(s)$, is due to Faurre [12]. It is based on results from the classical moment problem and the separating hyperplane theorem, guaranteeing the existence of a hyperplane separating two disjoint closed convex sets. How one might generalize this proof to the matrix $Z(s)$ case is not clear; this is unfortunate, since, as Section 5.5 shows, the scalar case is comparatively easily dealt with by other techniques, while the matrix case is difficult.

The Yakubovic proof implicitly contains a constructive procedure for obtaining solutions of the positive real lemma equation, although it would be very awkward to apply. The Faurre proof is essentially an existence proof.

Popov's proof, see [5], in rough terms is a variant on the proof of Section 5.5.

Applications of the Positive Real Lemma

Generalized Positive Real Matrices. There arise in some systems theory problems, including the study of instability [20], real rational functions or square matrices of real rational functions $Z(s)$ with $Z(\infty) < \infty$ for which

$$Z'(-j\omega) + Z(j\omega) \geq 0 \tag{5.6.1}$$

for all real ω with $j\omega$ not a pole of any element of $Z(s)$. Poles of elements of $Z(\cdot)$ are unrestricted. Such matrices have been termed *generalized positive real matrices*, and a "generalized positive real lemma" is available (see [21]). The main result is as follows:

Theorem 5.6.1. Let $Z(\cdot)$ be an $m \times m$ matrix of rational functions of a complex variable s such that $Z(\infty) < \infty$. Let $\{F, G, H, J\}$ be a minimal realization of $Z(s)$. Then a necessary and sufficient condition for (5.6.1) to hold for all real ω with $j\omega$ not a pole of any element of $Z(\cdot)$ is that there exist real matrices $P = P'$, L, and W_0, with P nonsingular, such that

$$\begin{aligned} PF + F'P &= -LL' \\ PG &= H - LW_0 \\ W_0'W_0 &= J + J' \end{aligned} \tag{5.6.2}$$

Problem 5.2.5 required proof that (5.6.2) imply (5.6.1). Proof of the converse takes a fair amount of work (see [21]). Problem 5.6.1 outlines a somewhat speedier proof than that of [21].

The use of Theorem 5.6.1 in instability problems revolves around construction of a Lyapunov function including the term $x'Px$; this Lyapunov function establishes instability as discussed in detail in [20].

Inverse Problem of Linear Optimal Control. This problem, treated in [9, 22, 23], is stated as follows. Suppose that there is associated a linear feedback law $u = -K'x$ with a system $\dot{x} = Fx + Gu$. When can one say that this law arises through minimization of a performance index of the type

$$V = \int_{t_0}^{\infty} (u'u + x'Qx)\,dt \qquad \lim_{T\to\infty} x(T) = 0 \tag{5.6.3}$$

where Q is some nonnegative definite symmetric matrix? Leaving aside the inverse problem for the moment, we recall a property of the solution of the basic optimization problem. This is that if $[F, G]$ is completely controllable, the control law K exists, and $\dot{x} = (F - GK')x$ is asymptotically stable. Also,

$$\begin{aligned} 0 \leq [I + G'(-sI - F')^{-1}K][I + K'(sI - F)^{-1}G] - I \\ = G'(-sI - F')^{-1}Q(sI - F)^{-1}G \end{aligned} \tag{5.6.4}$$

The inequality is known as the return difference inequality, since $I + K'(sI - F)^{-1}G$ is the return difference matrix associated with the closed-loop system. The solution of the inverse problem can now be stated. The control law K results from a quadratic loss problem if the inequality part of (5.6.4) holds, if $[F, G]$ is completely controllable, and if $\dot{x} = (F - GK')x$ is asymptotically stable.

Since the inequality part of (5.6.4) is essentially a variant on the positive real condition, it is not surprising that this result can be proved quite efficiently with the aid of the positive real lemma. Broadly speaking, one uses the lemma to exhibit the existence of a matrix L for which

$$\begin{aligned} [I + G'(-sI - F')^{-1}K][I + K'(sI - F)^{-1}G] \\ = I + G'(-sI - F')^{-1}LL'(sI - F)^{-1}G \end{aligned} \tag{5.6.5}$$

(Problem 5.6.2 requests a proof of this fact.) Then one shows that with $Q = LL'$ in (5.6.3), the associated control law is precisely K. It is at this point that the asymptotic stability of $\dot{x} = (F - GK')x$ is used.

Circle Criterion [24, 25]. Consider the single-input single-output system

$$\dot{x} = Fx + gu \qquad y = h'x \tag{5.6.6}$$

and suppose that u is obtained by the feedback law

$$u = -k(t)y \tag{5.6.7}$$

The circle criterion is as follows [25]:

Theorem 5.6.2. Let $\{F, g, h\}$ be a minimal realization of $W(s) = h'(sI - F)^{-1}g$, and suppose that F has p eigenvalues with positive real parts and no eigenvalues with zero real part. Suppose that $k(\cdot)$ is piecewise continuous and that $\alpha + \epsilon \leq k(t) \leq \beta - \epsilon$ for some small positive ϵ. Then the closed-loop system is asymptotically stable provided that

1. The Nyquist plot of $W(s)$ does not intersect the circle with diameter determined by $(-\alpha^{-1}, 0)$ and $(-\beta^{-1}, 0)$ and encircles it exactly p times in the counterclockwise direction, or, equivalently,

2. $$Z(s) = [W(s) + \alpha^{-1}][W(s) + \beta^{-1}]^{-1} \tag{5.6.8}$$

   is positive real, with $\text{Re}\, Z(j\omega) > 0$ for all finite real ω and possessing all poles in $\text{Re}\,[s] < 0$.

Part (1) of the theorem statement in effect explains the origin of the term circle criterion. The condition that the Nyquist plot of $W(s)$ stay outside a certain circle is of course a nonnegativity condition of the form $|W(j\omega) - c| \geq \rho$ or $[W(-j\omega) - c][W(j\omega) - c] - \rho^2 \geq 0$, and part (2) of the theorem serves to convert this to a positive real condition. Connection between (1) and (2) is requested in the problems, as is a proof of the theorem based on part (2). The idea of the proof is to use the positive real lemma to generate a Lyapunov function that will establish stability. This was done in [25, 26]. It should be noted that the circle criterion can be applied to infinite dimensional systems, as well as finite dimensional systems (see [24]). The easy Lyapunov approach to proving stability then fails.

Popov Criterion [24, 27, 28]. The Popov criterion is a first cousin to the circle criterion. Like the circle criterion, it applies to infinite-dimensional as well as finite-dimensional systems, with finite-dimensional systems being readily tackled via Lyapunov theory. Instead of (5.6.6), we will, however, have

$$\dot{x} = Fx + Gu \qquad y = H'x \tag{5.6.9}$$

We thereby permit multiple inputs and outputs. We shall assume that the dimensions of u and y are the same, and that

$$u_i = -\mu_i(y_i) \tag{5.6.10}$$

with

$$\epsilon \leq \frac{\mu_i(y_i)}{y_i} \leq k_i - \epsilon \tag{5.6.11}$$

for some set of positive constants k_i, for some small positive ϵ, and for all nonzero y_i. The reader should note that we are now considering nonlinear, but not time-varying feedback. In the circle criterion, we considered time-varying feedback (which, in the particular context, included nonlinear feedback).

The basic theorem, to which extensions are possible, is as follows [28]:

Theorem 5.6.3. Suppose that there exist diagonal matrices $A = \text{diag}\{a_1, \ldots, a_m\}$ and $B = \text{diag}\{b_1, \ldots, b_m\}$, where m is the dimension of u and y, $a_i \geq 0$, $b_i \geq 0$, $a_i + b_i > 0$, $-a_i/b_i$ is not a pole of any of the ith row elements of $H'(sI - F)^{-1}G$, and

$$Z(s) = AK^{-1} + (A + Bs)H'(sI - F)^{-1}G \tag{5.6.12}$$

is positive real, where $K = \text{diag}\{k_1, k_2, \ldots, k_m\}$. Then the closed-loop system defined by (5.6.9) and (5.6.10) is stable.

The idea behind the proof of this theorem is straightforward. One obtains a minimal realization for $Z(s)$, and assumes that solutions $\{P, L, W_0\}$ of the positive real lemma equations are available. A Lyapunov function

$$V(x) = x'Px + 2\sum_i \int_0^{y_i} \mu_i(\rho_i)b_i\, d\rho_i \tag{5.6.13}$$

is adopted, from which stability can be deduced. Details are requested in Problem 5.6.5.

Spectral Factorization by Algebra [29]. We have already explained that intimately bound up with the positive real lemma is the existence of a matrix $W(s)$ satisfying $Z'(-s) + Z(s) = W'(-s)W(s)$. Here, $Z(s)$ is of course positive real. In contexts other than network theory, a similar decomposition is important, e.g., in Wiener filtering [15, 16]. One is given an $m \times m$ matrix $\Phi(s)$ with $\Phi'(-s) = \Phi(s)$ and $\Phi(j\omega)$ nonnegative definite for all real ω. It is assumed that no element of $\Phi(s)$ possesses poles on $\text{Re}[s] = 0$, and that $\Phi(\infty) < \infty$. One is required to find a matrix $W(s)$ satisfying

$$\Phi(s) = W'(-s)W(s) \tag{5.6.14}$$

Very frequently, $W(s)$ and $W^{-1}(s)$ are not permitted to have elements with poles in $\text{Re}[s] \geq 0$.

When $\Phi(s)$ is rational, one may regard the problem of determining $W(s)$

as one of solving the positive real lemma equations—a topic discussed in much detail subsequently. To see this, observe that $\Phi(s)$ may be written as

$$\Phi(s) = Z'(-s) + Z(s) \tag{5.6.15}$$

where elements of $Z(s)$ are analytic in $\text{Re}[s] \geq 0$. The writing of $\Phi(s)$ in this way may be achieved by doing a partial fraction expansion of each entry of $\Phi(s)$ and grouping those summands together that are analytic in $\text{Re}[s] \geq 0$ to yield an entry of $Z(s)$. The remaining summands give the corresponding entry of $Z'(-s)$. Once $Z(s)$ is known, a minimal realization $\{F, G, H, J\}$ of $Z(s)$ may be found. [Note that $\Phi(\infty) < \infty$ implies $Z(\infty) < \infty$.] The problem of finding $W(s)$ with the properties described is then, as we know, a problem of finding a solution of the positive real lemma equations. Shortly, we shall study algebraic procedures for this task.

Problem 5.6.1 (Necessity proof for Theorem 5.6.1): Let K be a matrix such that $F_K = F - GK'$ possesses all its eigenvalues in $\text{Re}[s] < 0$; existence of K is guaranteed by complete controllability of $[F, G]$.

(a) Show that if $X(s) = I - K'(sI - F_K)^{-1}G$, then $Z(s)X(s) = J + (H - KJ')'(sI - F_K)^{-1}G$.

(b) Show that $X'(-s)[Z(s) + Z'(-s)]X(s) = Y'(-s) + Y(s)$, where $Y(s) = J + H_K'(sI - F_K)^{-1}G$ with H_K defined by

$$H_K = H - K(J + J') - P_K G$$
$$P_K F_K + F_K' P_K = -(KH' + HK' - KJK' - KJ'K')$$

(c) Observe that $Y(s)$ is positive real. Let P_Y, L_Y, and W_0 be solutions of the positive real lemma equation associated with $Y(s)$. (Invoke the result of Problem 5.2.2 if $[F_K, H_K]$ is not completely observable.) Show that $P = P_Y + P_K$, $L = L_Y + KW_0$, and W_0 satisfy (5.6.2).

(d) Suppose that P is singular. Change the coordinate basis to force $P = [I_r] \dotplus [-I_s] \dotplus [0_t]$. Then examine (5.6.2) to show that certain submatrices of F, L, and H are zero, and conclude that F, H is unobservable. Conclude that P must be nonsingular.

Problem 5.6.2 (Inverse problem of optimal control):

(a) Start with (5.6.4) and setting $F_K = F - GK'$, show that

$$I - [I - G'(-sI - F_K')^{-1}K][I - K'(sI - F_K)^{-1}G] \geq 0$$

(b) Take P_K as the solution of $P_K F_K + F_K' P_K = -KK'$, and $H_K = K - P_K G$. Show that $H_K'(sI - F_K)^{-1}G$ is positive real.

(c) Apply the positive real lemma to deduce the existence of a spectral factor of $H_K'(sI - F_K)^{-1}G + G'(-sI - F_K')^{-1}H_K$ of the form $L'(sI - F_K)^{-1}G$, and then trace the manipulations of (a) and (b) backward to deduce (5.6.5). (Invoke the result of Problem 5.2.2 if $[F_K, H_K]$ is not completely observable.)

(d) Suppose that the optimal control associated with (5.6.3) when $Q = LL'$ is $u = -K_0'x$; assume that the equality in (5.6.4) is satisfied with K replaced by K_0 and that $F - GK_0'$ has all eigenvalues with negative real parts. It follows, with the aid of (5.6.5), that

$$[I + K_0'(sI - F)^{-1}G][I + K'(sI - F)^{-1}G]^{-1} = [I + G'(-sI - F')^{-1}K_0]^{-1}[I + G'(-sI - F')^{-1}K]$$

Show that this equality leads to

$$(K_0 - K)'(sI - F_K)^{-1}G = G'[-sI - (F - GK_0')']^{-1}(K - K_0)$$

Conclude that $K = K_0$.

Problem 5.6.3 Show the equivalence between parts (1) and (2) of Theorem 5.6.2. (You may have to recall the standard Nyquist criterion, which will be found in any elementary control systems textbook.)

Problem 5.6.4 (Circle criterion): With $Z(s)$ as defined in Theorem 5.6.2, show that $Z(s)$ has a minimal realization $\{F - \beta gh', g, -(\beta/\alpha)(\beta - \alpha)h, \beta/\alpha\}$. Write down the positive real lemma equations, and show that $x'Px$ is a Lyapunov function for the closed-loop system defined by (5.6.6) and (5.6.7). (Reference [26] extends this idea to provide a result incorporating a degree of stability in the closed-loop system.)

Problem 5.6.5 Show that the matrix $Z(s)$ in Eq. (5.6.12) has a minimal realization $\{F, G, HA + F'HB, AK^{-1} + BH'G\}$. Verify that $V(x)$ as defined in (5.6.13) is a Lyapunov function establishing stability of the closed-loop system defined by (5.6.9) and (5.6.10).

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6

Computation of Solutions of the Positive Real Lemma Equations*

6.1 INTRODUCTION

If $Z(s)$ is an $m \times m$ positive real matrix of rational functions with minimal realization $\{F, G, H, J\}$, the positive real lemma guarantees the existence of a positive definite symmetric matrix P, and real matrices L and W_0, such that

$$\begin{aligned} PF + F'P &= -LL' \\ PG &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{6.1.1}$$

The logical question then arises: How may we compute all solutions $\{P, L, W_0\}$ of (6.1.1)?

We shall be concerned in this chapter with answering this question. We do not presume a knowledge on the part of the reader of the necessity proofs of the positive real lemma covered previously; we shall, however, quote results developed in the course of these proofs that assist in the computation process.

In developing techniques for the solution of (6.1.1), we shall need to derive theoretical results that we have not met earlier. The reader who is interested

*This chapter may be omitted at a first reading.

solely in the computation of P, L, and W_0 satisfying (6.1.1), as distinct from theoretical background to the computation, will be able to skip the proofs of the results.

A preliminary approach to solving (6.1.1) might be based on the following line of reasoning. We recall that if P, L, and W_0 satisfy (6.1.1), then the transfer-function matrix $W(s)$ defined by

$$W(s) = W_0 + L'(sI - F)^{-1}G \tag{6.1.2}$$

satisfies the equation

$$Z'(-s) + Z(s) = W'(-s)W(s) \tag{6.1.3}$$

Therefore, we might be tempted to seek all solutions $W(s)$ of (6.1.3) that are representable in the form (6.1.2), and somehow generate a matrix P such that P, together with the L and W_0 following from (6.1.2), might satisfy (6.1.1). There are two potential difficulties with this approach. First, it appears just as difficult to generate all solutions of (6.1.3) that have the form of (6.1.2) as it is to solve (6.1.1) directly. Second, even knowing L and W_0 it is sometimes very difficult to generate a matrix P satisfying (6.1.1). [Actually, if every element of $Z(s)$ has poles lying in Re $[s] < 0$, there is no problem; the only difficulty arises when elements of $Z(s)$ have pure imaginary poles.] On the grounds then that the generation of all $W(s)$ satisfying (6.1.3) that are expressible in the form of (6.1.2) is impractical, we seek other ways of solving (6.1.1).

The procedure we shall adopt is based on a division of the problem of solving (6.1.1) into two subproblems:

1. Find one solution triple of (6.1.1).
2. Find a technique for obtaining all other solutions of (6.1.1) *given this one solution*. This subproblem, 2, proves to be much easier to solve than finding directly all solutions of (6.1.1).

The remaining sections of this chapter deal separately with subproblems 1 and 2.

We shall note two distinct procedures for solving subproblem 1. The first procedure requires that $J + J'$ be nonsingular (a restriction discussed below), and deduces one solution of (6.1.1) by solving a matrix Riccati differential equation or by solving an algebraic quadratic matrix equation. In the next section we state how the solutions of (6.1.1) are defined in terms of the solutions of a Riccati equation or quadratic matrix equation, in Section 6.3 we indicate how to solve the Riccati equation, and in Section 6.4 how to solve the algebraic quadratic matrix equation. The material of Section 6.2 will be presented without proof, the proof being contained earlier in a necessity proof of the positive real lemma.

The second (and less favored) procedure for solving subproblem 1 uses a matrix spectral factorization. In Section 6.5 we first find a solution for (6.1.1) for the case when the eigenvalues of F are restricted to lying in the half-plane $\text{Re}[s] < 0$, and then for the case when all eigenvalues of F lie on $\text{Re}[s] = 0$; finally we tie these two cases together to consider the general case, permitting eigenvalues of F both in $\text{Re}[s] < 0$ and on $\text{Re}[s] = 0$. Much of this material is based on the necessity proof for the positive real lemma via spectral factorization, which was discussed earlier. Knowledge of this earlier material is however not essential for an understanding of Section 6.5.

Subproblem 2 is discussed in Section 6.6. We show that this subproblem is equivalent to the problem of finding all solutions of a quadratic matrix inequality; this inequality has sufficient structure to enable a solution. There is however a restriction that must be imposed in solving subproblem 2, identical with the restriction imposed in providing the first solution to subproblem 1: the matrix $J + J'$ must be nonsingular.

We have already commented on this restriction in one of the necessity proofs of the positive real lemma. Let us however repeat one of the two remarks made there. This is that given the problem of synthesizing a positive real $Z(s)$ for which $J + J'$ is singular, there exists (as we shall later see) a sequence of elementary manipulations on $Z(s)$, corresponding to minor synthesis steps, such as the shunt extraction of capacitors, which reduces the problem of synthesizing $Z(s)$ to one of synthesizing a second positive real matrix, $Z_1(s)$ say, for which $J_1 + J_1'$ is nonsingular. These elementary manipulations require no difficult calculations, and certainly not the solving of (6.1.1). Since our goal is to use the positive real lemma as a synthesis tool, in fact as a replacement for the difficult parts of classical network synthesis, we are not cheating if we choose to restrict its use to a subclass of positive real matrices, the synthesis of which is essentially equivalent to the synthesis of an arbitrary positive real matrix.

Summing up, one solution to subproblem 1—the determination of one solution triple for (6.1.1)—is contained in Sections 6.2–6.4. Solution of a matrix Riccati equation or an algebraic quadratic matrix equation is required; frequency-domain manipulations are not. A second solution to subproblem 1, based on frequency-domain spectral factorization, is contained in Section 6.5. This section also considers separately the important special case of F possessing only imaginary eigenvalues, which turns out to correspond with $Z(s)$ being lossless. Section 6.6 offers a solution to subproblem 2. Finally, although the material of Sections 6.2–6.4 and Section 6.6 applies to a restricted class of positive real matrices, the restriction is inessential.

The closest references to the material of Sections 6.2–6.4 are [1] for the material dealing with the Riccati equation (though this reference considers a related nonstationary problem), and the report [2]. Reference [3] considers the solution of (6.1.1) via an algebraic quadratic matrix equation. The material in Section 6.5 is based on [4], which takes as its starting point a theorem (stated in Section 6.5) from [5]. The material in Section 6.6 dealing with the solution of subproblem 2 is drawn from [6].

Problem 6.1.1 Suppose that $Z(s) + Z'(-s) = W'(-s)W(s)$, $Z(s)$ being positive real with minimal realization $\{F, G, H, J\}$ and $W(s)$ possessing a realization $\{F, G, L, W_0\}$. Suppose that F has all eigenvalues with negative real part. Show that $W_0'W_0 = J + J'$. Define P by

$$PF + F'P = -LL'$$

and note that P is nonnegative definite symmetric by the lemma of Lyapunov. Apply this definition of P, rewritten as $P(sI - F) + (-sI - F')P = -LL'$, to deduce that $W'(-s)W(s) = W_0'W_0 + (PG + LW_0)'(sI - F)^{-1}G + G'(-sI - F')^{-1}(PG + LW_0)$. Conclude that

$$PG = H - LW_0$$

Show that if $L'e^{Ft}x_0 = 0$ for some nonzero x_0 and all t, the defining equation for P implies that $Pe^{Ft}x_0 = 0$ for all t, and then that $H'e^{Ft}x_0 = 0$ for all t. Conclude that P is positive definite. Explain the difficulty in applying this procedure for the generation of P if any element of $Z(s)$ has a purely imaginary pole.

6.2 SOLUTION COMPUTATION VIA RICCATI EQUATIONS AND QUADRATIC MATRIX EQUATIONS

In this section we study the construction of a particular solution triple P, L, and W_0 to the equations

$$\begin{aligned} PF + F'P &= -LL' \\ PG &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{6.2.1}$$

under the restriction that $J + J'$ is nonsingular. The implications of this restriction were discussed in the last section.

The following theorem was proved earlier in Section 5.4. We do not assume the reader has necessarily read this material; however, if he has not read it, he must of course take this theorem on faith.

Theorem. Let $Z(s)$ be a positive real matrix of rational functions of s, with $Z(\infty) < \infty$. Suppose that $\{F, G, H, J\}$ is a minimal realization of $Z(s)$, with $J + J' = R$, a nonsingular matrix. Then there exists a negative definite matrix $\bar{\Pi}$ satisfying the equation

$$\begin{aligned}\bar{\Pi}(F - GR^{-1}H') + (F' - HR^{-1}G')\bar{\Pi} \\ - \bar{\Pi}GR^{-1}G'\bar{\Pi} - HR^{-1}H' = 0\end{aligned} \tag{6.2.2}$$

Moreover

$$\bar{\Pi} = \lim_{t_1 \to \infty} \Pi(t, t_1) = \lim_{t \to -\infty} \Pi(t, t_1)$$

where $\Pi(\cdot, t_1)$ is the solution of the Riccati equation

$$\begin{aligned}-\dot{\Pi} = \Pi(F - GR^{-1}H') + (F' - HR^{-1}G')\Pi \\ - \Pi GR^{-1}G'\Pi - HR^{-1}H'\end{aligned} \tag{6.2.3}$$

with boundary condition $\Pi(t_1, t_1) = 0$.

The significance of this result in solving (6.2.1) is as follows. We define

$$P = -\bar{\Pi} \qquad W_0 = VR^{1/2} \qquad L = (\bar{\Pi}G + H)R^{-1/2}V' \tag{6.2.4}$$

where V is an arbitrary real orthogonal matrix; i.e., $V'V = VV' = I$. Then we claim that (6.2.1) is satisfied. To see this, rearrange (6.2.2) as

$$-(\bar{\Pi}F + F'\bar{\Pi}) = -(\bar{\Pi}G + H)R^{-1}(\bar{\Pi}G + H)'$$

and use (6.2.4) to yield the first of Eqs. (6.2.1). The second of equations (6.2.1) follows from (6.2.4) immediately, as does the third.

Thus one way to achieve a particular solution of (6.2.1) is to solve (6.2.3) to determine $\bar{\Pi}$, and then define P, L, and W_0 by (6.2.4). Technical procedures for solving (6.2.3) are discussed in the following sections.

We recall that the matrices L and W_0 define a transfer-function matrix $W(s)$ via

$$\begin{aligned}W(s) &= W_0 + L'(sI - F)^{-1}G \\ &= V[R^{1/2} + R^{-1/2}(\bar{\Pi}G + H)'(sI - F)^{-1}G]\end{aligned} \tag{6.2.5}$$

[Since V is an arbitrary real orthogonal matrix, (6.2.5) actually defines a family of $W(s)$.] This transfer-function matrix $W(s)$ satisfies

$$Z'(-s) + Z(s) = W'(-s)W(s) \tag{6.2.6}$$

Although infinitely many $W(s)$ satisfy (6.2.6), the family defined by (6.2.5) has a distinguishing property. In conjunction with the proof of the theorem just quoted, we showed that the inverse of $W(s)$ above with $V = I$ was defined throughout Re $[s] > 0$. Multiplication by the constant orthogonal V does not affect the conclusion. In other words, with $W(s)$ as in (6.2.5),

$$\det W(s) \neq 0 \qquad \text{Re}[s] > 0 \tag{6.2.7}$$

Youla, in reference [5], proves that *spectral factors $W(s)$ of (6.2.6) with the property (6.2.7), and with poles of all elements restricted to* Re $[s] \leq 0$, *are uniquely defined to within left multiplication by an arbitrary real constant orthogonal matrix.* Therefore, in view of our insertion of the matrix V in (6.2.5), we can assert that the Riccati equation approach generates all spectral factors of (6.2.6) that have the property noted by Youla.

Now we wish to note a minor variation of the above procedure. If the reader reviews the earlier argument, he will note that the essential fact we used in proving that (6.2.4) generates a solution to (6.2.1) was that $\bar{\Pi}$ satisfied (6.2.2). The fact that $\bar{\Pi}$ was computable from (6.2.3) was strictly incidental to the proof that (6.2.4) worked. Therefore, if we can solve the algebraic equation (6.2.2) directly—without the computational aid of the differential equation (6.2.3)—we still obtain a solution of (6.2.1). Direct solution of (6.2.2) is possible and is discussed in Section 6.4.

There is one factor, though, that we must not overlook. *In general, the algebraic equation (6.2.2) will possess more than one negative definite solution.* Just as a scalar quadratic equation may have more than one solution, so may a matrix quadratic equation. The exact number of solutions will generally be greater than two, but will be finite.

Of course, only one solution of the algebraic equation (6.2.2) is derivable by solving the Riccati differential equation (6.2.3). But the fact that a solution of the algebraic equation (6.2.2) is not derivable from the Riccati differential equation (6.2.3) does not debar it from defining a solution to the positive real lemma equations (6.2.1) via the defining equations (6.2.4).

Any solution of the algebraic equation (6.2.2) will generate a spectral factor family (6.2.5), different members of the family resulting from different choices of the orthogonal V. However, since our earlier proof for the invertibility of $W(s)$ in Re$[s] > 0$ relied on using that particular solution $\bar{\Pi}$ of the algebraic equation (6.2.2) that was derived from the Riccati equation (6.2.3), *it will not in general be true that a spectral factor $W(s)$ formed as in (6.2.5) by using any solution of the quadratic equation (6.2.2) will be invertible in* Re$[s] > 0$. Only that $W(s)$ formed by using the particular solution of the quadratic equation (6.2.2) derivable as the limit of the solution of the Riccati equation (6.2.3) will have the invertibility property.

In Section 6.4 when we consider procedures for solving the quadratic matrix equation (6.2.2) directly, we shall note how to isolate that particular solution of (6.2.2) obtainable from the Riccati equation, without of course having to solve the Riccati equation itself.

Finally, we note a simple property, sometimes important in applications, of the matrices L and W_0 generated by solving the algebraic or Riccati equations (6.2.4). This property is inherited by the spectral factor $W(s)$ defined by (6.2.5), and is that the number of rows of L', of W_0, and of $W(s)$ is m, where $Z(s)$ is an $m \times m$ matrix.

Examination of (6.2.1) or (6.2.6) shows that the number of rows of L', W_0, and $W(s)$ are not intrinsically determined. However, a lower bound is imposed. Thus with $J + J'$ nonsingular in (6.2.1), W_0 must have at least m rows to ensure that the rank of $W_0'W_0$ is equal to m, the rank of $J + J'$.

Since L' and $W(s)$ must have the same number of rows as W_0, it follows that *the procedures for solving (6.2.1) specified hitherto lead to L' and W_0 [and, as a consequence, $W(s)$] possessing the minimum number of rows.*

That there exist L' and W_0 with a nonminimal number of rows is easy to see. Indeed, let V be a matrix of $m' > m$ rows and m columns satisfying $V'V = I_m$ (such V always exist). Then (6.2.4) now defines L' and W_0 with a nonminimal number of rows, and (6.2.5) defines $W(s)$ also with a nonminimal number of rows. This is a somewhat trivial example. Later we shall come upon instances in which the matrix P in (6.2.1) is such that $PF + F'P$ has rank greater than m. [Hitherto, this has not been the case, since rank LL', with L as in (6.2.4), and V $m' \times m$ with $V'V = I_m$, is independent of V.] With $PF + F'P$ possessing rank greater than m, it follows that L' must have more than m rows for (6.2.1) to hold.

Problem 6.2.1 Assume available a solution triple for (6.2.1), with $J + J'$ nonsingular and W_0 possessing m rows, where m is the size of $Z(s)$. Show that L and W_0 may be eliminated to get an equation for P that is equivalent to (6.2.2) given the substitution $P = -\bar{\Pi}$.

Problem 6.2.2 Assuming that F has all eigenvalues in $\mathrm{Re}[s] < 0$, show that any solution $\bar{\Pi}$ of (6.2.2) is negative definite. First rearrange (6.2.2) as $-(\bar{\Pi}F + F'\bar{\Pi}) = -(\bar{\Pi}G + H)R^{-1}(\bar{\Pi}G + H)'$. Use the lemma of Lyapunov to show that $\bar{\Pi}$ is nonpositive definite. Assume the existence of a nonzero x_0 such that $(\bar{\Pi}G + H)'e^{Ft}x_0 = 0$ for all t, show that $\bar{\Pi}e^{Ft}x_0 = 0$ for all t, and deduce a contradiction. Conclude that $\bar{\Pi}$ is negative definite.

Problem 6.2.3 For the positive real impedance $Z(s) = \frac{1}{2} + 1/(s + 1)$, find a minimal realization $\{F, G, H, J\}$ and set up the equation for $\bar{\Pi}$. This equation, being scalar, is easily solved. Find two spectral factors by solving this equation, and verify that only one is nonzero throughout $\mathrm{Re}[s] > 0$.

6.3 SOLVING THE RICCATI EQUATION

In this section we wish to comment on techniques for solving the Riccati differential equation

$$-\dot{\Pi} = \Pi(F - GR^{-1}H') + (F' - HR^{-1}G')\Pi - \Pi GR^{-1}G'\Pi - HR^{-1}H' \tag{6.3.1}$$

where F, G, H, and R are known, $\Pi(t, t_1)$ is to be found for $t \leq t_1$, and the boundary condition is $\Pi(t_1, t_1) = 0$. In reality we are seeking $\lim_{t_1 \to \infty} \Pi(t, t_1) = \lim_{t \to -\infty} \Pi(t, t_1)$; the latter limit is obviously much easier to compute than the former.

One direct approach to solving (6.3.1) is simply to program the equation directly on a digital or analog computer. It appears that in practice the equation is computationally stable, and a theoretical analysis also predicts computational stability. Direct solution is thus quite feasible.

An alternative approach to the solution of (6.3.1) requires the solution of linear differential equations, instead of the nonlinear equation (6.3.1).

Riccati Equation Solution via Linear Equations

The computation of $\Pi(\cdot, t_1)$ is based on the following result, established in, for example, [7, 8].

Constructive Procedure for Obtaining $\Pi(t, t_1)$. Consider the equations

$$\frac{d}{dt}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} F - GR^{-1}H' & -GR^{-1}G' \\ HR^{-1}H' & -F' + HR^{-1}G' \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = M\begin{bmatrix} X \\ Y \end{bmatrix} \tag{6.3.2}$$

where $X(t)$ and $Y(t)$ are $n \times n$ matrices, n being the size of F, the initial condition is $X(t_1) = I$, $Y(t_1) = 0$, and the matrix M is defined in the obvious way. Then the solution of (6.3.1) exists on $[t, t_1]$ if and only if X^{-1} exists on $[t, t_1]$, and

$$\Pi(t, t_1) = Y(t)X^{-1}(t) \tag{6.3.3}$$

We shall not derive this result here, but refer the reader to [7, 8]. Problem 6.3.1 asks for a part derivation.

Note that when F, G, H, and $R = J + J'$ are derived from a positive real $Z(s)$, *existence of $\Pi(t, t_1)$ as the solution of (6.3.1) is guaranteed for all $t \leq t_1$*; this means that $X^{-1}(t)$ exists for all $t \leq t_1$, and accordingly the formula (6.3.3) is valid for all $t \leq t_1$.

An alternative expression for $\Pi(t, t_1)$ is available, which expresses $\Pi(t, t_1)$ in terms of $\exp M(t - \tau)$. *Call this $2n \times 2n$ matrix $\Phi(t - \tau)$ and partition it as*

$$\Phi(t - \tau) = \begin{bmatrix} \Phi_{11}(t - \tau) & \Phi_{12}(t - \tau) \\ \Phi_{21}(t - \tau) & \Phi_{22}(t - \tau) \end{bmatrix} \tag{6.3.4}$$

where each $\Phi_{ij}(t - \tau)$ is $n \times n$. Then the definitions of $X(t)$ and $Y(t)$ imply that

$$X(t) = \Phi_{11}(t - t_1) \qquad Y(t) = \Phi_{21}(t - t_1)$$

and so

$$\Pi(t - t_1) = \Phi_{21}(t - t_1)\Phi_{11}^{-1}(t - t_1) \tag{6.3.5}$$

Because of the constancy of F, G, H, and R, there is an analytic formula available for $\Phi(t - t_1)$. Certainly this formula involves matrix exponentiation, which may be difficult. In many instances, though, the exponentiation may be straightforward or perhaps amenable to one of the many numerical techniques now being developed for matrix exponentiation. It would appear that (6.3.2) is computationally unstable when solved on a digital computer. Therefore, it only seems attractive when analytic formulas for the $\Phi_{ij}(t - \tau)$ are used—and even then, difficulties may be encountered.

Example 6.3.1. Consider the positive real $Z(s) = \frac{1}{2} + 1/(s + 1)$. A minimal realization is provided by $F = -1$, $G = H = 1$, and $J = \frac{1}{2}$, and the equation for Φ becomes accordingly

$$\frac{d}{dt}\Phi(t - \tau) = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}\Phi(t - \tau)$$

Now Φ is given by exponentiating $\begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$. One way to compute the exponential is to diagonalize the matrix by a similarity transformation. Thus

$$\begin{bmatrix} \sqrt{3} - 2 & \sqrt{3} + 2 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{3} - 2 & \sqrt{3} + 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix}$$

Consequently

$$\begin{aligned} \Phi(t - \tau) &= \begin{bmatrix} \sqrt{3} - 2 & \sqrt{3} + 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{\sqrt{3}(t-\tau)} & 0 \\ 0 & e^{-\sqrt{3}(t-\tau)} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \sqrt{3} - 2 & \sqrt{3} + 2 \\ 1 & -1 \end{bmatrix}^{-1} \\ &= \frac{1}{2\sqrt{3}}\begin{bmatrix} (\sqrt{3} - 2)e^{\sqrt{3}(t-\tau)} + (\sqrt{3} + 2)e^{-\sqrt{3}(t-\tau)} & -e^{\sqrt{3}(t-\tau)} + e^{-\sqrt{3}(t-\tau)} \\ e^{\sqrt{3}(t-\tau)} - e^{-\sqrt{3}(t-\tau)} & (\sqrt{3} + 2)e^{\sqrt{3}(t-\tau)} + (\sqrt{3} - 2)e^{-\sqrt{3}(t-\tau)} \end{bmatrix} \end{aligned}$$

From (6.3.5),

$$\Pi(t, t_1) = \frac{e^{\sqrt{3}(t-t_1)} - e^{-\sqrt{3}(t-t_1)}}{(\sqrt{3} - 2)e^{\sqrt{3}(t-t_1)} + (\sqrt{3} + 2)e^{-\sqrt{3}(t-t_1)}}$$

and then

$$\lim_{t\to-\infty} \Pi(t, t_1) = -\frac{1}{\sqrt{3}+2} = -(2 - \sqrt{3})$$

In Section 6.4 we shall see that the operations involved in diagonalizing the matrix $\begin{bmatrix} F - GR^{-1}H' & -GR^{-1}G' \\ HR^{-1}H' & -F' + HR^{-1}G' \end{bmatrix}$ may be used to compute $\bar{\Pi} = \lim_{t\to-\infty} \Pi(t, t_1)$ directly, avoiding the step of computing $\Phi(t - t_1)$ and $\Pi(t, t_1)$ by the formula (6.3.5).

Problem 6.3.1 Consider (6.3.1) and (6.3.2). Show that if $X^{-1}(t)$ exists for all $t \leq t_1$, then $\Pi(t, t_1)$ satisfies (6.3.3).

Problem 6.3.2 For the example discussed in the text, the Riccati equation becomes

$$-\dot{\Pi} = -4\Pi - \Pi^2 - 1$$

Solve this equation directly by using

$$\int_{\Pi(t)}^{\Pi(t_1)} \frac{d\Pi}{\Pi^2 + 4\Pi + 1} = (t_1 - t)$$

and check the limiting value.

6.4 SOLVING THE QUADRATIC MATRIX EQUATION

In this section we study three techniques for solving the equation

$$\bar{\Pi}(F - GR^{-1}H') + (F' - HR^{-1}G')\bar{\Pi} - \bar{\Pi}GR^{-1}G'\bar{\Pi} - HR^{-1}H' = 0 \tag{6.4.1}$$

The first technique involves the calculation of the eigenvalues and eigenvectors of a $2n \times 2n$ matrix, n being the size of F. It allows determination of all solutions of (6.4.1), and also allows us to isolate that particular solution of (6.4.1) that is the limit of the solution of the associated Riccati equation. The basic ideas appeared in [9, 10], with their application to the problem in hand in [3]. The second procedure is closely related to the first, but represents a potential saving computationally, since eigenvalues alone, rather than eigenvalues and eigenvectors, of a certain matrix must be calculated.

The third technique is quite different. Only one solution of (6.4.1) is calculated, viz., that which is the limit of the solution of the associated Riccati equation, and the calculation is via a *recursive difference equation.* This technique appeared in [11]. A fourth technique appears in the problems and involves replacing (6.4.1) by a sequence of linear matrix equations.

Method Based on Eigenvalue and Eigenvector Computation

From the coefficient matrices in (6.4.1) we construct the matrix

$$M = \begin{bmatrix} F - GR^{-1}H' & -GR^{-1}G' \\ HR^{-1}H' & -F' + HR^{-1}G' \end{bmatrix} \tag{6.4.2}$$

This can be shown to have the property that if λ is an eigenvalue, $-\lambda$ is also an eigenvalue, and if λ is pure imaginary, λ has even multiplicity (Problem 6.4.1 requests this verification).

To avoid complications, we shall assume that the matrix M is diagonalizable. Accordingly, there exists a matrix T such that

$$T^{-1}MT = \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \tag{6.4.3}$$

where Λ is the direct sum of 1×1 matrices $[\lambda_i]$ with λ_i real, and 2×2 matrices $\begin{bmatrix} \lambda_i & -\mu_i \\ \mu_i & \lambda_i \end{bmatrix}$ with λ_i, μ_i real.

Let T be partitioned into four $n \times n$ submatrices as

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{6.4.4}$$

Then if T_{11} is nonsingular, a solution of (6.4.1) is provided by

$$\bar{\Pi} = T_{21}T_{11}^{-1} \tag{6.4.5}$$

Let us verify this result. Multiplying both sides of (6.4.3) on the left by T, we obtain

$$\begin{aligned} (F - GR^{-1}H')T_{11} - GR^{-1}G'T_{21} &= -T_{11}\Lambda \\ HR^{-1}H'T_{11} - (F' - HR^{-1}G')T_{21} &= -T_{21}\Lambda \end{aligned} \tag{6.4.6}$$

From these equations we have

$$\begin{aligned} T_{21}T_{11}^{-1}(F - GR^{-1}H') - T_{21}T_{11}^{-1}GR^{-1}G'T_{21}T_{11}^{-1} &= -T_{21}\Lambda T_{11}^{-1} \\ HR^{-1}H' - (F' - HR^{-1}G')T_{21}T_{11}^{-1} &= -T_{21}\Lambda T_{11}^{-1} \end{aligned}$$

The right-hand sides of both equations are the same. On equating the left sides and using (6.4.5), Eq. (6.4.1) is recovered.

Notice that we have not restricted the matrix Λ in (6.4.3) to having all nonnegative or all nonpositive diagonal entries. This means that there are many different Λ that can arise in (6.4.3). If $+2$ (and thus -2) is an eigenvalue of M, then we can assign $+2$ to either $+\Lambda$ or $-\Lambda$; so there can be up to 2^n different Λ, with this number being attained only when M has all its eigenvalues real, with no two the same. Of course, different Λ yield different T and thus different $\bar{\Pi}$, provided T_{11}^{-1} exists.

One might ask whether there are solutions of (6.4.1) not obtainable in the manner specified above. The answer is no. Reference [9] shows for a related problem that any $\bar{\Pi}$ satisfying (6.4.1) must have the form of (6.4.5).

The above technique can be made to yield that particular $\bar{\Pi}$ which is the limit of the associated Riccati equation solution. *Choose* Λ *to have nonnegative diagonal entries. We claim that* (6.4.5) *then defines the required* $\bar{\Pi}$. To see this, observe from the first of Eq. (6.4.6) that

$$F - GR^{-1}H' - GR^{-1}G'T_{21}T_{11}^{-1} = -T_{11}\Lambda T_{11}^{-1}$$

or

$$F - GR^{-1}(H' + G'\bar{\Pi}) = -T_{11}\Lambda T_{11}^{-1} \tag{6.4.7}$$

Since Λ has nonnegative diagonal entries, all eigenvalues of $F - GR^{-1}(H' + G'\bar{\Pi})$ must have nonpositive real parts.

Now the spectral factor family defined by $\bar{\Pi}$ is, from an earlier section,

$$W(s) = V[R^{1/2} + R^{-1/2}(\bar{\Pi}G + H)'(sI - F)^{-1}G] \tag{6.4.8}$$

where $VV' = V'V = I$. Problem 6.4.2 requests proof that if $F - GR^{-1}(H' + G'\bar{\Pi})$ has all eigenvalues with nonpositive real parts, $W^{-1}(s)$ exists in $\text{Re}[s] > 0$. (We have noted this result also in an earlier necessity proof of the positive real lemma.) As we know, the family of $W(s)$ of the form of (6.4.8) with the additional property that $W^{-1}(s)$ exists in $\text{Re}[s] > 0$ is unique, and is also determinable via a $\bar{\Pi}$ that is the limit of a Riccati equation solution. Clearly, different $\bar{\Pi}$ could not yield the same family of $W(s)$. Hence the $\bar{\Pi}$ defined by the special choice of Λ is that $\bar{\Pi}$ which is the limit of the solution of the associated Riccati equation.

Example 6.4.1 Consider $Z(s) = \frac{1}{2} + 1/(s + 1)$, for which $F = -1$, $G = H = R = 1$. The matrix M is

$$M = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} \sqrt{3}-2 & \sqrt{3}+2 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{3}-2 & \sqrt{3}+2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{bmatrix}$$

Hence $T_{11} = \sqrt{3} - 2$ and $T_{21} = 1$, so

$$\bar{\Pi} = \frac{1}{\sqrt{3}-2} = -(\sqrt{3}+2).$$

Alternatively, to obtain that $\bar{\Pi}$ which is the limit of the Riccati equation solution, we have

$$\begin{bmatrix} \sqrt{3}+2 & \sqrt{3}-2 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{3}+2 & \sqrt{3}-2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

(Thus $\Lambda = \sqrt{3}$.) But now

$$\bar{\Pi} = -\frac{1}{\sqrt{3}+2} = -(2-\sqrt{3})$$

This agrees with the calculation obtained in the previous section.

Method Based on Eigenvalue Computation Only

As before, we form the matrix M. But now we require only its eigenvalues. If $p(s)$ is the characteristic polynomial of M, the already mentioned properties of the eigenvalues of M guarantee the existence of a monic polynomial $r(s)$ such that

$$p(s) = r(-s)r(s) \tag{6.4.9}$$

If desired, $r(s)$ can be taken to have no zeros in $\mathrm{Re}[s] > 0$.

Now with $r(s) = s^n + a_1 s^{n-1} + \cdots + a_n$, let $r(M) = M^n + a_1 M^{n-1} + \cdots + a_n I$. We claim that a solution $\bar{\Pi}$ of the quadratic equation (6.4.1) is defined by

$$r(M)\begin{bmatrix} I \\ \bar{\Pi} \end{bmatrix} = 0 \tag{6.4.10}$$

Moreover, that solution which is also the limit of the solution of the associated Riccati equation is achieved by taking $r(s)$ to have no zeros in $\mathrm{Re}[s] > 0$.

We shall now justify the procedure. The argument is straightforward; for convenience, we shall restrict ourselves to the special case when $r(s)$ has no zeros in $\text{Re}[s] > 0$.

Let T be as in (6.4.3), with diagonal entries of Λ possessing nonnegative real parts. Then

$$T^{-1}r(M)T = r(T^{-1}MT) = \begin{bmatrix} r(-\Lambda) & 0 \\ 0 & r(\Lambda) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & r(\Lambda) \end{bmatrix}$$

The last step follows because the zeros of $r(s)$ are the nonpositive real part eigenvalues of M, which are also eigenvalues of $-\Lambda$. The Cayley–Hamilton theorem guarantees that $r(-\Lambda)$ will equal zero. Now

$$r(M)\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & r(\Lambda) \end{bmatrix}$$

whence

$$r(M)\begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} = 0$$

or

$$r(M)\begin{bmatrix} I \\ T_{21}T_{11}^{-1} \end{bmatrix} = 0$$

The first method discussed established that $T_{21}T_{11}^{-1} = \bar{\Pi}$, with $\bar{\Pi}$ the limit of the Riccati equation solution.

Equation (6.4.10) is straightforward to solve, being a linear matrix equation.

For large n, the computational difficulties associated with finding eigenvalues and eigenvectors increase enormously, and almost certainly direct solution of the Riccati equation or use of the next method to be noted are the only feasible approaches to obtaining $\bar{\Pi}$.

Example 6.4.2 With the same positive real function as in Example 6.4.1, viz., $Z(s) = \frac{1}{2} + 1/(s+1)$, with $F = -1$, $G = H = R = 1$, we have

$$M = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$$

and $p(s) = s^2 - 3$. Then $r(s) = s + \sqrt{3}$ and

$$r(M) = \begin{bmatrix} -2+\sqrt{3} & -1 \\ 1 & 2+\sqrt{3} \end{bmatrix}$$

Consequently

$$\begin{bmatrix} -2+\sqrt{3} & -1 \\ 1 & 2+\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ \bar{\Pi} \end{bmatrix} = 0$$

yielding, as before,

$$\bar{\Pi} = -2 + \sqrt{3}$$

Method Based on Recursive Difference Equation

The proof of the method we are about to state relies for its justification on concepts from optimal-control theory that would take us too long to present. Therefore, we refer the interested reader to [11] for background and content ourselves here with a statement of the procedure.

Define the matrices A, B, C, and U by

$$A = (I+F)(I-F)^{-1} \qquad B = \frac{1}{\sqrt{2}}(A+I)G$$
$$C = \frac{1}{\sqrt{2}}(A'+I)H \qquad U = R + C'(A+I)^{-1}B + B'(A'+I)^{-1}C \tag{6.4.11}$$

and form the recursive difference equation

$$\hat{\Pi}(n+1) = A'\hat{\Pi}(n)A - [A'\hat{\Pi}(n)B + C][U + B'\hat{\Pi}(n)B]^{\#}[B'\hat{\Pi}(n)A + C'] \tag{6.4.12}$$

with initial condition $\hat{\Pi}(0) = 0$. The significance of (6.4.12) is that

$$\bar{\Pi} = \lim_{n\to\infty} \hat{\Pi}(n) \tag{6.4.13}$$

where $\bar{\Pi}$ is that solution of (6.4.1) which is also the limit of the associated Riccati equation solution. This method however extends to singular R.

As an example, consider again $Z(s) = \frac{1}{2} + 1/(s+1)$, with $F = -1$, $G = H = R = 1$. It follows that $A = 0$, $B = C = 1/\sqrt{2}$, $U = 2$, and (6.4.12) becomes

$$\hat{\Pi}(n+1) = -\frac{1}{4+\hat{\Pi}(n)}$$

The iterates are $\hat{\Pi}(0) = 0$, $\hat{\Pi}(1) = -.25$, $\hat{\Pi}(2) = -.26667$, $\hat{\Pi}(3) = -.26785$, $\hat{\Pi}(4) = -.26794$.

As we have seen, $\bar{\Pi} = -2 + \sqrt{3} = -.26795$ to five figures. Thus four or five iterations suffice in this case.

Essentially, Eqs. (6.4.11) and (6.4.12) define an optimal-control problem and its solution in discrete time. The limiting solution of the problem is set up to be the same as $\bar{\Pi}$.

The reader will notice that if F has an eigenvalue at $+1$, the matrix A in (6.4.11) will not exist. The way around this difficulty is to modify (6.4.11), defining A, B, C, and U by

$$A=(\alpha I+F)(\alpha I-F)^{-1} \qquad B=\frac{1}{\sqrt{2\alpha}}(A+I)G$$
$$C=\frac{1}{\sqrt{2\alpha}}(A'+I)H \qquad U=\frac{1}{\alpha}[R+\frac{1}{\alpha}C'(A+I)^{-1}B+\frac{1}{\alpha}B'(A'+I)^{-1}C] \tag{6.4.14}$$

Here α is a real constant that may be selected arbitrarily, subject to the restriction that $\alpha I - F$ must be nonsingular. With these new definitions, (6.4.12) and (6.4.13) remain unchanged.

Experimental simulations suggest that a satisfactory value of α is n^{-1} tr F, where F is $n \times n$. Variation in α causes variation in the rate of convergence of the sequence of iterates, and variation in the amount of roundoff error.

Problem 6.4.1 Show that if λ is an eigenvalue of M in (6.4.2), so is $-\lambda$. (From an eigenvector u satisfying $Mu = \lambda u$, construct a row vector v' such that $v'M = -\lambda v'$.) Also show that if λ is an eigenvalue that is pure imaginary, it has even multiplicity. Do this by first proving

$$sI-M=\begin{bmatrix} sI-F & 0 \\ 0 & sI+F'\end{bmatrix}\left[I+\begin{bmatrix}(sI-F)^{-1}GR^{-1} \\ (-sI-F')^{-1}HR^{-1}\end{bmatrix}\right][H' G']$$

Take determinants and use the nonnegativity of $Z(s)+Z'(-s)$ for $s = j\omega$.

Problem 6.4.2 Suppose that $W(s) = R^{1/2} + R^{-1/2}(\bar{\Pi}G + H)'(sI - F)^{-1}G$. Prove that if $F - GR^{-1}(H' + G'\bar{\Pi})$ has no eigenvalues with positive real parts, then $W^{-1}(s)$ exists throughout $\text{Re}[s] > 0$.

Problem 6.4.3 Choose a degree 2 positive real $Z(s)$ and construct a minimal realization for it. Apply the three methods of this section to solve (6.4.1). Compare the computational efficiencies of each method.

Problem 6.4.4 Use the eigenvalue-eigenvector computation method to construct a spectral factor $W(s)$ for which $W^{-1}(s)$ exists everywhere in $\text{Re}[s] < 0$.

Problem 6.4.5 With a positive real matrix $Z(s)$ of realization $\{F, G, H, J\}$ with $R = J + J'$ nonsingular we can associate a quadratic equation for the matrix $\bar{\Pi}$. Show that the inverse of each such $\bar{\Pi}$ satisfies the quadratic equation associated with the realization $\{F', H, G, J'\}$ of $Z'(s)$. If $F' - HR^{-1}(G' + H'\bar{\Pi}^{-1})$ has eigenvalues with negative real parts [which would be the case if $\bar{\Pi}^{-1}$ were determined as the limiting solution of the Riccati equation associated with $Z'(s)$], show that $F - GR^{-1}(H' + G'\bar{\Pi})$ has eigenvalues with positive real parts.

Problem 6.4.6 Show that an attempted solution of the quadratic equation (6.4.1) with a Newton-Raphson procedure leads to the iterative equation

$$\bar{\Pi}_{n+1}[F - GR^{-1}(H' + G'\bar{\Pi}_n)] + [F' - (H + \bar{\Pi}_n G)R^{-1}G']\bar{\Pi}_{n+1} = HR^{-1}H' - \bar{\Pi}_n GR^{-1}G'\bar{\Pi}_n$$

(If you are unfamiliar with Newton-Raphson procedures, pass immediately to the remainder of the problem.) The algorithm is initialized by taking $\bar{\Pi}_0 = 0$. Define

$$F_n = F - GR^{-1}(H' + G'\bar{\Pi}_n)$$

As the basis of an inductive argument to establish convergence, show that Re $\lambda_i[F_0] < 0$ and assume that Re $\lambda_i[F_n] \leq 0$. Evaluate $(\bar{\Pi}_{n+1} - \bar{\Pi}_n) F_n + F_n'(\bar{\Pi}_{n+1} - \bar{\Pi}_n)$ and show that $\bar{\Pi}_{n+1} - \bar{\Pi}_n \leq 0$. Show also that $\bar{\Pi} - \bar{\Pi}_{n+1} \leq 0$ by evaluating $(\bar{\Pi} - \bar{\Pi}_{n+1})F_n + F_n'(\bar{\Pi} - \bar{\Pi}_{n+1})$ where $\bar{\Pi}$ is that solution of (6.4.1) derivable by solving the Riccati equation (6.3.1). Use the inequality $\bar{\Pi} - \bar{\Pi}_{n+1} \leq 0$ and an expression for $(\bar{\Pi} - \bar{\Pi}_{n+1}) F_{n+1} + F_{n+1}'(\bar{\Pi} - \bar{\Pi}_{n+1})$ to show that Re $\lambda_i[F_{n+1}] \leq 0$. Conclude that $\bar{\Pi}_n$ converges monotonically to $\bar{\Pi}$.

6.5 SOLUTION COMPUTATION VIA SPECTRAL FACTORIZATION

In this section we study the derivation, via spectral factorization, of a particular solution triple P, L, and W_0 to the positive real lemma equations

$$\begin{aligned} PF + F'P &= -LL' \\ PG &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{6.5.1}$$

We consider first the case when F has all negative real part eigenvalues, and then the case when F has pure imaginary eigenvalues. Finally, we link both cases together. As will be seen, the case when F has pure imaginary eigenvalues corresponds to the matrix $Z(s) = H'(sI - F)^{-1}G$ being lossless (the presence or absence of J is irrelevant), and the computation of solutions to (6.5.1) is especially straightforward. This case is therefore important in its own right.

Case of F Possessing Negative Real Part Eigenvalues

In considering this case we shall restate a number of results, here without proof, that we have already discussed in presenting a necessity proof for the positive real lemma based on spectral factorization.

First note that the eigenvalue restriction on F implies that no element of $Z(s)$ can have a pole on $\text{Re}[s] = 0$ (or, of course, in $\text{Re}[s] > 0$). A procedure specified in [5] then yields a matrix $W(s)$ such that

$$Z'(-s) + Z(s) = W'(-s)W(s) \tag{6.5.2}$$

with $W(s)$ possessing the following additional properties:

1. $W(s)$ is $r \times m$, where $Z(s)$ is $m \times m$ and $Z'(-s) + Z(s)$ has rank r almost everywhere.
2. $W(s)$ has no element with a pole in $\text{Re}[s] \geq 0$.
3. $W(s)$ has constant rank r in $\text{Re}[s] > 0$; i.e., there exists a right inverse for $W(s)$ defined throughout $\text{Re}[s] > 0$.
4. $W(s)$ is unique to within left multiplication by an arbitrary real constant orthogonal $r \times r$ matrix.

In view of property 4 we can think of the result of [5] as specifying a *family* of $W(s)$ satisfying (6.5.2) and properties 1, 2, and 3, members of the family differing by left multiplication by an arbitrary real constant orthogonal $r \times r$ matrix. In the case when $Z(s)$ is scalar, the orthogonal matrix degenerates to being a scalar $+1$ or -1.

To solve (6.5.1), we shall proceed in two steps: first, find $W(s)$ as described above; second, from $W(s)$ define P, L, and W_0.

For the execution of the first step, we refer the reader to [5] in the case of matrix $Z(s)$, and indicate briefly here the procedure for scalar $Z(s)$. This procedure was discussed more fully in Chapter 5 in the necessity proof of the positive real lemma by spectral factorization.

We represent $Z(s)$ as the ratio of two polynomials, with the denominator polynomial monic; thus

$$Z(s) = \frac{q(s)}{p(s)} \tag{6.5.3}$$

Then we form $q(s)p(-s) + p(s)q(-s)$ and factor it in the following fashion:

$$q(s)p(-s) + p(s)q(-s) = r(-s)r(s) \tag{6.5.4}$$

where $r(s)$ is a real polynomial with no zero in $\text{Re}[s] > 0$ (the earlier discussion explains why this factorization is possible). Then we set

$$W(s) = \frac{r(s)}{p(s)} \tag{6.5.5}$$

Problem 6.5.1 asks for verification that this $W(s)$ satisfies all the requisite conditions.

The second step required to obtain a solution of (6.5.1) is to pass from a $W(s)$ satisfying (6.5.2) and the associated additional constraints to matrices P, L, and W_0 satisfying (6.5.1).

We now quote a further result from the earlier necessity proof of the positive real lemma by spectral factorization:

> If $W(s)$ satisfies (6.5.2) and the associated additional conditions, then $W(s)$ possesses a minimal realization of the form $\{F, G, L, W_0\}$ for some matrices L and W_0. Further, if the first of Eqs. (6.5.1) is used to *define* P, the second is automatically satisfied. The third is also satisfied.

Accordingly, to solve (6.5.1) we must

1. Construct a minimal realization of $W(s)$ with the same F and G matrix as $Z(s)$; this defines L and W_0 by $W(s) = W_0 + L'(sI - F)^{-1}G$.
2. Solve the first of Eqs. (6.5.1) to give P. Then P, L, and W_0 jointly satisfy (6.5.1).

Let us outline each of these steps in turn. Both turn out to be simple.

From $W(s)$ we can construct a minimal realization by any known method. In general, this realization will not have the same F and G matrix as $Z(s)$. Denote it therefore by $\{F_w, G_w, L_w, W_0\}$. We are assured that there exists a minimal realization of the form $\{F, G, L, W_0\}$, and therefore there must exist a nonsingular T such that

$$TF_wT^{-1} = F \qquad TG_w = G \qquad (T^{-1})'L_w = L \tag{6.5.6}$$

If we can find T, then construction of L is immediate. So our immediate problem is to find T, the data at our disposal being knowledge of F_w, G_w, L_w, and W_0 from the minimal realization of $W(s)$, and F and G from the minimal realization of $Z(s)$.

But now Eqs. (6.5.6) imply that

$$T[G_w \; F_wG_w \cdots F_w^{n-1}G_w] = [G \; FG \cdots F^{n-1}G] \tag{6.5.7}$$

where n is the dimension of F. Equation (6.5.7) is solvable for T, because the matrix $[G_w \; F_wG_w \cdots F_w^{n-1}G_w]$ has rank n through the minimality, and thus complete controllability, of $\{F_w, G_w, L_w, W_0\}$. Writing (6.5.7) as $TV_w = V$, where V_w and V are obviously defined, we have

$$T = VV_w'[V_wV_w']^{-1} \tag{6.5.8}$$

the inverse existing because V_w has rank n, equal to the number of its rows.

In summary, therefore, to obtain a minimal realization for $W(s)$ of the form $\{F, G, L, W_0\}$, first form any minimal realization $\{F_w, G_w, L_w, W_0\}$.

Then with $V = [G\ FG \cdots F^{n-1}G]$ and V_w similarly defined, define T by (6.5.8). Then the first two equations of (6.5.6) are satisfied, and the third yields L.

Having obtained a minimal realization for $W(s)$ of the form $\{F, G, L, W_0\}$, all that remains is to solve the first of Eqs. (6.5.1). This, being a linear matrix equation for P, is readily solved by procedures outlined in, e.g., [12].

The most difficult part computationally of the whole process of finding a solution triple P, L, W_0 to (6.5.1) is undoubtedly the determination of $W(s)$ in frequency-domain form, especially when $Z(s)$ is a matrix. The manipulations involved in passing from $W(s)$ to P, L, and W_0 are by comparison straightforward, involving operations no more complicated than matrix inversion or solving linear matrix equations.

It is interesting to compare the different solutions of (6.5.1) that follow from different members of the family of $W(s)$ satisfying (6.5.2) and properties 1, 2, and 3. As we know, such $W(s)$ differ by left multiplication by a real constant orthogonal $r \times r$ matrix. Let V be the matrix relating two particular $W(s)$, say $W_1(s)$ and $W_2(s)$. With $W_1(s)$ and $W_2(s)$ possessing minimal realizations $\{F, G, L_1, W_{01}\}$ and $\{F, G, L_2, W_{02}\}$, it follows that $L_1' = VL_2'$ and $W_{01} = VW_{02}$. Now observe that $L_1L_1' = L_2L_2'$ and $L_1W_{01} = L_2W_{02}$ because V is orthogonal. It follows that the matrix P in (6.5.1) is the same for $W_1(s)$ and $W_2(s)$, and thus *the matrix P is independent of the particular member of the family of $W(s)$ satisfying (6.5.2) and the associated properties 1, 2, and 3*. Put another way, P is associated purely with the family.

Example 6.5.1 Consider $Z(s) = (s + 2)(s^2 + 4s + 3)^{-1}$, which can be verified to be positive real. This function possesses a minimal realization

$$\left\{\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

Now observe that

$$\begin{aligned} Z(s) + Z(-s) &= \frac{-4s^2 + 12}{(s^2 - 4s + 3)(s^2 + 4s + 3)} \\ &= \frac{-2s + 2\sqrt{3}}{s^2 - 4s + 3} \frac{2s + 2\sqrt{3}}{s^2 + 4s + 3} \end{aligned}$$

Evidently, $W(s) = (2s + 2\sqrt{3})(s^2 + 4s + 3)^{-1}$ and a minimal realization for $W(s)$, with the same F and G matrices as the minimal realization for $Z(s)$, is given by

$$\left\{\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2\sqrt{3} \\ +2 \end{bmatrix}\right\}$$

The equation for P is then

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}+\begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$= -\begin{bmatrix} 2\sqrt{3} \\ +2 \end{bmatrix}[2\sqrt{3} \quad +2]$$

That is,

$$P = \begin{bmatrix} 11 - 4\sqrt{3} & 2 \\ 2 & 1 \end{bmatrix}$$

Of course, $L' = [2\sqrt{3} \quad +2]$ and $W_0 = 0$.

Case of F Possessing Pure Imaginary Eigenvalues—Lossless $Z(s)$

We wish to tackle the solution of (6.5.1) with F possessing pure imaginary eigenvalues only. We shall show that this is essentially the same as the problem of solving (6.5.1) when $Z(s)$ is lossless. We shall then note that the solution is very easy to obtain and is the only one possible.

Because F has pure imaginary eigenvalues, $Z(s)$ has pure imaginary poles. Then, as outlined in an earlier chapter, we can write

$$Z(s) = J + \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} + \frac{C}{s} \tag{6.5.9}$$

where the ω_i are real, and the matrices J, C, A_i, and B_i satisfy constraints that we have listed earlier. The matrix

$$Z_L(s) = \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} + \frac{C}{s} \tag{6.5.10}$$

is lossless positive real and has a minimal realization $\{F, G, H\}$.

Now because F has pure imaginary eigenvalues and P is positive definite, the only L for which the equation $PF + F'P = -LL'$ can be valid is $L = 0$, by an extension of the lemma of Lyapunov described in an earlier chapter. Consequently, our search for solutions of (6.5.1) becomes one for solutions of

$$\begin{aligned} PF + F'P &= 0 \\ PG &= H \\ J + J' &= W_0'W_0 \end{aligned} \tag{6.5.11}$$

Now observe that we have two decoupled problems: one requires the determination of W_0 satisfying the last of Eqs. (6.5.11); this is trivial and will not be further discussed. The other requires the determination of a positive definite P such that

$$\begin{aligned} PF + F'P &= 0 \\ PG &= H \end{aligned} \tag{6.5.12}$$

These equations, which exclude J, are the positive real lemma equations associated with the lossless positive real matrix $Z_L(s)$, since $Z_L(s)$ has minimal realization $\{F, G, H\}$.

Thus we have shown that essentially the problem of solving (6.5.1) with F possessing pure imaginary eigenvalues is one of solving (6.5.12), the positive real lemma equations associated with a lossless $Z(s)$. We now solve (6.5.12), and in so doing, establish that P is unique.

The solution of (6.5.12) is very straightforward. One way to achieve it is as follows. (Problem 6.5.2 asks for a second technique, which leads to the same value of P.) From (6.5.12) we have

$$\begin{aligned} PFG &= -F'PG = -F'H \\ PF^2G &= -F'PFG = (F')^2H \\ PF^3G &= -F'PF^2G = -(F')^3H \\ &\text{etc.} \end{aligned}$$

So

$$\begin{aligned} &P[G\ FG\ F^2G \cdots F^{n-1}G] \\ &\quad = [H\ -F'H\ (F')^2H \cdots (-1)^{n-1}(F')^{n-1}H] \end{aligned} \tag{6.5.13}$$

Set

$$V_c = [G\ FG \cdots F^{n-1}G] \qquad V_0 = [H\ -F'H\ (F')^2H \cdots (-1)^{n-1}(F')^{n-1}H] \tag{6.5.14}$$

Then V_c has rank n if F is $n \times n$ because of complete controllability. Consequently $(V_cV_c')^{-1}$ exists and

$$P = V_0V_c'(V_cV_c')^{-1} \tag{6.5.15}$$

The lossless positive real lemma guarantees existence of a positive definite P satisfying (6.5.12). Also, (6.5.15) is a direct consequence of (6.5.12). Therefore, (6.5.15) defines a solution of (6.5.12) that must be positive definite symmetric, and because (6.5.15) defines a unique P, *the solution of (6.5.12) must be unique.*

Case of F Possessing Nonpositive Real Part Eigenvalues

We now consider the solution of (6.5.1) when F may have negative or zero real part eigenvalues. Thus the entries of $Z(s)$ may have poles in $\text{Re}[s] < 0$ or on $\text{Re}[s] = 0$.

The basic idea is to apply an earlier stated decomposition property to break up the problem of solving (6.5.1) into two separate problems, both requiring the solutions of equations like (6.5.1) but for cases that we know how to handle. Then we combine the two separate solutions together.

Accordingly we write $Z(s)$ as

$$Z(s) = Z_0(s) + \sum_i \frac{A_i s + B_i}{s^2 + \omega_i^2} + \frac{C}{s} \tag{6.5.16}$$

Here the ω_i are real, and the matrices C, A_i, and B_i satisfy constraints that were listed earlier and do not concern us here, but which guarantee that $Z_1(s) = \sum_i (A_i s + B_i)(s^2 + \omega_i^2)^{-1} + s^{-1}C$ is lossless positive real. The matrix $Z_0(s)$ is also positive real, and each element has all poles restricted to $\text{Re}[s] < 0$.

Let $\{F_0, G_0, H_0, J_0\}$ be a minimal realization for $Z_0(s)$ and $\{F_1, G_1, H_1\}$ a minimal realization for $Z_1(s)$. Evidently, a realization for $Z(s)$ is provided by

$$\left\{\begin{bmatrix} F_0 & 0 \\ 0 & F_1 \end{bmatrix}, \begin{bmatrix} G_0 \\ G_1 \end{bmatrix}, \begin{bmatrix} H_0 \\ H_1 \end{bmatrix}, J_0\right\}$$

This realization is moreover minimal, since no element of $Z_0(s)$ has a pole in common with any element of $Z_1(s)$. In general, an arbitrary realization $\{F, G, H, J\}$ for $Z(s)$ is such that $J = J_0$, although it will not normally be true that $F = \begin{bmatrix} F_0 & 0 \\ 0 & F_1 \end{bmatrix}$, etc. But there exists a T, such that

$$F = T^{-1}\begin{bmatrix} F_0 & 0 \\ 0 & F_1 \end{bmatrix}T \qquad G = T^{-1}\begin{bmatrix} G_0 \\ G_1 \end{bmatrix} \qquad H = T'\begin{bmatrix} H_0 \\ H_1 \end{bmatrix} \tag{6.5.17}$$

Further, T is readily computable from the controllability matrices

$$V = [G \; FG \cdots F^{n-1}G]$$

and

$$\hat{V} = \left[\begin{bmatrix} G_0 \\ G_1 \end{bmatrix}, \begin{bmatrix} F_0G_0 \\ F_1G_1 \end{bmatrix}, \cdots \begin{bmatrix} F_0^{n-1}G_0 \\ F_1^{n-1}G_1 \end{bmatrix}\right]$$

which both have full rank. In fact, $TV = \hat{V}$ and so

$$T = \hat{V}V'(VV')^{-1} \tag{6.5.18}$$

And now the determination of P, L, and W_0 satisfying (6.5.1) for an arbitrary realization $\{F, G, H, J\}$ is straightforward. Obtain P_0, L_0, and W_0 satisfying

$$P_0F_0 + F_0'P_0 = -L_0L_0' \qquad P_0G_0 = H_0 - L_0W_0 \qquad J + J' = W_0'W_0 \tag{6.5.19}$$

and P_1 satisfying

$$P_1F_1 + F_1'P_1 = 0 \qquad P_1G_1 = H_1 \tag{6.5.20}$$

Of course, P_0 and P_1 are obtained by the procedures already discussed in this section. Then it is straightforward to verify that solutions of (6.5.1) are provided by W_0 and

$$P = T'\begin{bmatrix} P_0 & 0 \\ 0 & P_1 \end{bmatrix}T \qquad L = T'\begin{bmatrix} L_0 \\ 0 \end{bmatrix} \tag{6.5.21}$$

Of course, the transfer-function matrix $W(s) = W_0 + L'(sI - F)^{-1}G$ is a spectral factor of $Z(s) + Z'(-s)$. In fact, slightly more is true. Use of (6.5.21) and (6.5.17) shows that

$$W(s) = W_0 + L_0'(sI - F_0)^{-1}G_0 \tag{6.5.22}$$

Thus $W(s)$ is a spectral factor of both $Z'(-s) + Z(s)$ and $Z_0'(-s) + Z_0(s)$. This is hardly surprising, since the fact that $Z_1(s)$ is lossless guarantees that

$$Z'(-s) + Z(s) = Z_0'(-s) + Z_0(s) \tag{6.5.23}$$

The realization $\{F, G, L, W_0\}$ of $W(s)$ is clearly nonminimal because $\{F_0, G_0, L_0, W_0\}$ is a realization of $W(s)$ and F_0 has smaller dimension than F. Since $[F, G]$ is completely controllable, $[F, L]$ must not be completely observable (this is easy to check directly). We still require this nonminimal realization of $W(s)$, however, in order to generate a P, L, and W_0 satisfying the positive real lemma equations. The matrices P_0, L_0, and W_0 associated with a minimal realization of $W(s)$ simply are not solutions of the positive real lemma equations. This is somewhat surprising—usually minimal realizations are the only ones of interest.

The spectral factor $W(s) = W_0 + L'(sI - F)^{-1}G$ of course can be calculated to have the three properties 1, 2, and 3 listed in conjunction with Eq. (6.5.2); it is unique to within multiplication on the left by an arbitrary real constant orthogonal matrix.

We recall that the computation of solutions of (6.5.1) via a Riccati equation solution allowed us to define a family of spectral factors satisfying (6.5.2) and properties 1, 2, and 3. It follows therefore that the procedures just presented and the earlier procedure lead to the same family of solutions of (6.5.1). As we have already noted, the matrix P is the same for all members of the family, while L' and W_0 differ through left multiplication by an orthogonal matrix. Therefore, the technique of this section and Section 6.2 lead to the same matrix P satisfying (6.5.1).

Example 6.5.2 We shall consider the positive real function

$$Z(s) = \frac{1}{2} + \frac{1}{s+1} + \frac{s}{s^2+1}$$

which has a minimal realization

$$F = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad J = \frac{1}{2}$$

Because $Z(s)$ has a pole with zero real part, we shall consider separately the positive real functions

$$Z_0(s) = \frac{1}{2} + \frac{1}{s+1} \qquad Z_1(s) = \frac{s}{s^2+1}$$

For the former, we have as a minimal realization

$$[-1, 1, 1, \tfrac{1}{2}]$$

Now

$$\begin{aligned} Z_0(s) + Z_0(-s) &= 1 + \frac{1}{s+1} + \frac{1}{-s+1} \\ &= \frac{3-s^2}{1-s^2} \\ &= \frac{\sqrt{3}-s}{1-s}\frac{\sqrt{3}+s}{1+s} \end{aligned}$$

We take as the spectral factor $(\sqrt{3}+s)/(1+s)$, whose inverse exists throughout $\mathrm{Re}[s] > 0$. Now

$$W(s) = \frac{\sqrt{3}+s}{1+s} = 1 + \frac{\sqrt{3}-1}{s+1}$$

and so $W(s)$ has a realization

$$[-1, 1, \sqrt{3}-1, 1]$$

Thus $L_0 = \sqrt{3} - 1$, $W_0 = 1$, and a solution P_0 of $P_0F_0 + F_0'P_0 = -L_0L_0'$ is readily checked to be $2 - \sqrt{3}$.

For $Z_1(s)$, we have a minimal realization

$$\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

and the solution of $P_1F_1 + F_1'P_1 = 0, P_1G_1 = H_1$ is easily checked to be

$$P_1 = I$$

Combining the individual results for $Z_0(s)$ and $Z_1(s)$ together, it follows that matrices P, L, and W_0, which work for the particular minimal realization of $Z(s)$ quoted above, are

$$P = \begin{bmatrix} 2 - \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} \sqrt{3} - 1 \\ 0 \\ 0 \end{bmatrix} \qquad W_0 = 1$$

Problem 6.5.1 Verify that Eqs. (6.5.3) through (6.5.5) and the associated remarks define a spectral factor $W(s)$ satisfying (6.5.2) and the additional properties listed following this equation.

Problem 6.5.2 Suppose that P satisfies the lossless positive real lemma equations $PF + F'P = 0$ and $PG = H$. From these two equations form a single equation of the form $P\hat{F} + \hat{F}'P = -\hat{L}\hat{L}'$, where $[\hat{F}, \hat{L}]$ is completely observable. The positive definite nature of P implies that $\hat{F}$ has all eigenvalues in the left half-plane, which means that the linear equation for P is readily solvable to give a unique solution.

6.6 FINDING ALL SOLUTIONS OF THE POSITIVE REAL LEMMA EQUATIONS

In this section we tackle the problem of finding all, rather than one, solution of the positive real lemma equations

$$\begin{aligned} PF + F'P &= -LL' \\ PG &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{6.6.1}$$

We shall solve this problem under the assumption that we know one solution triple P, L, W_0 of (6.6.1).

A general outline of our approach is as follows:

1. We shall show that the problem of solving (6.6.1) is equivalent to the problem of solving a quadratic matrix inequality (see Theorem 6.6.1 for the main result).
2. We shall show that, knowing one solution of (6.6.1), the quadratic matrix inequality may be replaced by another quadratic matrix inequality of simpler structure (see Theorem 6.6.2 for the main result).
3. We shall show how to find all solutions of this second quadratic matrix inequality and relate them to solutions of (6.6.1).

Throughout, we shall assume that $J + J'$ is nonsingular.

This section contains derivations that establish the validity of the computational procedure, as distinct from forming an integral part of the computational procedure. They may therefore be omitted on a first reading of this chapter. The derivations falling into this category are the proofs of Theorem 6.6.1 and the corollary to Theorem 6.6.2.

The technique discussed here for solving (6.6.1) first appeared in [6].

A Quadratic Matrix Inequality

The following theorem provides a quadratic matrix inequality equivalent to Eqs. (6.6.1).

Theorem 6.6.1. With $\{F, G, H, J\}$ a minimal realization of a positive real $Z(s)$ and with $J + J'$ nonsingular, solution of (6.6.1) is equivalent to the solution of

$$PF + F'P \leq -(PG - H)(J + J')^{-1}(PG - H)' \tag{6.6.2}$$

in the sense that if $\{P, L, W_0\}$ satisfies (6.6.1), then P satisfies (6.6.2), and if a positive definite symmetric P satisfies (6.6.2), then there exist associated real matrices L and W_0 (not unique) such that P, L, and W_0 satisfy (6.6.1). Moreover, the set of all associated L and W_0 are determinable as follows. Let N be a real matrix with minimum number of columns such that

$$PF + F'P + (PG - H)(J + J')^{-1}(PG - H)' = -NN' \tag{6.6.3}$$

Then the set of L and W_0 is defined by

$$\begin{aligned} L &= [-(PG - H)(J + J')^{-1/2} \mid N]V \\ W_0' &= [(J + J')^{1/2} \mid 0]V \end{aligned} \tag{6.6.4}$$

where V ranges over the set of real matrices for which $VV' = I$, and the zero block in the definition of W_0' is chosen so that L and W_0' have the same number of columns.

Proof.* First we shall show that Eqs. (6.6.1) imply (6.6.2).

As a preliminary, observe that $W_0'W_0$ is nonsingular, and so $M = W_0(W_0'W_0)^{-1}W_0' \leq I$. For if x is an eigenvector of M, $\lambda x = Mx$ for some λ. Then $\lambda W_0'x = W_0'Mx = W_0'x$ since $W_0'M = W_0'$ by inspection. Hence either $\lambda = 1$ or $W_0'x = 0$, which implies $Mx = 0$ and thus $\lambda = 0$. Thus the eigenvalues of M are all either 0 or 1. The eigenvalues of $I - M$ are therefore all either 1 or 0. Since $I - M$ is symmetric, it follows that $I - M \geq 0$ or $M \leq I$. Next it follows that $LML' \leq LL'$ or $-LL' \leq -LML'$. Then

$$\begin{aligned} PF + F'P = -LL' &\leq -LW_0(W_0'W_0)^{-1}W_0'L' \\ &= -(PG - H)(J + J')^{-1}(PG - H)' \end{aligned}$$

which is Eq. (6.6.2).

Next we show that (6.6.2) implies (6.6.1).

From (6.6.2) it follows that

$$PF + F'P + (PG - H)(J + J')^{-1}(PG - H)' \leq 0$$

and so there exists a real constant matrix N such that

$$PF + F'P + (PG - H)(J + J')^{-1}(PG - H)' = -NN' \quad (6.6.3)$$

We assume that N has the minimum number of columns—a number equal to the rank, q say, of the left-hand side of (6.6.3). This determines N uniquely to within right multiplication by an arbitrary real constant orthogonal matrix V.

Then in order that (6.6.1) hold, it is clearly sufficient, and as we prove below also necessary, that

$$\begin{aligned} L &= [-(PG - H)(J + J')^{-1/2} \mid N]V \\ W_0' &= [(J + J')^{1/2} \mid 0]V \end{aligned} \quad (6.6.4)$$

where V is any real constant matrix for which $VV' = I$, and the zero block in W_0' is chosen so that L and W_0' have the same number of columns. Note that V is not necessarily square.

To see that Eqs. (6.6.4) are necessary, observe from (6.6.1) that any L and W_0 satisfying (6.6.1) must satisfy

$$\begin{bmatrix} -(PF + F'P) & PG - H \\ (PG - H)' & J + J' \end{bmatrix} = \begin{bmatrix} -L \\ W_0' \end{bmatrix} [-L' \quad W_0] \quad (6.6.5)$$

*This proof may be omitted in a first reading of this chapter.

Let L_1 and W'_{01} denote the values of L and W'_0 achieved in (6.6.4) by taking $V = I$. Now (6.6.5) implies that any L and W'_0 must have a number of columns equal to or greater than the rank, call it p, of the left-hand side of (6.6.5); if L_1 and W'_{01} have p columns, then (6.6.4) will define all solutions of (6.6.1) for the prescribed P as V ranges over the set of matrices for which $VV' = I_p$, with V possessing a number of columns greater than or equal to p.

Let us now check that L_1 and W'_{01} do actually have p columns. The following equality is easily verified:

$$\begin{bmatrix} I & -(PG-H)(J+J')^{-1} \\ 0 & (J+J')^{-1/2} \end{bmatrix} \begin{bmatrix} -(PF+F'P) & PG-H \\ (PG-H)' & J+J' \end{bmatrix}$$
$$\times \begin{bmatrix} I & 0 \\ -(J+J')^{-1}(PG-H)' & (J+J')^{-1/2} \end{bmatrix}$$
$$= \begin{bmatrix} -[PF+F'P+(PG-H)(J+J')^{-1}(PG-H)'] & 0 \\ 0 & I \end{bmatrix}$$

Therefore, the rank of the left side of (6.6.5) is rank $NN' + m$, where m is the size of the square matrix J; i.e., $p = q + m$. Now recall that N has a number of columns equal to the rank of NN', viz., q. Then from (6.6.4) it is immediate that L_1 and W'_{01} have a number of columns equal to $q + m = p$, as required. $\nabla\nabla\nabla$

We wish to make two points concerning the above theorem; the first is sufficiently important to be given status as a corollary.

Corollary. If $\{P, L, W_0\}$ is a triple satisfying (6.6.1), J is $m \times m$, and W_0 has m rows and columns, then P satisfies the inequality (6.6.2) with equality. Conversely, if P satisfies (6.6.2) with equality, among the family of L and W_0 satisfying together with P Eq. (6.6.1), there exist L' and W_0 with m rows.

Proof is by immediate verification. Notice from the third of Eqs. (6.6.1) that it is never possible for W_0 to have fewer than m rows with $J + J'$ nonsingular.

The second point is this: essentially, Theorem 6.6.1 explains how to eliminate the matrices L and W_0 from (6.6.1) to obtain a single equation, actually an inequality, for the matrix P. [Of course, it is logical to try to replace the three equations involving three unknowns by a single equation, although we have yet to show that this inequality is any simpler to solve than the original equations.] The inequality (6.6.2) has the general form

$$PAP + PB + B'P + C \leq 0$$

and is apparently very difficult to solve. However, were the matrix C equal to zero, one would expect by analogy with the scalar situation that solution would be an easier task. In the next subsection we discuss such a simplification.

A Simpler Quadratic Matrix Inequality

Instead of attempting to solve (6.6.2) directly, we shall instead try to find all values of $Q = P - \bar{P}$, where $\bar{P}$ is a known solution of (6.6.2) actually satisfying (6.6.2) with equality, and P is any other solution of the inequality. The defining relation for Q is the subject of the next theorem.

Theorem 6.6.2. Let a matrix $\bar{P}$ satisfy (6.6.2) with equality, and let P be any other symmetric matrix satisfying the inequality (6.6.2). Define the matrix Q by $Q = P - \bar{P}$. Then Q satisfies

$$Q\bar{F} + \bar{F}'Q \leq -QG(J + J')^{-1}G'Q \tag{6.6.6}$$

where

$$\bar{F} = F + G(J + J')^{-1}(\bar{P}G - H)' \tag{6.6.7}$$

Conversely, if Q satisfies (6.6.6) and $\bar{P}$ satisfies (6.6.2) with equality, $P = Q + \bar{P}$ satisfies (6.6.2). Finally, Q satisfies (6.6.6) with equality if and only if P satisfies (6.6.2) with equality.

The proof of this theorem is straightforward, proceeding by simple manipulation. The details are requested in Problem 6.6.1.

Evidently, if we can find all solutions of the inequality (6.6.6), and if we know one solution of (6.6.2) with equality, then we can completely solve the positive real lemma equations (6.6.1). (Recall that from Theorem 6.6.1 the generation of all the possible L and W_0 associated with a given P is straightforward, knowing P.)

The previous sections of this chapter have discussed techniques for the computation of a matrix $\bar{P}$ satisfying (6.6.2) with equality. In the next subsection, we discuss the solution of (6.6.6).

Let us note now an interesting property, provable using (6.6.6).

Corollary. Under the condition of Theorem 6.6.2, suppose that $\bar{P}$ is taken as that particular solution of (6.6.2) which satisfies the equality and is the negative of the limit of the solution of the associated Riccati equation. Then the eigenvalues of $\bar{F}$ all have nonpositive real parts, and $Q \geq 0$. Moreover, if $\bar{F} = \begin{bmatrix} \bar{F}_1 & 0 \\ 0 & \bar{F}_2 \end{bmatrix}$, where $\bar{F}_1$ has all eigenvalues with zero real parts and $\bar{F}_2$ all eigenvalues with negative real parts, and if Q is partitioned conformably as $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q'_{12} & Q_{22} \end{bmatrix}$, then Q_{11} and Q_{12} are zero.

Proof.* Because $\bar{P}$ is the negative of the limit of the Riccati equation solution, it follows, as we have noted earlier, that the spectral factor family

$$W(s) = V'[(J + J')^{1/2} - (J + J')^{-1/2}(\bar{P}G - H)'(sI - F)^{-1}G]$$

where $V'V = VV' = I$ is such that $W^{-1}(s)$ exists throughout $\text{Re}[s] > 0$.

This means that $\bar{F}$ has all eigenvalues in $\text{Re}[s] \leq 0$, because†

$$\begin{aligned} &\det \{sI - [F + G(J + J')^{-1}(\bar{P}G - H)']\} \\ &\quad = \det (sI - F) \det [I - (sI - F)^{-1}G(J + J')^{-1}(\bar{P}G - H)'] \\ &\quad = \det (sI - F) \det [I - (J + J')^{-1}(\bar{P}G - H)'(sI - F)^{-1}G] \\ &\quad = \det (sI - F) \det (J + J')^{-1/2} \det [(J + J')^{1/2} - (J + J')^{-1/2} \\ &\qquad \times (\bar{P}G - H)'(sI - F)^{-1}G] \\ &\quad = \pm\det (sI - F) \det (J + J')^{-1/2} \det W(s) \end{aligned}$$

Thus eigenvalues of $\bar{F}$ are either eigenvalues of F or zeros of $\det W(s)$, although not necessarily conversely. Since both the eigenvalues of F and the zeros of $\det W(s)$ lie in $\text{Re}[s] \leq 0$, the first part of the corollary is proved.

If the eigenvalues of $\bar{F}$ are in $\text{Re}[s] < 0$ rather than just $\text{Re}[s] \leq 0$, Eq. (6.6.6) implies immediately by the lemma of Lyapunov that $Q \geq 0$, since $Q\bar{F} + \bar{F}'Q \leq 0$. The conclusion extends to the case when eigenvalues of $\bar{F}$ are in $\text{Re}[s] \leq 0$ as follows.

There is no loss of generality in assuming that

$$\bar{F} = \begin{bmatrix} \bar{F}_1 & 0 \\ 0 & \bar{F}_2 \end{bmatrix}$$

with $\bar{F}_1$ possessing pure imaginary poles, $\bar{F}_2$ possessing poles in

*The proof may be omitted at a first reading of this chapter.

†The following equalities make use of the results $\det [I + AB] = \det [I + BA]$ and $\det CD = \det DC$.

Re $[s] < 0$. [For there is clearly no loss of generality in replacing an arbitrary $\bar{F}$ by $T\bar{F}T^{-1}$, provided we simultaneously replace G by TG and Q by $(T^{-1})'QT^{-1}$ for some nonsingular T—if after these replacements (6.6.6) is valid, then before the replacements it is valid, and conversely. Certainly there exists a T such that $T\bar{F}T^{-1}$ is the direct sum of matrices with the eigenvalue restrictions of $\bar{F}_1$ and $\bar{F}_2$.] By the same argument as used above to justify our special choice of $\bar{F}$, it follows that there is no loss of generality in considering $\bar{F}_1$ skew.

Equation (6.6.6) can be written as

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}\begin{bmatrix} \bar{F}_1 & 0 \\ 0 & \bar{F}_2 \end{bmatrix} + \begin{bmatrix} \bar{F}_1' & 0 \\ 0 & \bar{F}_2' \end{bmatrix}\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}$$
$$= -\begin{bmatrix} Q_{11}\bar{G}_1 + Q_{12}\bar{G}_2 \\ Q_{12}'\bar{G}_1 + Q_{22}\bar{G}_2 \end{bmatrix}\begin{bmatrix} Q_{11}\bar{G}_1 + Q_{12}\bar{G}_2 \\ Q_{12}'\bar{G}_1 + Q_{22}\bar{G}_2 \end{bmatrix}' - \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{bmatrix} \tag{6.6.8}$$

where

$$\begin{bmatrix} \bar{G}_1 \\ \bar{G}_2 \end{bmatrix} = \bar{G}(J + J')^{-1/2} \qquad \text{and} \qquad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22} \end{bmatrix}$$

is nonnegative definite symmetric. We shall show that Q_{11} and Q_{12} are zero, and that $Q_{22} \geq 0$. This will complete the proof of all the remaining claims of the corollary.

Now consider the 1–1 block of this equation. The trace of the 1–1 block of the right side of the equation is nonpositive, since the right side itself is nonpositive definite. Also

$$\operatorname{tr}(Q_{11}\bar{F}_1 + \bar{F}_1'Q_{11}) = \operatorname{tr}(Q_{11}\bar{F}_1 - \bar{F}_1Q_{11}) = 0$$

These equalities follow from the skewness of $\bar{F}_1$, and the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for arbitrary A, B for which AB and BA exist.

This means that $S_{11} = 0$ and $Q_{11}\bar{G}_1 + Q_{12}\bar{G}_2 = 0$, for otherwise the trace of the 1–1 block on the right side would be negative. With $S_{11} = 0$, it follows that $S_{12} = 0$ to ensure that S is nonnegative.

Now consider the 2–1 block, recalling that $Q_{11}\bar{G}_1 + Q_{12}\bar{G}_2$ and S_{12} are zero. We have

$$Q_{12}'\bar{F}_1 + \bar{F}_2'Q_{12}' = 0$$

Since $\bar{F}_1$ and $\bar{F}_2'$ have no eigenvalues in common, a standard property of linear matrix equations implies that $Q_{12}' = 0$. Since

$Q_{11}\bar{G}_1 + Q_{12}\bar{G}_2 = 0$, it follows that $Q_{11}\bar{G}_1 = 0$. Now we can argue that $Q_{11} = 0$. For the complete controllability of $[F, G]$ implies complete controllability of $[F - GK', G]$ for any K, and thus of $[\bar{F}, \bar{G}]$. In turn, this implies the complete controllability of $[\bar{F}_1, \bar{G}_1]$ and $[\bar{F}_2, \bar{G}_2]$. Now we have

$$Q_{11}\bar{F}_1 + \bar{F}_1'Q_{11} = 0 \qquad \text{and} \qquad Q_{11}\bar{G}_1 = 0$$

It follows that

$$Q_{11}\bar{F}_1\bar{G}_1 = -\bar{F}_1'Q_{11}\bar{G}_1 = 0$$

and generally

$$Q_{11}\bar{F}_1^i\bar{G}_1 = 0$$

Complete controllability of $\bar{F}_1$, $\bar{G}_1$ now implies that $Q_{11} = 0$.

Consequently, (6.6.8) becomes simply

$$Q_{22}\bar{F}_2 + \bar{F}_2'Q_{22} = -Q_{22}\bar{G}_2\bar{G}_2'Q_{22} - S_{22}$$

for some nonnegative definite S_{22}, and with $\bar{F}_2$ possessing eigenvalues in $\text{Re}[s] < 0$. As we have already argued, $Q_{22} \geq 0$. Therefore, $Q = \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix} \geq 0$, as required. $\nabla\nabla\nabla$

Solution of the Quadratic Matrix Inequality— Nonsingular Solutions

Now we shall concentrate on solving the inequality (6.6.6) for Q. Our basic technique will be to somehow replace (6.6.6) by a linear equation, and in one particular instance this is very straightforward. Because it may help to illustrate the general approach to be discussed, we shall present this special case now.

We shall find all nonsingular solutions of (6.6.6). (*Note:* From the corollary to Theorem 6.6.2 it follows that if $\bar{F}$ possesses a pure imaginary eigenvalue, there exist no nonsingular solutions. Therefore, we assume that all eigenvalues of $\bar{F}$ possess negative real part.)

All nonsingular solutions of (6.6.6) are generated from all nonsingular solutions of

$$Q\bar{F} + \bar{F}'Q = -QG(J + J')^{-1}G'Q - S$$

where S ranges over the set of all nonnegative definite matrices. Because we are assuming nonsingularity of Q, it follows that the set of nonsingular solutions of (6.6.6) may be generated by solving

$$\bar{F}Q^{-1} + Q^{-1}\bar{F}' = -G(J+J')^{-1}G' - Q^{-1}SQ^{-1} \tag{6.6.9}$$

where S ranges over the set of all nonnegative definite matrices. But if S ranges over the set of all nonnegative definite matrices, so does $Q^{-1}SQ^{-1}$. Therefore, all nonsingular solutions of (6.6.6) are found by solving

$$\bar{F}Q^{-1} + Q^{-1}\bar{F}' = -G(J+J')^{-1}G' - \hat{S} \tag{6.6.10}$$

where $\hat{S}$ ranges over the set of all nonnegative definite matrices. This equation is linear in Q^{-1}, and procedures exist for its solution.

With the constraint on $\bar{F}$ that its eigenvalues all have negative real parts, it is known from the lemma of Lyapunov that

$$Q^{-1} = \int_0^\infty e^{\bar{F}t}[G(J+J')^{-1}G' + \hat{S}]e^{\bar{F}'t}\,dt \tag{6.6.11}$$

This matrix is always nonsingular for the following reason. Observe that

$$\int_0^\infty e^{\bar{F}t}[G(J+J')^{-1}G' + \hat{S}]e^{\bar{F}'t}\,dt \geq \int_0^\infty e^{\bar{F}t}G(J+J')^{-1}G'e^{\bar{F}'t}\,dt$$

If the matrix on the right side of the inequality is nonsingular, so is that on the left. But the matrix on the right is nonsingular if (and only if) $[\bar{F}, G]$ is completely controllable. Since $[F, G]$ is completely controllable, and since $\bar{F} = F - GK'$ for a certain K, $[\bar{F}, G]$ must be completely controllable.

In summary, Eq. (6.6.11) defines, as $\hat{S}$ ranges over the set of all nonnegative definite symmetric matrices, all nonsingular solutions to (6.6.6); for such to exist, $\bar{F}$ must have all negative real part eigenvalues.

Solution of the Quadratic Matrix Inequality—Singular Solutions

To find singular solutions of (6.6.6), it is necessary to delve deeper into the structure of $\bar{F}$ and to determine its eigenvalues. (*Note:* This was not necessary to compute nonsingular solutions.)

For convenience we suppose that all eigenvalues of $\bar{F}$ with negative real parts are distinct, and that $\bar{F}$ is diagonalizable. Then there exists a matrix T such that

$$T^{-1}\bar{F}T = \hat{F} = [-\lambda_1] \dotplus \cdots \dotplus [-\lambda_i] \dotplus \begin{bmatrix} -\lambda_{i+1} & \mu_{i+1} \\ -\mu_{i+1} & -\lambda_{i+1} \end{bmatrix} \dotplus \cdots \dotplus \begin{bmatrix} -\lambda_j & \mu_j \\ -\mu_j & -\lambda_j \end{bmatrix} \dotplus \hat{F}_2$$

where $\hat{F}_2$ has purely imaginary eigenvalues and is skew. In this equation $\dotplus$ denotes direct sum, and $\lambda_1, \ldots, \lambda_l, \mu_{l+1}, \ldots, \mu_j$ are real positive numbers.

With $\hat{Q} = T'QT$ and $\hat{G} = T^{-1}G(J + J')^{-1/2}$, the inequality (6.6.6) is equivalent to

$$\hat{Q}\hat{F} + \hat{F}'\hat{Q} \leq -\hat{Q}\hat{G}\hat{G}'\hat{Q} \tag{6.6.12}$$

As we have already noted, $[\bar{F}, G]$ is completely controllable. It follows easily from this fact that $[\hat{F}, \hat{G}]$ is also completely controllable.

We have already shown in the corollary to Theorem 6.6.2 that certain parts of $\hat{Q}$ must be zero. In particular, if we write

$$\hat{F} = \hat{F}_1 \dotplus \hat{F}_2$$

so that $\hat{F}_1$ contains negative real part eigenvalues only, and $\hat{F}_2$ zero real part eigenvalues only, and if we partition $\hat{Q}$ as

$$\begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{12}' & \hat{Q}_{22} \end{bmatrix}$$

so that $\hat{Q}_{11}$ has the same dimension as $\hat{F}_1$, then $\hat{Q}_{12}$ and $\hat{Q}_{22}$ are zero.

Therefore, we can restrict attention to the equation

$$\hat{Q}_{11}\hat{F}_1 + \hat{F}_1'\hat{Q}_{11} \leq -\hat{Q}_{11}\hat{G}_1\hat{G}_1'\hat{Q}_{11} \tag{6.6.13}$$

where $\hat{F}_1$ has negative real part eigenvalues, $\hat{G}_1$ is defined in an obvious fashion, and $[\hat{F}_1, \hat{G}_1]$ is completely controllable because $[\hat{F}, \hat{G}]$ is completely controllable.

Let us drop the subscripts on (6.6.13), to write again

$$\hat{Q}\hat{F} + \hat{F}'\hat{Q} \leq -\hat{Q}\hat{G}\hat{G}'\hat{Q} \tag{6.6.12}$$

where we now understand that $\hat{F}$ has all negative real part eigenvalues, and of course $[\hat{F}, \hat{G}]$ is completely controllable.

The inequality (6.6.12) can be rewritten as

$$\hat{Q}\hat{F} + \hat{F}'\hat{Q} = -\hat{Q}\hat{G}\hat{G}'\hat{Q} - S \tag{6.6.14}$$

where S is an arbitrary nonnegative definite symmetric matrix.

Now observe that if x is a vector such that $\hat{Q}x = 0$, then $Sx = 0$ and $\hat{Q}\hat{F}^i x = 0$ for all i. For multiplication of (6.6.14) on the left by x' and on the right by x establishes that $x'Sx = 0$, and thus $Sx = 0$. Multiplication on the right by x then establishes that $\hat{Q}\hat{F}x = 0$, and $\hat{Q}\hat{F}^i x = 0$ follows by iterating the argument.

The fact that $\hat{Q}x = 0$ implies that $\hat{Q}\hat{F}^i x = 0$, for all i, implies that the vector x must have some zero entries if $\hat{Q}$ is not identically zero. For if x has no zero entry, the set of vectors $x, \hat{F}x, \hat{F}^2x, \ldots$, spans the whole space of vectors of dimension equal to the size of $\hat{Q}$ [13]; then we would have to have $\hat{Q} = 0$.

Now let $x_1, \ldots, x_m$ be vectors spanning the null space of $\hat{Q}$. By reordering rows and columns of $\hat{Q}$, $\hat{F}$, $\hat{G}$, and S if necessary, we may suppose that the first $r > 0$ entries of $x_1, \ldots, x_m$ are zero, while at least one of $x_1, \ldots, x_m$ has an $(r + s)$th entry nonzero for each s greater than zero.

Let q be the dimension of $\hat{Q}$. It may be checked that the set $x_1, \hat{F}x_1, \ldots, x_2, \hat{F}x_2, \ldots, \ldots, x_m, \hat{F}x_m, \ldots$, spans the space of vectors whose first r entries are zero and whose last $(q - r)$ entries can be arbitrary. This space is identical with the nullspace of $\hat{Q}$. Therefore, the last $(q - r)$ columns of $\hat{Q}$ and S are zero. The symmetry of $\hat{Q}$ and S then implies that the last $(q - r)$ rows are zero, so that (6.6.14) becomes

$$\hat{Q}_1\hat{F}_1 + \hat{F}_1'\hat{Q}_1 = -\hat{Q}_1(\hat{G}\hat{G}')_1\hat{Q}_1 - S_1 \tag{6.6.15}$$

where the subscript 1 denotes deletion of the last $(q - r)$ rows and columns of the matrix with which it is associated. The matrix $\hat{Q}_1$ must be *nonsingular*, or else we would have a contradiction of the fact that $x_1, \ldots, x_m$ span the nullspace of $\hat{Q}_1$.

Notice that $\hat{F}_1$ will be a direct sum of blocks of the type $[-\lambda]$ and $\begin{bmatrix} -\lambda & \mu \\ -\mu & -\lambda \end{bmatrix}$ for λ, μ real and positive; i.e., the blocks making up $\hat{F}_1$ are a subset of the blocks making up $\hat{F}$. It is not hard to check that it is impossible for a block of the form $\begin{bmatrix} -\lambda & \mu \\ -\mu & -\lambda \end{bmatrix}$ in the group making up $\hat{F}$ to be "split," so that part of the block is included in $\hat{F}_1$ and part not.

The solution of (6.6.15) with $\hat{Q}_1$ known to be nonsingular is straightforward. Following the earlier theory for constructing all nonsingular solutions to the original inequality, we note that solution of (6.6.15) for all nonnegative definite S_1 is equivalent to solution for all nonnegative definite $\hat{S}_1$ of

$$\hat{F}_1\hat{Q}_1^{-1} + \hat{Q}_1^{-1}\hat{F}_1' = -(\hat{G}\hat{G}')_1 - \hat{S}_1 \tag{6.6.16}$$

That this equation always has a nonsingular solution $\hat{Q}_1^{-1}$ is readily checked. Thus $\hat{Q}_1$ can be formed.

The corresponding solution of (6.6.12) is

$$\hat{Q} = \begin{bmatrix} \hat{Q}_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.6.17}$$

The above analysis implicitly describes how to compute all singular solutions of (6.6.12), without determining vectors x such that $\hat{Q}x = 0$. *One simply drops blocks from the direct sum making up $\hat{F}$, and at the same time constructs a submatrix of $\hat{G}\hat{G}'$ by dropping the same rows and columns as were dropped from $\hat{F}$. Denote by $\hat{F}_1$ the matrix $\hat{F}$ with certain blocks dropped (equivalently, certain rows and columns dropped). Denote by $(\hat{G}\hat{G}')_1$ the matrix $\hat{G}\hat{G}'$ with corresponding rows and columns dropped. Then Eqs. (6.6.15) and (6.6.16) yield a whole family of solutions to (6.6.12), or equivalently the original (6.6.6). All solutions are obtained by dropping different blocks of $\hat{F}$ and proceeding in the above manner.*

Properties of the Solutions

In this subsection we wish to note briefly properties of the solutions P of (6.6.1). We shall describe these properties in terms of solutions to (6.6.6).

As we have already noted in the corollary to Theorem 6.6.2, there is a minimum solution to (6.6.1), which is the matrix (termed $\bar{P}$ in this section) associated with that family of spectral factors whose inverse exists in $\text{Re}\,[s] > 0$.

It is also possible to write down the maximum solution to (6.6.1). For simplicity, suppose that $\bar{F}$ has eigenvalues restricted to $\text{Re}\,[s] < 0$. Then there exist nonsingular solutions to (6.6.6), one of them being defined by

$$Q\bar{F} + \bar{F}'Q = -QG(J + J')^{-1}G'Q \tag{6.6.18}$$

or

$$\bar{F}Q^{-1} + Q^{-1}\bar{F}' = -G(J + J')^{-1}G' \tag{6.6.19}$$

If $\tilde{Q}$ denotes this particular solution and $\tilde{P} = \bar{P} + \tilde{Q}$ the corresponding solution of (6.6.1), then $\tilde{P}$ is the maximum solution of (6.6.1); i.e., if P is any other solution, $\tilde{P} - P \geq 0$. This is proved in [6]. The spectral factor family generated by $\tilde{P}$ is interesting; it has the property that the inverse of a member of the family exists throughout $\text{Re}\,[s] < 0$. (Problem 6.6.2 asks for a derivation of this fact. See also [6].)

As noted in the last subsection, solutions of (6.6.6) fall into families, each family derivable by deleting certain rows and columns in the matrix we have termed $\hat{F}$. There is a maximum member for each family computed as follows. The general equation determining a family is of the form

$$\hat{F}_1\hat{Q}_1^{-1} + \hat{Q}_1^{-1}\hat{F}_1' = -(\hat{G}\hat{G}')_1 - \hat{S}_1 \tag{6.6.16}$$

where $\hat{F}_1$ is $\hat{F}$ with some of its rows and columns deleted and $\hat{S}_1$ is nonnegative definite. The smallest $\hat{Q}_1^{-1}$ satisfying (6.6.16) is easily checked to be that obtained by setting $\hat{S}_1 = 0$, since in general

$$\hat{Q}_1^{-1} = \int_0^\infty e^{\hat{F}_1 t}(\hat{G}\hat{G})_1 e^{\hat{F}'_1 t}\,dt + \int_0^\infty e^{\hat{F}_1 t}\hat{S}_1 e^{\hat{F}'_1 t}\,dt$$

Therefore, the $\hat{Q}_1$ derived from $\hat{S}_1 = 0$ will be the largest, or maximum, member of the associated family.

The associated Q can be checked to satisfy (6.6.6) with equality, so the maximum members of the families are precisely the solutions of (6.6.6) with the equality constraint.

Example 6.6.1 We wish to illustrate the preceding theory with a very simple example. We take $Z(s) = \frac{1}{2} + 1/(s+1)$, for which $[-1, 1, 1, \frac{1}{2}]$ is a minimal realization. As we have calculated earlier, $\bar{P} = 2 - \sqrt{3}$. Then

$$\bar{F} = -\sqrt{3}$$

and the inequality for Q becomes

$$-2\sqrt{3}Q \leq -Q^2$$

There is only one family of solutions. The maximum member of that family is obtained from a nonsingular Q satisfying

$$-2\sqrt{3}Q = -Q^2$$

That is,

$$Q = 2\sqrt{3}$$

Any Q in the interval $[0, 2\sqrt{3}]$ satisfies the inequality.

With $Q = 2\sqrt{3}$, $P = 2 + \sqrt{3}$, and (6.6.1) yields $L = -1 - \sqrt{3}$. The associated spectral factor is

$$W(s) = 1 - \frac{1+\sqrt{3}}{s+1} = \frac{s-\sqrt{3}}{s+1}$$

and is evidently maximum phase.

If, for the sake of argument, we take $Q = \sqrt{3}$, then P becomes 2. It follows that

$$PF + F'P + (PG - H)(J + J')^{-1}(PG - H)' = -3$$

so that, following (6.6.3) and the associated remarks, $N = \sqrt{3}$. The associated family of spectral factors is defined [see (6.6.4)] by

$$L = [-1 \quad \sqrt{3}]V$$
$$W_0' = [1 \quad 0]V$$

where V is an arbitrary matrix satisfying $VV' = I$. It follows that

$$W(s) = V'\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}\frac{1}{s+1}\right\}$$
$$= V'\begin{bmatrix} \dfrac{s}{s+1} \\ \dfrac{\sqrt{3}}{s+1} \end{bmatrix}$$

Observe that, as required,

$$W'(-s)W(s) = \begin{bmatrix} \dfrac{-s}{-s+1} & \dfrac{\sqrt{3}}{-s+1} \end{bmatrix} VV' \begin{bmatrix} \dfrac{s}{s+1} \\ \dfrac{\sqrt{3}}{s+1} \end{bmatrix}$$
$$= \frac{-s^2}{-s^2+1} + \frac{3}{-s^2+1}$$
$$= \frac{-s^2+3}{-s^2+1}$$
$$= Z(s) + Z(-s)$$

This example emphasizes the fact that spectral factors $W(s)$ readily arise that have a nonminimal number of rows.

Problem 6.6.1 Prove Theorem 6.6.2.

Problem 6.6.2 Suppose that $\bar{P}$ is that particular solution of (6.6.2) which satisfies the equality and is the negative of the limit of a Riccati equation solution. Suppose also that the eigenvalues of $\bar{F}$ all have negative real parts. Let $\tilde{Q}$ be the nonsingular solution of $Q\bar{F} + \bar{F}'Q = -QG(J+J')^{-1}G'Q$ and let $\tilde{P} = \bar{P} + \tilde{Q}$. Show that if $W(s)$ is a member of the spectral factor family defined by $\tilde{P}$, then det $W(s)$ is nonzero in Re $[s] < 0$. [Thus $W(s)$ is a "maximum phase" spectral factor.]

Problem 6.6.3 Find a general formula for all degree 1 spectral factors associated with $\frac{1}{2} + 1/(s+1)$ by extending the analysis of Example 6.6.1 to cope with the case in which Q is arbitrary, within the interval $[0, 2\sqrt{3}]$.

Problem 6.6.4 Study the generation of solutions of the positive real lemma equations with $Z(s) = (s^2 + 3.5s + 2)(s^2 + 3s + 2)^{-1}$.

Problem 6.6.5 Show that the set of matrices P satisfying the positive real lemma equations for a prescribed positive real $Z(s)$ is convex, i.e. if P_1 and P_2 satisfy the equations, so does $\alpha P_1 + (1-\alpha)P_2$ for all α in the range $[0, 1]$.

Problem 6.6.6 Show that the maximum solution of

$$PF + F'P + (PG - H)(J + J')^{-1}(PG - H)' = 0$$

is the inverse of the minimum solution of

$$\bar{P}F' + F\bar{P} + (\bar{P}H - G)(J + J')^{-1}(\bar{P}H - G)' = 0$$

What is the significance of this fact?

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7

Bounded Real Matrices and the Bounded Real Lemma; the State-space Description of Reciprocity

7.1 INTRODUCTION

We have two main aims in this chapter. First, we shall discuss bounded real matrices, and present an algebraic criterion for a rational matrix to be bounded real. Second, we shall present conditions on a minimal state-space realization of a transfer-function matrix for that transfer-function matrix to be symmetric. In later chapters these results will be put to use in synthesizing networks, i.e., in passing from a set of prescribed port characteristics of a network to a network possessing these port characteristics.

As we know, an $m \times m$ matrix of real rational functions $S(s)$ of the complex variable s is bounded real if and only if

1. All elements of $S(s)$ are analytic in $\text{Re}[s] \geq 0$.
2. $I - S'^*(s)S(s)$ is nonnegative definite Hermitian in $\text{Re}[s] > 0$, or equivalently, $I - S'^*(j\omega)S(j\omega)$ is nonnegative definite Hermitian for all real ω.

   Further, $S(s)$ is lossless bounded real if it is bounded real and
3. $I - S'(-s)S(s) = 0$ for all s.

These conditions are analytic in nature; the ***bounded real lemma*** seeks to replace them by an equivalent set of algebraic conditions on the matrices of a state-space realization of $S(s)$. In Section 7.2 we state with proofs both the bounded real lemma and the lossless bounded real lemma, which are of course counterparts of the positive real lemma and lossless positive real lemma, being translations into state-space terms of the bounded real property. Just as there arises out of the positive real lemma a set of equations that we are interested in solving, so arising out of the bounded real lemma there is a set of equations, which again we are interested in solving. Section 7.3 is devoted to discussing the solution of these equations. Numerous parallels with the earlier results for positive real matrices will be observed; for example, the equations are explicitly and easily solvable when derived from a lossless bounded real matrix.

In Section 7.4 we turn to the question of characterizing the symmetry of a transfer-function matrix in terms of the matrices of a state-space realization of that matrix. As we know, if a network is composed of passive resistors, inductors, capacitors, and transformers, but no gyrators, it is reciprocal, and immittance and scattering matrices associated with the network are symmetric. Now in our later examination of synthesis problems we shall be interested in reciprocal and nonreciprocal synthesis of networks from port descriptions. Since reciprocal syntheses can only result from transfer-function-matrix descriptions with special properties, in general, the property of symmetry, and since our synthesis procedures will be based on state-space descriptions, we need a translation into state-space terms of the property of symmetry of a transfer-function matrix. This is the rationale for the material in Section 7.4.

Problem 7.1.1 Let $S(s)$ be a scalar lossless bounded real function of the form $p(s)/q(s)$ for relatively prime polynomials $p(s)$ and $q(s)$. Show that $S(s)$ is an all-pass function, i.e., has magnitude unity for all $s = j\omega$, ω real, and has zeros that are reflections in the right half-plane $\text{Re}\,[s] > 0$ of its poles.

7.2 THE BOUNDED REAL LEMMA

Statement and Proof of the Bounded Real Lemma

We now wish to reduce the bounded real conditions for a real rational $S(s)$ to conditions on the matrices of a minimal state-space realization of $S(s)$. The lemma we are about to present was stated without proof in [1], while a proof appears in [2]. The proof we shall give differs somewhat from that of [2], but shares in common with [2] the technique of appealing to the positive real lemma at an appropriate part of the necessity proof.

Bounded Real Lemma. Let $S(s)$ be an $m \times m$ matrix of real rational functions of a complex variable s, with $S(\infty) < \infty$, and let $\{F, G, H, J\}$ be a minimal realization of $S(s)$. Then $S(s)$ is bounded real if and only if there exist real matrices P, L, and W_0 with P positive definite symmetric such that

$$\begin{aligned} PF + F'P &= -HH' - LL' \\ -PG &= HJ + LW_0 \\ I - J'J &= W_0'W_0 \end{aligned} \tag{7.2.1}$$

Note that the restriction $S(\infty) < \infty$ is even more inessential here than the corresponding restriction in the positive real lemma: If $S(s)$ is to be bounded real, no element may have a pole in $\text{Re}\,[s] \geq 0$, which, inter alia, implies that no element may have a pole at $s = \infty$; i.e., $S(\infty) < \infty$.

Proof of Sufficiency. The proof is analogous to that of the positive real lemma. First we must check analyticity of all elements of $S(s)$ in $\text{Re}\,[s] \geq 0$. This follows from the first of Eqs. (7.2.1), for the positive definiteness of P and the complete observability of $[F, H]$ guarantee by the lemma of Lyapunov that $\text{Re}\,\lambda_i[F] < 0$ for all i. This is the same as requiring analyticity of all elements of $S(s)$ in $\text{Re}\,[s] \geq 0$.

Next we shall check the nonnegative definite Hermitian nature of $I - S'^*(j\omega)S(j\omega) = I - S'(-j\omega)S(j\omega)$. We have

$$\begin{aligned} &G'(-j\omega I - F')^{-1}(HH' + LL')(j\omega I - F)^{-1}G \\ &\quad = G'(-j\omega I - F')^{-1}[P(j\omega I - F) + (-j\omega I - F')P] \\ &\qquad \times (j\omega I - F)^{-1}G \qquad \text{using the first of Eqs. (7.2.1)} \\ &\quad = G'(-j\omega I - F')^{-1}PG + G'P(j\omega I - F)^{-1}G \\ &\quad = -G'(-j\omega I - F')^{-1}(HJ + LW_0) - (HJ + LW_0)' \\ &\qquad \times (j\omega I - F)^{-1}G \qquad \text{using the second of eqs. (7.2.1)} \end{aligned}$$

Now rewrite this equality, using the third of Eqs. (7.2.1), as

$$\begin{aligned} I - J'J - W_0'W_0 - G'(-j\omega I - F')^{-1}(HJ + LW_0) \\ - (HJ + LW_0)'(j\omega I - F)^{-1}G \\ - G'(-j\omega I - F')^{-1}(HH' + LL')(j\omega I - F)^{-1}G = 0 \end{aligned}$$

By setting

$$W(s) = W_0 + L'(sI - F)^{-1}G \tag{7.2.2}$$

it follows that

$$I - S'(-j\omega)S(j\omega) - W'(-j\omega)W(j\omega) = 0 \tag{7.2.3}$$

or

$$I - S'(-j\omega)S(j\omega) \geq 0$$

as required. ∇ ∇ ∇

Before turning to a proof of necessity, we wish to make one important point. The above proof of sufficiency includes within it the definition of a transfer-function matrix $W(s)$, via (7.2.2), formed with the same F and G as $S(s)$, but with the remaining matrices in its realization, L and W_0, defined from the bounded real lemma equations. Moreover, $W(s)$ satisfies (7.2.3), and, in fact, with the most minor of adjustments to the argument above, can be shown to satisfy Eq. (7.2.4) below. Thus $W(s)$ is a spectral factor of $I - S'(-s)S(s)$. Let us sum up these remarks, in view of their importance.

Existence of Spectral Factor. Let $S(s)$ be a bounded real matrix of real rational functions, and let $\{F, G, H, J\}$ be a minimal realization of $S(s)$. Assume the validity of the bounded real lemma (despite necessity having not yet been proved). In terms of the matrices L and W_0 mentioned therein, the transfer-function matrix $W(s) = W_0 + L'(sI - F)^{-1}G$ is a spectral factor of $I - S'(-s)S(s)$ in the sense that

$$I - S'(-s)S(s) = W'(-s)W(s) \tag{7.2.4}$$

The fact that there are many $W(s)$ satisfying (7.2.4) suggests that we should expect many triples P, L, and W_0 to satisfy the bounded real lemma equations. Later in the chapter we shall discuss means for their determination.

Proof of Necessity. Our strategy will be to convert the bounded real constraint to a positive real constraint. Then we shall apply the positive real lemma, and this will lead to the existence of $P, L,$ and W_0 satisfying (7.2.1).

First, define P_1 as the unique positive definite symmetric solution of

$$P_1F + F'P_1 = -HH' \tag{7.2.5}$$

The stated properties of P_1 follow from the complete observability of $[F, H]$ and the analyticity restriction on the bounded real $S(s)$, which implies that $\operatorname{Re} \lambda_i[F] < 0$.

Equation (7.2.5) allows a rewriting of $I - S'(-s)S(s)$:

$$\begin{aligned} I - S'(-s)S(s) &= (I - J'J) - (HJ)'(sI - F)^{-1}G \\ &\quad - G'(-sI - F')^{-1}HJ \\ &\quad - G'(-sI - F')^{-1}HH'(sI - F)^{-1}G \\ &= (I - J'J) - (HJ + P_1G)'(sI - F)^{-1}G \\ &\quad - G'(-sI - F')^{-1}(HJ + P_1G) \end{aligned} \tag{7.2.6}$$

where we have used an argument in obtaining the last equality, which should be by now familiar, involving the rewriting of HH' as $P_1(sI - F) + (-sI - F')P_1$. Now define

$$Z(s) = \frac{I - J'J}{2} - (HJ + P_1G)'(sI - F)^{-1}G \qquad (7.2.7)$$

Then $Z(s)$ has all elements analytic in $\text{Re}[s] \geq 0$, and evidently $Z(s) + Z'(-s) = I - S'(-s)S(s)$, which is nonnegative definite Hermitian for all $s = j\omega$, ω real, by the bounded real constraint. It follows that $Z(s)$ is positive real. Moreover, $\{F, G, -(HJ + P_1G), (I - J'J)/2\}$ is a completely controllable realization of $Z(s)$. This realization of $Z(s)$ is not necessarily completely observable, and so the positive real lemma is not immediately applicable; a modification is applicable, however. This modification was given in Problem 5.2.2, and also appears as a lemma in [3]; it states that there exist real matrices P_2, L, and W_0 with P_2 *nonnegative definite* symmetric satisfying

$$\begin{aligned} P_2F + F'P_2 &= -LL' \\ P_2G &= -(HJ + P_1G) - LW_0 \\ I - J'J &= W_0'W_0 \end{aligned} \qquad (7.2.8)$$

Now set $P = P_1 + P_2$; P is positive definite symmetric because P_1 is positive definite symmetric and P_2 is nonnegative definite symmetric. Combining the first of Eqs. (7.2.8) with (7.2.5) we obtain

$$PF + F'P = -HH' - LL'$$

while

$$\begin{aligned} -PG &= HJ + LW_0 \\ I - J'J &= W_0'W_0 \end{aligned}$$

are immediate from (7.2.8). These are precisely the equations of the bounded real lemma, and accordingly the proof is complete. ∇ ∇ ∇

Other proofs not relying on application of the positive real lemma are of course possible. In fact, we can generate a proof corresponding to each necessity proof of the positive real lemma. The problems at the end of this section seek to outline development of two such proofs, one based on a network synthesis result and energy-balance arguments, the other based on a quadratic variational problem.

Example 7.2.1 Let $S(s) = s/(s + 1)$. It is easy to verify that $S(s)$ has a minimal realization $\{-1, 1, -1, 1\}$. Further, one may check that Eqs. (7.2.1) are satisfied with $P = 1$, $L = 1$, and $W_0 = 0$. Thus $S(s)$ is bounded real and

$$W(s) = \frac{1}{s+1}$$

is such that $1 - S(-s)S(s) = W(-s)W(s)$.

The Lossless Bounded Real Lemma

The lossless bounded real lemma is a specialization of the bounded real lemma to lossless bounded real matrices. Recall that a lossless bounded real matrix is a bounded real matrix $S(s)$ satisfying the additional restriction

$$I - S'(-s)S(s) = 0 \tag{7.2.9}$$

The lemma is as follows.

Lossless Bounded Real Lemma. Let $S(s)$ be an $m \times m$ matrix of real rational functions of a complex variable s, with $S(\infty) < \infty$, and let $\{F, G, H, J\}$ be a minimal realization of $S(s)$. Then $S(s)$ is lossless bounded real if and only if there exists a (real) positive definite symmetric matrix P such that

$$\begin{aligned} PF + F'P &= -HH' \\ -PG &= HJ \\ I - J'J &= 0 \end{aligned} \tag{7.2.10}$$

Proof. Suppose first that $S(s)$ is lossless bounded real. By the bounded real lemma and the fact that $S(s)$ is simply bounded real, there exist matrices P, L, and W_0 satisfying the regular bounded real lemma equations and such that

$$I - S'(-s)S(s) = W'(-s)W(s) \tag{7.2.4}$$

where

$$W(s) = W_0 + L'(sI - F)^{-1}G \tag{7.2.2}$$

Applying the lossless constraint, it follows that $W'(-s)W(s) = 0$, and setting $s = j\omega$ we see that $W(j\omega) = 0$ for all ω. Hence $W(s) = 0$ for all s, implying $W_0 = 0$, and, because $[F, G]$ is completely controllable, $L = 0$. When $W_0 = 0$ and $L = 0$ are inserted in the regular bounded real lemma equations, (7.2.10) result.

Conversely, suppose that (7.2.10) hold. These equations are a special case of the bounded real lemma equations, corresponding to $W_0 = 0$ and $L = 0$, and imply that $S(s)$ is bounded real and, by (7.2.2) and (7.2.4), that $I - S'(-s)S(s) = 0$; i.e., $S(s)$ is lossless bounded real. $\nabla\nabla\nabla$

Example 7.2.2 Consider $S(s) = (s-1)/(s+1)$, which has a minimal realization $\{-1, 1, -2, 1\}$. We see that (7.2.10) are satisfied by taking $P = [2]$, a positive definite matrix. Therefore, $S(s)$ is lossless bounded real.

Problem 7.2.1 Using the bounded real lemma, prove that $S'(s)$ is bounded real if $S(s)$ is bounded real, and lossless bounded real if $S(s)$ is lossless bounded real.

Problem 7.2.2 This problem requests a necessity proof of the bounded real lemma via an advanced network theory result. This result is as follows. Suppose that $S(s)$ is bounded real with minimal state-space realization $\{F, G, H, J\}$. Then there exists a network of passive components synthesizing $S(s)$, where, in the state-space equations $\dot{x} = Fx + Gu$, $y = H'x + Ju$, one can identify $u = \frac{1}{2}(v + i)$, v and i being port voltage and current, and $y = \frac{1}{2}(v - i)$. Check that the instantaneous power flow into the network is given by $u'u - y'y$. Then proceed as in the corresponding necessity proof of the positive real lemma to generate a necessity proof of the bounded real lemma.

Problem 7.2.3 This problem requests a necessity proof of the bounded real lemma via a quadratic variational problem. The time-domain statement of the fact that $S(s)$ with minimal realization $\{F, G, H, J\}$ is bounded real is

$$\int_0^{t_1}\bigg\{u'(t)(I - J'J)u(t) - 2u'(t)J'H'\int_0^t e^{F(t-\tau)}Gu(\tau)\,d\tau - \left[\int_0^t e^{F(t-\tau)}Gu(\tau)\,d\tau\right]'HH'\left[\int_0^t e^{F(t-\tau)}Gu(\tau)\,d\tau\right]\bigg\}\,dt \geq 0$$

for all t_1 and $u(\cdot)$ for which the integrals exist. Put another way, if $x(0) = 0$ and $\dot{x} = Fx + Gu$,

$$\int_0^{t_1} [u'(I - J'J)u - 2u'J'H'x - x'HH'x]\,dt \geq 0$$

for all t_1 and $u(\cdot)$ for which the integrals exist.

To set up the variational problem, it is necessary to assume that $I - J'J = R$ is positive definite, as distinct from merely nonnegative definite. The variational problem becomes one of minimizing

$$V(x_0, u(\cdot), t_1) = \int_0^{t_1} (u'Ru - 2u'J'H'x - x'HH'x)\,dt$$

subject to $\dot{x} = Fx + Gu$, $x(0) = x_0$.

(a) Prove that the optimal performance index is bounded above and below if $S(s)$ is bounded real, and that the optimal performance index is $x_0'\Pi(0, t_1)x_0$, where $\Pi(\cdot, t_1)$ satisfies a certain Riccati equation.

(b) Show that $\lim_{t_1 \to \infty} \Pi(t, t_1)$ exists, is independent of t, and satisfies a quadratic matrix equation. Generate a solution of the bounded real lemma equations.

Problem 7.2.4 Let $S(s) = \sqrt{2}/(s + 2)$. Show from the bounded real definition that $S(s)$ is bounded real, and find a solution of the bounded real lemma equations.

7.3 SOLUTION OF THE BOUNDED REAL LEMMA EQUATIONS*

In this section our aim is to indicate how triples P, L, and W_0 may be found that satisfy the bounded real lemma equations

$$\begin{aligned} PF + F'P &= -HH' - LL' \\ -PG &= HJ + LW_0 \\ I - J'J &= W_0'W_0 \end{aligned} \tag{7.3.1}$$

We shall consider first the special case of a lossless bounded real matrix, corresponding to L and W_0 both zero. This case turns out subsequently to be important and is immediately dealt with. Setting L and W_0 to zero in (7.3.1), we get

$$\begin{aligned} PF + F'P &= -HH' \\ -PG &= HJ \\ I - J'J &= 0 \end{aligned}$$

Because Re $\lambda_i(F) < 0$ and $[F, H]$ is completely observable, the first equation is solvable to yield a unique solution P. The second equation has to automatically be satisfied, while the third does not involve P. Thus solution of the lossless bounded real lemma equations is equivalent to solution of

$$PF + F'P = -HH' \tag{7.3.2}$$

We now turn to the more general case. We shall state most of the results as theorems without proofs, the proofs following closely on corresponding results associated with the positive real lemma equations.

A Special Solution via Spectral Factorization

The spectral factorization result of Youla [4], which we have referred to previously, will establish the following result.

*This section may be omitted at a first reading.

Theorem 7.3.1. Consider the solution of (7.3.1) when $\{F, G, H, J\}$ is a minimal realization of an $m \times m$ bounded real matrix $S(s)$. Then there exists a matrix $W(s)$ such that

$$I - S'(-s)S(s) = W'(-s)W(s) \tag{7.3.3}$$

with $W(s)$ computable by procedures specified in [4]. Moreover, $W(s)$ satisfies the following properties:

1. $W(s)$ is $r \times m$, where r is the normal rank of $I - S'(-s)S(s)$.
2. All entries of $W(s)$ are analytic in $\text{Re}\,[s] \geq 0$.
3. $W(s)$ has rank r throughout $\text{Re}\,[s] > 0$; equivalently, $W(s)$ possesses a right inverse whose entries are analytic in $\text{Re}\,[s] > 0$.
4. $W(s)$ is unique to within left multiplication by an arbitrary $r \times r$ real orthogonal matrix.
5. There exist matrices W_0 and L such that

$$W(s) = W_0 + L'(sI - F)^{-1}G \tag{7.3.4}$$

with L and W_0 readily computable from $W(s)$, F, and G.

The content of this theorem is the existence and computability via a technique discussed in [4] of a certain transfer-function matrix $W(s)$. Among various properties possessed by $W(s)$ is that of the existence of the decomposition (7.3.4).

The way all this relates to the solution of (7.3.1) is simple.

Theorem 7.3.2. Let $W(s) = W_0 + L'(sI - F)^{-1}G$ be as described in Theorem 7.3.1. Let P be the solution of the first of Eqs. (7.3.1). Then the second and third of Eqs. (7.3.1) also hold; i.e., P, L, and W_0 satisfy (7.3.1). Further, P is positive definite symmetric and is independent of the particular $W(s)$ of the family described in Theorem 7.3.1.

Therefore, the computations involved in obtaining one solution of (7.3.1) break into three parts: the determination of $W(s)$, the determination of L and W_0, and the determination of P. While P is uniquely defined by this procedure, L and W_0 are not; in fact, L and W_0 may be replaced by LV' and VW_0 for any orthogonal V.

In the case when $W(s)$ is scalar, the computations are not necessarily difficult; the "worst" operation is polynomial factorization. In the case of matrix $W(s)$, however, the computational burden is much greater. For this reason we note another procedure, again analogous to a procedure valid for solving the positive real lemma equations.

A Special Solution via a Riccati Equation

A preliminary specialization must be made before this method is applied. We require

$$R = I - J'J \tag{7.3.5}$$

to be nonsingular. As with the analogous positive real lemma equations, we can justify imposition of this restriction in two ways:

1. With much greater complexity in the theory, we can drop the restriction. This is essentially done in [5].
2. We can show that the task of synthesizing a prescribed $S(s)$—admittedly a notion for which we have not yet given a precise definition, but certainly the main application of the bounded real lemma—for the case when R is singular may be reduced by simple transformations, described in detail subsequently in our discussion of the synthesis problem, to the case when R is nonsingular.

In a problem at the end of the last section, the reader was requested to prove Theorem 7.3.3 below. The background to this theorem is roughly as follows. A quadratic variational problem may be defined using the matrices F, G, H, and J of a minimal state-space realization of $S(s)$. This problem is solvable precisely when $S(s)$ is bounded real. The solution is obtained by solving a Riccati equation, and the solution of the equation exists precisely when $S(s)$ is bounded real.

Theorem 7.3.3. Let $\{F, G, H, J\}$ be a minimal realization of a bounded real $S(s)$, with $R = I - J'J$ nonsingular. Then $\Pi(t, t_1)$ exists for all $t \leq t_1$, where

$$-\frac{d\Pi}{dt} = \Pi(F + GR^{-1}J'H') + (F + GR^{-1}J'H')'\Pi \\ - \Pi GR^{-1}G'\Pi - HH' - HJR^{-1}J'H' \tag{7.3.6}$$

and $\Pi(t_1, t_1) = 0$. Further, there exists a constant negative definite symmetric matrix $\bar{\Pi}$ given by

$$\bar{\Pi} = \lim_{t_1 \to \infty} \Pi(t, t_1) = \lim_{t \to -\infty} \Pi(t, t_1) \tag{7.3.7}$$

and $\bar{\Pi}$ satisfies a limiting version of (7.3.6), viz.,

$$\bar{\Pi}(F + GR^{-1}J'H') + (F + GR^{-1}J'H')'\bar{\Pi} - \bar{\Pi}GR^{-1}G'\bar{\Pi} \\ - HH' - HJR^{-1}J'H' = 0 \tag{7.3.8}$$

The significance of $\bar{\Pi}$ is that it defines a solution of (7.3.1) as follows:

Theorem 7.3.4. Let $\bar{\Pi}$ be as above. Then $P = -\bar{\Pi}$, $L = (\bar{\Pi}G - HJ)R^{-1/2}$, and $W_0 = R^{1/2}$ are a solution triple of (7.3.1).

The proof of this theorem is requested in the problems. It follows simply from (7.3.8). Note that, again, L and W_0 may be replaced by LV' and VW_0 for any orthogonal V.

With the interpretation of Theorem 7.3.4, Theorem 7.3.3 really provides two ways of solving (7.3.1); the first way requires the solving of (7.3.6) backward in time till a steady-state solution is reached. The second way requires direct solution of (7.3.8).

We have already discussed techniques for the solution of (7.3.6) and (7.3.8) in connection with solving the positive real lemma equations. The ideas are unchanged. Recall too that the quadratic matrix equation

$$X(F + GR^{-1}J'H') + (F + GR^{-1}J'H')'X - XGR^{-1}G'X \\ - HH' - HJR^{-1}J'H' = 0 \qquad (7.3.9)$$

in general has a number of solutions, only one of which is the limit of the associated Riccati equation (7.3.6). Nevertheless, any solution of (7.3.9) will define solutions of (7.3.1), since the application of Theorem 7.3.4 is not dependent on using that solution of (7.3.9) which is also derivable from the Riccati equation.

The definitions of P, L, and W_0 provided by Theorem 7.3.4 always generate a spectral factor $W(s) = W_0 + L'(sI - F)^{-1}G$ that is of dimension $m \times m$; this means $W(s)$ has the smallest possible number of rows.

Connection of Theorems 7.3.2 and 7.3.3 is possible, as for the positive real lemma case. The connection is as follows:

Theorem 7.3.5. Let P be determined by the procedure outlined in Theorem 7.3.2 and $\bar{\Pi}$ by the procedure outlined in Theorem 7.3.3. Then $P = -\bar{\Pi}$.

In other words, the spectral factor generated using the formulas of Theorem 7.3.4, when $\bar{\Pi}$ is determined as the limiting solution of the Riccati equation (7.3.6), is the spectral factor defined in [4] with the properties listed in Theorem 7.3.1 (to within left multiplication by an arbitrary $m \times m$ real orthogonal matrix).

Example 7.3.1 Consider $S(s) = \sqrt{2}/(s + 2)$, which can be verified to be bounded real. Let us first proceed via Theorem 7.3.1 to compute solutions to the bounded real lemma equations. We have

$$1 - S(-s)S(s) = 1 - \frac{2}{-s^2 + 4} = \frac{-s^2 + 2}{-s^2 + 4} = \frac{-s + \sqrt{2}}{-s + 2}\,\frac{s + \sqrt{2}}{s + 2}$$

We see that $W(s) = (s + \sqrt{2})(s + 2)^{-1}$ satisfies all the conditions listed in the statement of Theorem 7.3.1. Further, a minimal realization for $S(s)$ is $\{-2, 1, \sqrt{2}, 0\}$, and for $W(s)$ one is readily found to be $\{-2, 1, -2 + \sqrt{2}, 1\}$. The equation for P is simply

$$-4P = -2 - (-2 + \sqrt{2})^2$$

or

$$P = 2 - \sqrt{2}$$

As Theorem 7.3.2 predicts, we can verify simply that the second and third of Eqs. (7.3.1) hold.

Let us now follow Theorem 7.3.3. The Riccati equation is

$$-\dot{\Pi} = -4\Pi - \Pi^2 - 2$$

or

$$\int_{\Pi(t)}^{\Pi(t_1)} \frac{d\Pi}{\Pi^2 + 4\Pi + 2} = t_1 - t$$

from which

$$\bar{\Pi}(t, t_1) = \frac{-(2 - \sqrt{2})[1 - e^{2\sqrt{2}(t_1 - t)}]}{1 - [(2 - \sqrt{2})/(2 + \sqrt{2})]e^{2\sqrt{2}(t_1 - t)}}$$

Clearly $\Pi(t, t_1)$ exists for all $t \leq t_1$, and

$$\bar{\Pi} = \lim_{t \to -\infty} \Pi(t, t_1) = -(2 - \sqrt{2})$$

This is the correct value, as predicted by Theorem 7.3.5. We can also solve (7.3.1) by using any solution of the quadratic equation (7.3.8), which here becomes

$$\bar{\Pi}^2 + 4\bar{\Pi} + 2 = 0$$

One solution is $-(2 - \sqrt{2})$, as we have found. The other is $-(2 + \sqrt{2})$. This leads to $P = 2 + \sqrt{2}$, $L = -2 - \sqrt{2}$, while still $W_0 = 1$. The spectral factor is

$$W(s) = 1 - \frac{2 + \sqrt{2}}{s + 2} = \frac{s - \sqrt{2}}{s + 2}$$

So far, we have indicated how to obtain special solution P of (7.3.1) and associated L and W_0; also, we have shown how to find a limited number of other P and associated L and W_0 by computing solutions of a quadratic matrix equation differing from the limiting solution of a Riccati equation. Now we wish to find all solutions of (7.3.1).

General Solution of the Bounded Real Lemma Equations

We continue with a policy of stating results in theorem form without proofs, on account of the great similarity with the analogous calculations based on the positive real lemma. As before, we convert in two steps the problem of solving (7.3.1) to the problem of solving a more or less manageable quadratic matrix inequality.

We continue the restriction that $R = I - J'J$ be nonsingular. Then we have

Theorem 7.3.6. Solution of (7.3.1) is equivalent, in a sense made precise below, to the solution of the quadratic matrix inequality

$$PF + F'P \leq -HH' - (PG + HJ)R^{-1}(PG + HJ)' \qquad (7.3.10)$$

which is the same as

$$P(F + GR^{-1}J'H') + (F + GR^{-1}J'H')'P + PGR^{-1}G'P + HH' + HJR^{-1}J'H' \leq 0 \qquad (7.3.11)$$

If $P, L,$ and W_0 satisfy (7.3.1), then P satisfies (7.3.10). If P satisfies (7.3.10), then P together with some matrices L and W_0 satisfies (7.3.1). The set of all such L and W_0 is given by

$$L = [-(PG + HJ)R^{-1/2} \mid N]V \qquad W_0' = [R^{1/2} \mid 0]V \qquad (7.3.12)$$

where V ranges over the set of real matrices for which $VV' = I$, the zero block in the definition of W_0' contains the same number of columns as N, and N is a real matrix with minimum number of columns such that

$$PF + F'P = -HH' - (PG + HJ)R^{-1}(PG + HJ)' - NN' \qquad (7.3.13)$$

A partial proof of this theorem is requested in the problems. Essentially, it says that P satisfies (7.3.1) if and only if P satisfies a quadratic matrix inequality, and L and W_0 can be computed by certain formulas if P is known.

Observe that the matrices P satisfying (7.3.10) with equality are the negatives of the solutions of (7.3.9). In particular, $\bar{P} = -\bar{\Pi}$ satisfies (7.3.10) with equality, where $\bar{\Pi}$ is the limiting solution as t approaches minus infinity of the Riccati equation (7.3.6).

The next step is to replace (7.3.10) by a more manageable inequality. Recalling that $\bar{P}$ satisfies (7.3.10) with equality, it is not difficult to prove the following:

Theorem 7.3.7. Let $Q = P - \bar{P}$, where $\bar{P}$ is defined as above. Then P satisfies (7.3.10) if and only if Q satisfies

$$Q\bar{F} + \bar{F}'Q \leq -QGR^{-1}G'Q \tag{7.3.14}$$

where

$$\bar{F} = F + GR^{-1}J'H' + GR^{-1}G'\bar{P} \tag{7.3.15}$$

Solution procedures for (7.3.14) have already been discussed. As one might imagine, the choice of $\bar{P}$ as the negative of the limiting solution of the Riccati equation (7.3.6) leads to $\bar{F}$ possessing nonpositive real part eigenvalues, and thus to Q being nonnegative definite.

At this point, we conclude our discussion. The main point that we hope the reader has grasped is the essential similarity between the problem of solving the positive real lemma equations and the bounded real lemma equations.

Problem 7.3.1 Prove Theorem 7.3.4.

Problem 7.3.2 Prove that if P satisfies Eqs. (7.3.1), then P satisfies the inequality (7.3.10). Verify that if P satisfies (7.3.10) and L and W_0 are defined by (7.3.12), then (7.3.1) holds.

Problem 7.3.3 Let $S(s) = \frac{1}{2} - \frac{1}{2}/(s + 1)$. Find a minimal realization $\{F, G, H, J\}$ for $S(s)$. Find a $W(s)$ with all the properties listed in Theorem 7.3.1 and construct a realization $\{F, G, L, W_0\}$; determine a P such that (7.3.1) holds. Also set up a quadratic equation determining two P satisfying (7.3.1); find these P and the associated $W(s)$. Finally, find all P satisfying (7.3.1).

7.4 RECIPROCITY IN STATE-SPACE TERMS

As we have noted in the introduction to this chapter, there are advantages from the point of view of solving synthesis problems in stating the constraints applying to the matrices in a state-space realization of a symmetric transfer function. Without further ado, we state the first main result. For the original proof, see [6].

Theorem 7.4.1. Let $W(s)$ be an $m \times m$ matrix of real rational functions of s, with $W(\infty) < \infty$. Let $\{F, G, H, J\}$ be a minimal realization of $W(s)$. Then $W(s) = W'(s)$ if and only if

$$J = J' \tag{7.4.1}$$

and there exists a nonsingular symmetric matrix P such that

$$\begin{aligned} PF &= F'P \\ PG &= -H \end{aligned} \tag{7.4.2}$$

Proof. First suppose that $W(s) = W'(s)$. Equation (7.4.1) is immediate, by setting $s = \infty$. Symmetry of $W(s)$ implies then that

$$H'(sI - F)^{-1}G = G'(sI - F')^{-1}H = (-G)'(sI - F')^{-1}(-H)$$

so that $\{F, G, H\}$ and $\{F', -H, -G\}$ are two minimal realizations of $W(s) - J$. It follows that there exists a nonsingular matrix P, not necessarily symmetric, such that

$$PFP^{-1} = F' \qquad PG = -H \qquad H'P^{-1} = -G' \tag{7.4.3}$$

The first two of these equations imply Eqs. (7.4.2).

It also follows that P is unique. This is a general property of a coordinate-basis transformation relating prescribed minimal realizations of the same transfer-function matrix, but may be seen in this particular case by the following argument. Suppose that P_1 and P_2 both satisfy (7.4.2). Set $P_3 = P_1 - P_2$, so that $P_3F = F'P_3$, and $P_3G = 0$. Then $P_3FG = F'P_3G = 0$, and more generally $P_3F^iG = 0$, or $P_3[G \; FG \cdots F^{n-1}G] = 0$, where n is the dimension of F. Complete controllability implies that $P_3 = 0$ or $P_1 = P_2$, establishing uniqueness.

Having proved the uniqueness of P, to establish symmetry is straightforward. Transposing the first and last equations of (7.4.3) yields

$$\begin{aligned} F'P' &= P'F \\ P'G &= -H \end{aligned} \tag{7.4.4}$$

Now compare Eqs. (7.4.4) with (7.4.2). We see that if P satisfies (7.4.2), so does P'. By the uniqueness of matrices P satisfying (7.4.2), $P = P'$ as required.

Now we prove the converse. Assume (7.4.1) and (7.4.2). Then

$$\begin{aligned} W(s) &= J + H'(sI - F)^{-1}G \\ &= J' + (H'P^{-1})(sI - PFP^{-1})^{-1}PG \\ &= J' + (-G)'(sI - F')^{-1}(-H) \\ &= J' + G'(sI - F')^{-1}H \\ &= W'(s) \end{aligned}$$

The third equality follows by using (7.4.2); the remainder is evident, and the proof is complete. ∇ ∇ ∇

In the statement of the above theorem we restricted $W(s)$ to being such that $W(\infty) < \infty$. If $W(s)$ is real rational but does not satisfy this constraint, we may write $W(s) = \cdots + W_2 s^2 + W_1 s + J + H'(sI - F)^{-1}G$. Then Theorem 7.4.1 applies with the addition of $W_1 = W_1'$, $W_2 = W_2', \cdots$, to Eq. (7.4.1).

Calculation of P

Just as we have foreshadowed the need to use solutions of the positive real lemma equation in applications, so we make the point now that it will be necessary to use the matrix P in later synthesis calculations. For this reason, we note here how P may be calculated.

For notational convenience, we define

$$V_c = [G \quad FG \cdots F^{n-1}G] \qquad V_o = [H \quad F'H \cdots (F')^{n-1}H]$$

as the controllability and observability matrices associated with $W(s)$. It will be recalled that F is assumed to be $n \times n$, and that V_c and V_o will have rank n by virtue of the minimality of $\{F, G, H, J\}$.

From (7.4.2) we have

$$PG = -H$$
$$PFG = F'PG = -F'H$$
$$PF^2G = F'PFG = -(F')^2H$$

and generally

$$PF^iG = -(F')^iH$$

It follows that

$$PV_c = -V_o$$

or

$$P = (-V_oV_c')(V_cV_c')^{-1} \tag{7.4.5}$$

with the inverse existing by virtue of the full rank property of V_c.

Example 7.4.1 Consider the scalar transfer function $W(s) = (s + 1)(s^3 + s^2 + s + 1)^{-1}$ for which a minimal realization is

$$F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad J = 0$$

The controllability and observability matrices are

$$V_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad V_o = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$P = (-V_o V_c')(V_c V_o')^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ +1 & -2 & 0 \\ 0 & -1 & +1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

Observe that P, as required, is symmetric. The satisfaction of (7.4.2) is easily checked.

Developments from Theorem 7.4.1

We wish to indicate certain consequences of Theorem 7.4.1, including a coordinate-basis change that results in Eqs. (7.4.2) taking a specially simple form.

Because P is symmetric and nonsingular, there exists a nonsingular matrix T such that

$$P = T'\Sigma T \tag{7.4.6}$$

where Σ is a diagonal matrix of the form

$$\Sigma = I_{k_1} \dotplus (-1)I_{k_2} \tag{7.4.7}$$

Here, if F is $n \times n$, we have $k_1 + k_2 = n$. We note the following important result:

Theorem 7.4.2. The integers k_1 and k_2 above are uniquely defined by $W(s)$ and are independent of the particular minimal realization of $W(s)$ used to generate P.

Proof. Suppose that $\{F_1, G_1, H_1, J\}$ and $\{F_2, G_2, H_2, J\}$ are two minimal realizations of the same $W(s) = W'(s)$, with T a nonsingular matrix such that $TF_1T^{-1} = F_2$, etc. Then if P_1 and P_2 are the solutions of (7.4.2) corresponding to the two realizations, it is easily checked that $P_1 = T'P_2T$. It follows that P_1 and P_2 have identical numbers of positive and negative real eigenvalues. But k_1 and k_2 are, respectively, the number of positive and the number of negative eigenvalues of P_1 or P_2; thus k_1 and k_2 are uniquely defined by $W(s)$. $\nabla\nabla\nabla$

In the sequel the computation of T from P will be required. This is actually quite straightforward and can proceed by simple rational operations on the entries of P. Reference [7] includes two techniques, due to Lagrange and

Jacobi. The matrix T is not unique and may be taken to be triangular. If k_1 and k_2 above are required rather than T, these numbers can be computed from the signs of the principal leading minors of P without calculation of T. As shown in [7], if D_i is the value of the minor of P obtained by deleting all but the first i rows and columns, and if $D_i \neq 0$ for all i, then k_1 is the number of permanences of sign of the sequence $1, D_1, D_2, \ldots, D_n$. (If one or more D_i is zero, a more complicated rule applies.) Given k_1, k_2 is immediately obtained as $n - k_1$.

Now let us note a coordinate-basis change that yields an interesting form for Eqs. (7.4.2).

With T such that $P = T'\Sigma T$, define a new realization $\{F_1, G_1, H_1, J\}$ of $W(s)$ by

$$F_1 = TFT^{-1} \qquad G_1 = TG \qquad H_1 = (T^{-1})'H \tag{7.4.8}$$

Then it is a simple matter to check that Eqs. (7.4.2) in the new coordinate basis become

$$\Sigma F_1 = F_1'\Sigma \qquad \Sigma G_1 = -H_1 \tag{7.4.9}$$

so that

$$F_1 = \begin{bmatrix} F_{11} & F_{12} \\ -F_{12}' & F_{22} \end{bmatrix} \qquad F_{11}' = F_{11} \qquad F_{22}' = F_{22}$$

and if $G_1' = [G_{11}' \quad G_{12}']$, then $H_1' = [-G_{11}' \quad G_{12}']$

Since T is not unique, F_{11}, F_{12}, etc., are also not unique, although k_1 and k_2 are unique.

We sum up this result as follows:

Theorem 7.4.3. Let $W(s)$ be a symmetric matrix of real rational functions of s with $W(\infty) < \infty$. Let $\{F, G, H, J\}$ be a minimal realization. Then there exists a nonunique minimal realization $\{F_1, G_1, H_1, J\}$ with $F_1 = TFT^{-1}$, $G_1 = TG$, and $H_1 = (T^{-1})'H$, the matrix T being defined below, such that

$$\Sigma F_1 = F_1'\Sigma \qquad \Sigma G_1 = -H_1$$

Here the matrix Σ is of the form $I_{k_1} \dotplus (-I_{k_2})$, with k_1 and k_2 determined uniquely by $W(s)$. With P the unique symmetric solution of $PF = F'P$, $PG = -H$, any decomposition of P as $P = T'\Sigma T$ yields T.

Problem 7.4.1 Suppose that $W(s)$ is a scalar with minimal realization $\{F, G, H, J\}$, with F of dimension $n \times n$. Suppose also that $W(s)$ is expanded in the form

$$W(s) = J + \frac{w_0}{s} + \frac{w_1}{s^2} + \frac{w_2}{s^3} + \cdots$$

so that $w_i = H'F^iG$. By examining $V_c'PV_c$, where P is a solution of $PF = F'P$ and $PG = -H$, conclude that k_1 and k_2 are the number of negative eigenvalues and positive eigenvalues respectively of the matrix

$$\begin{bmatrix} w_0 & w_1 & \cdots & w_{n-1} \\ w_1 & w_2 & \cdots & w_n \\ w_2 & w_3 & \cdots & w_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ w_{n-1} & w_n & \cdots & w_{2n-2} \end{bmatrix}$$

Problem 7.4.2 Suppose that $W(s) = W'(s)$, and that $\{F, G, H, J\}$ is a realization for $W(s)$ with the property that $\Sigma F = F'\Sigma$ and $\Sigma G = -H$ for some $\Sigma = I_{k_1} \dotplus (-1)I_{k_2}$. Show that a realization $\{F_1, G_1, H_1, J\}$ with $F_1 = TFT^{-1}$, etc., will satisfy $\Sigma F_1 = F_1'\Sigma$ and $\Sigma G_1 = -H_1$ if and only if $T'\Sigma T = \Sigma$.

Problem 7.4.3 Suppose that $W(s) = (s + 1)/(s + 2)(s + 3)$. Compute the integers k_1 and k_2 of Theorem 7.4.2.

Problem 7.4.4 Let

$$F = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Find P such that $PF = F'P$, $PG = -H$, and then T such that $T'\Sigma T = P$. Finally, determine $F_1 = TFT^{-1}$, $G_1 = TG$, $H_1 = (T^{-1})'H$, and check that $\Sigma F_1 = F_1'\Sigma$ and $\Sigma G_1 = -H_1$ for some appropriate Σ.

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Part V

NETWORK SYNTHESIS

In this part our aim is to apply the algebraic descriptions of the positive real, bounded real, and reciprocity properties to the problem of synthesis. We shall present algorithms for passing from prescribed positive real immittance and bounded real scattering matrices to networks whose immittance or scattering matrices are identical with those prescribed. We shall also pay attention to the problem of reciprocal (gyratorless) synthesis and to synthesizing transfer functions.

8

Formulation of State-Space Synthesis Problems

8.1 AN INTRODUCTION TO SYNTHESIS PROBLEMS

In the early part of this book we were largely preoccupied with obtaining analysis results. Now our aim is to reverse some of these results. The reader will recall, for example, that the impedance matrix $Z(s)$ of a passive network N is positive real, and that $\delta[Z(s)]$ is less than or equal to the number of inductors and capacitors in the network N.* A major synthesis result we shall establish is that, given a prescribed positive real $Z(s)$, one can find a passive network with impedance $Z(s)$ containing precisely the minimum possible number $\delta[Z(s)]$ of reactive elements. Again, the reader will recall that the impedance matrix of a reciprocal network N is symmetric. The corresponding synthesis result we shall establish is that, given a prescribed symmetric positive real $Z(s)$, we can find a passive network with impedance $Z(s)$ containing precisely $\delta[Z(s)]$ reactive elements and no gyrators.

Statements similar to the above can be made regarding hybrid-matrix synthesis and scattering-matrix synthesis. We leave the details to later sections.

Generally speaking, we shall attempt to solve synthesis problems by converting them in some way to a problem of matrix algebra, and then solving

*This second point was established in a section that may have been omitted at a first reading.

the problem of matrix algebra. In this chapter our main aims are to formulate, but not solve, the problems of matrix algebra and to indicate some preliminary and minor, but also helpful, steps that can be used to simplify a synthesis problem. Sections 8.2 and 8.3 are concerned with converting synthesis problems to matrix algebra problems for, respectively, impedance (actually including hybrid) matrices and scattering matrices. Section 8.4 is concerned with presenting the preliminary synthesis steps.

Solutions to the matrix algebra problems—which amount to solutions of the synthesis problems themselves—will be obtained in later chapters. In obtaining these solutions, we shall make great use of the positive real and bounded real lemmas and the state-space characterization of reciprocity.

The reader will probably appreciate that impedance (or hybrid) matrix synthesis and scattering-matrix synthesis constitute two sides of the same coin, in the following sense. Associated with an impedance $Z(s)$ there is a scattering matrix

$$S(s) = [Z(s) - I][Z(s) + I]^{-1} \tag{8.1.1}$$

A synthesis of $Z(s)$ automatically yields a synthesis of $S(s)$, and conversely. It might then be argued that the separate consideration of impedance- and scattering-matrix synthesis is redundant; however, the insights to be gained from separate considerations are great, and we have no hesitation in offering separate consideration. This is entirely consonant with the ideas expressed in books dealing with network synthesis via classical procedures (see, e.g., [1]).

8.2 THE BASIC IMPEDANCE SYNTHESIS PROBLEM

In this section we shall examine from a state-space viewpoint broad aspects of the impedance synthesis problem. In the past, many classical multiport synthesis methods made use of the underlying idea of an early classical synthesis called the Darlington synthesis (see [1]), the idea being that of *resistance extraction*, which we define below. With recent developments in the theory of state-space characterization of a real-rational matrix, a new synthesis concept has been discovered—the *reactance-extraction* technique, again defined below. The technique appears to have originated in a paper by Youla and Tissi [2], which deals with the rational bounded real scattering matrix synthesis problem. Later Anderson and Newcomb extended use of the concept to the rational positive real impedance synthesis problem [3]. Indeed, the technique plays a prominent role in the modern theory of state-space n-port network synthesis, as we shall see subsequently. We shall now formulate the impedance synthesis problem via (1) the resistance-extraction approach, and (2) the reactance-extraction approach.

We shall assume throughout this section that there is prescribed a rational positive real $Z(s)$ that is to be synthesized, with $Z(s)$ $m \times m$, and with $Z(\infty) < \infty$. Although the property that $Z(\infty) < \infty$ is not always possessed by rational positive real matrices, we have noted in Section 5.1 that a rational positive real $Z(s)$ may be expressed as

$$Z(s) = sL + Z_0(s) \tag{8.2.1}$$

where L is nonnegative definite symmetric, and $Z_0(s)$ is rational positive real with $Z_0(\infty) < \infty$. [Of course, the computations involved in breaking up $Z(s)$ this way are straightforward.] It follows then that a synthesis of $Z(s)$ is achieved by a series connection of syntheses of sL and $Z_0(s)$. The synthesis of sL can be achieved in a trivial manner using transformers and inductors; the details are developed in one of the problems. So even if the original $Z(s)$ does not have $Z(\infty) < \infty$, we can always reduce in a simple manner the problem of synthesizing a positive real $Z(s)$ to one of synthesizing another positive real $Z_0(s)$ with the bounded-at-infinity property.

The Resistance-Extraction Approach to Synthesis

The key idea of the resistance-extraction technique lies in viewing the network N synthesizing the prescribed $Z(s)$ as a *cascade interconnection of a lossless subnetwork and positive resistors.* Since any positive resistor r is equivalent to a transformer of turns ratio $\sqrt{r} : 1$ terminated in a unit resistor, we can therefore assume all resistance values to be 1 Ω by absorbing in the lossless subnetwork the transformers used for normalization. On collecting these unit resistors, p in number, into a subnetwork N_l, we see that N_l loads a lossless $(m + p)$ port N_L to yield N, as shown in Fig. 8.2.1.

The synthesis problem falls into two parts. The first part requires the derivation of a lossless positive real hybrid matrix $\mathcal{H}(s)$ of size $(m + p) \times$

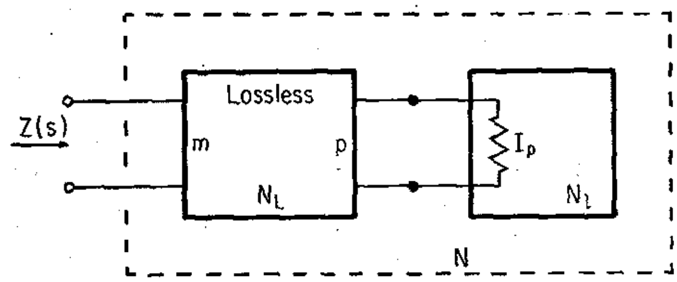

FIGURE 8.2.1. Resistance Extraction to Obtain a Lossless Coupling Network.

$(m + p)$ from the given $m \times m$ positive real $Z(s)$ such that termination in unit resistors of the last p ports of a network synthesizing $\mathcal{H}(s)$ results in $Z(s)$ being observed at the unterminated ports. The second part requires the synthesis of the lossless positive real $\mathcal{H}(s)$. As we shall later see, the lossless character of $\mathcal{H}(s)$ makes the problem of synthesizing $\mathcal{H}(s)$ an easy one. The difficult problem is to pass from $Z(s)$ to $\mathcal{H}(s)$, and we shall now discuss this in a little more detail.

Let us now translate the basic concept of the resistance-extraction technique into a true *state-space* impedance synthesis problem. Consider an $(m + p)$-port lossless network N_L, as shown in Fig. 8.2.2. Let us identify

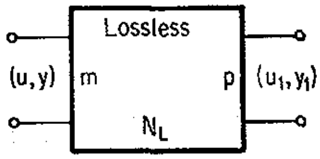

FIGURE 8.2.2. An $(m + p)$-port Lossless Network.

an input m vector u with the currents and a corresponding output m vector y with the voltages at the left-hand m ports of N_L. Let a p vector u_1 denote the inputs at the right-hand p ports, where the i^{th} component u_{1i} of u_1 is *either* a current *or* a voltage, and let a p vector y_1 denote the corresponding outputs with the ith component y_{1i} being the quantity dual to u_{1i}, i.e., either a voltage or a current. Assume that we know the hybrid matrix $\mathcal{H}(s)$ describing the port behavior of N_L, and that $\{F_L, G_L, H_L, J_L\}$ is a minimal realization of $\mathcal{H}(s)$. Then we can describe the port behavior of N_L by the following time-domain state-space equations:

$$\dot{x} = F_L x + G_L \begin{bmatrix} u \\ u_1 \end{bmatrix} \tag{8.2.2a}$$

$$\begin{bmatrix} y \\ y_1 \end{bmatrix} = H'_L x + J_L \begin{bmatrix} u \\ u_1 \end{bmatrix} \tag{8.2.2b}$$

We shall denote the dimension of the realization $\{F_L, G_L, H_L, J_L\}$ by n.

Because N_L is lossless, the quadruple $\{F_L, G_L, H_L, J_L\}$ satisfies the lossless positive real lemma equations, i.e., the equations

$$\begin{aligned} PF_L + F'_L P &= 0 \\ PG_L &= H_L \\ J_L + J'_L &= 0 \end{aligned} \tag{8.2.3}$$

should hold for a symmetric positive definite $n \times n$ matrix P.

Now suppose that termination of the right-hand p ports of N_L in unit resistors leads to the impedance matrix $Z(s)$ being observed at the left-hand ports. The operation of terminating these p ports in unit resistors is equivalent to forcing

$$u_1 = -y_1 \tag{8.2.4}$$

Therefore, the matrices F_L, G_L, H_L, and J_L, besides satisfying (8.2.3), are further restricted to those with the property that by setting (8.2.4) into (8.2.2) and on eliminating u_1 and y_1, (8.2.2) must then reduce simply to

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= H'x + Ju \end{aligned} \tag{8.2.5}$$

with F, G, H, and J such that $J + H'(sI - F)^{-1}G = Z(s)$.

If N_L is reciprocal, as will be the case when a symmetric $Z(s)$ is synthesized by a reciprocal network, there is imposed on $\mathcal{H}(s)$ (and thus on the matrices F_L, G_L, H_L, and J_L) a further constraint, which is that $\mathcal{H}(s) = J_L + H_L'(sI - F_L)^{-1}G_L$ should satisfy

$$\Sigma\, \mathcal{H}(s) = \mathcal{H}'(s)\, \Sigma \tag{8.2.6}$$

for some diagonal matrix Σ with only $+1$ and -1 entries, in which the $+1$ entries correspond to the current-excited ports and the -1 entries to the voltage-excited ports in the definition of $\mathcal{H}(s)$.

We may summarize these remarks by saying that if a network N synthesizing $Z(s)$ is viewed as a lossless network N_L terminated in unit resistors, and if the lossless network has a hybrid matrix $\mathcal{H}(s)$ with minimal realization $\{F_L, G_L, H_L, J_L\}$, then (8.2.3) hold for some positive P, and under the substitution of (8.2.4) in (8.2.2), the state-space equations (8.2.5) will be recovered, where $\{F, G, H, J\}$ is a realization of $Z(s)$. The additional constraint (8.2.6) applies in case N is reciprocal.

Now we reverse the line of reasoning just taken. Assuming for the moment the ready solvability of the lossless synthesis problem, it should be clear that impedance synthesis via the resistance-extraction technique requires the following: *Find a minimal state-space realization* $\{F_L, G_L, H_L, J_L\}$ *that characterizes* N_L *via* (8.2.2) *such that*

1. *For nonreciprocal networks,*
   (a) *the conditions of the lossless positive real lemma, (8.2.3), are fulfilled, and*
   (b) *under the constraint of (8.2.4), the set of equations (8.2.2) simplifies to (8.2.5).*

2. *For reciprocal networks,*
   (a) *the conditions of the lossless positive real lemma and the reciprocity property, corresponding to (8.2.3) and (8.2.6), respectively, are met, and*
   (b) *under the constraint of (8.2.4), the set of equations (8.2.2) simplifies to (8.2.5).*

Before we proceed to the next subsection, we wish to make two comments.

First, the entries of the vector u_1 have not been specified up to this point as being voltages or currents. Actually, we can demand that they be all currents, or all voltages, or some definite combination of currents and voltages, and nonreciprocal synthesis is still possible for any assignation of variables. However, as we shall see later, reciprocal synthesis requires us to label certain entries of u_1 as currents and the others as voltages according to a pattern determined by the particular $Z(s)$ being synthesized.

The second point is that we have for the moment sidestepped the question of synthesizing a lossless hybrid matrix $\mathcal{H}(s)$. Now lossless synthesis problems have been easily solved in classical network synthesis, and therefore it is not surprising to anticipate similar ease of synthesis via state-space procedures. However, when a lossless hybrid matrix $\mathcal{H}(s)$ is derived *in the course of a resistance-extraction synthesis of a positive real $Z(s)$*, it turns out that synthesis of $\mathcal{H}(s)$ is even easier than if $\mathcal{H}(s)$ were simply an arbitrary lossless hybrid matrix that was to be synthesized. We shall defer illustration of this point to a subsequent chapter dealing with actual synthesis methods.

The Reactance-Extraction Problem

In a similar fashion to the resistance-extraction idea, we again suppose that we have a network N synthesizing $Z(s)$, but now isolate and separate out all reactive elements (inductors and capacitors), instead of resistors. Thus an m-port network N realizing a rational positive real $Z(s)$ is viewed as an interconnection of two subnetworks N_r and N_l, as illustrated in Fig. 8.2.3. The subnetwork N_r is a nondynamic $(m + n)$ port, while the n port N_l consists of, say, n_1 1-H inductors and n_2 1-F capacitors with $n_1 + n_2 = n$. Note that all inductors and capacitors may always be assumed unity in value. This is because an l-H inductor may be replaced by a transformer of turns ratio $\sqrt{l}:1$ terminated in a 1-H inductor with a similar replacement possible for capacitors; the normalizing transformers can then be incorporated into N_r.

In formulating the synthesis problem via the reactance-extraction technique, it is more convenient to consider the nonreciprocal and reciprocal cases separately.

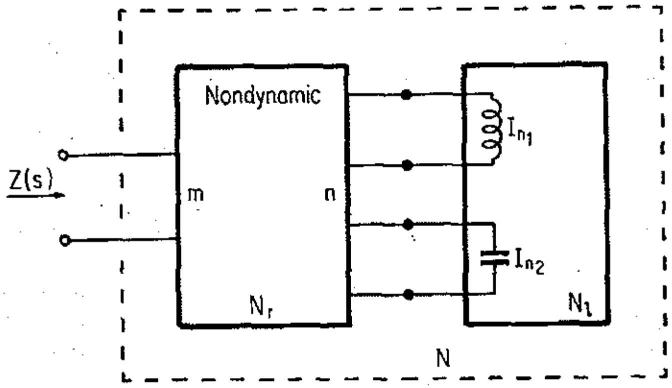

FIGURE 8.2.3. Reactance Extraction to Obtain a Nondynamic Coupling Network.

Reactance Extraction—Nonreciprocal Networks

In the nonreciprocal case, gyrators are allowable as circuit elements. We can simplify the situation by assuming that all reactive elements present in the network N synthesizing $Z(s)$ are of the same kind, i.e., either inductors or capacitors. This is always possible since one type of element may be replaced by the other with the aid of gyrators, as illustrated by the equivalences of Fig. 8.2.4. (These equivalences may easily be verified from the basic direct element definitions.) Assume for the sake of argument that only inductors are used; then the proposed realization arrangement in Fig. 8.2.3 simplifies to that in Fig. 8.2.5.

Although N possesses an impedance matrix $Z(s)$ by assumption, there is however no guarantee that N_r will possess an impedance matrix, though, as

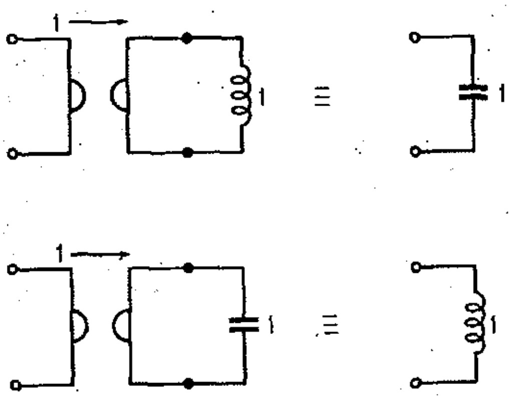

FIGURE 8.2.4. Element Replacements.

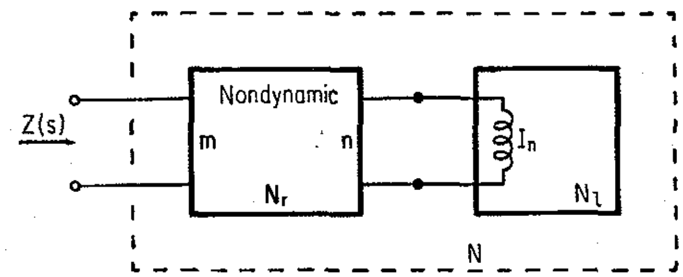

FIGURE 8.2.5. Inductor Extraction.

remarked earlier, a scattering matrix or a hybrid matrix must exist for N_r. For the present purpose in discussing an approach to the synthesis problem, it will be sufficient to assume tentatively that an impedance matrix does exist for N_r; this will be shown later to be true* when we actually tackle the synthesis of an arbitrary rational positive real matrix.

Now because N_r consists of purely nondynamic elements, its impedance matrix M is real and constant; it must also be positive real since N_r consists of purely passive elements. Application of the positive real definition yields a necessary and sufficient condition for M to be positive real as

$$M + M' \geq 0 \tag{8.2.7}$$

Let the $(m + n) \times (m + n)$ constant positive real impedance matrix M be partitioned similarly to the ports of N_r as

$$M = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \tag{8.2.8}$$

where Z_{11} is $m \times m$, Z_{22} is $n \times n$, etc. To relate these submatrices Z_{ij} to the input impedance $Z(s)$ when N_r is loaded by N_l, consisting of n unit inductors, we note that the impedance of N_l is simply sI_n. It follows simply that

$$Z(s) = Z_{11} - Z_{12}(sI_n + Z_{22})^{-1}Z_{21} \tag{8.2.9}$$

Comparison with the standard equation

$$Z(s) = J + H'(sI - F)^{-1}G$$

*Strictly speaking, the existence of an impedance matrix for N_r is guaranteed only if the number of reactive elements used in N_l in realizing $Z(s)$ is a minimum (as shown earlier in Section 5.3). However, for the sake of generality, such a restriction on reactive-element minimality will not be imposed at this stage.

reveals immediately that a realization of $Z(s)$ is given by

$$\{F, G, H, J\} = \{-Z_{22}, Z_{21}, -Z'_{12}, Z_{11}\} \tag{8.2.10}$$

We may conclude from (8.2.10) therefore that *if we have on hand a nonreciprocal synthesis of $Z(s)$, and if, on isolating all reactive elements (normalized) and with all capacitors replaced by inductors as shown in Fig. 8.2.5, M becomes the constant impedance of the nondynamic network N_r, then this impedance matrix M determines one particular realization of $Z(s)$ through (8.2.10).*

Conversely, then, if we have any realization $\{F, G, H, J\}$ of a prescribed $Z(s)$, we might think of trying to synthesize $Z(s)$ by terminating a nondynamic N_r in unit inductors, with the impedance matrix M of N_r given by

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} \tag{8.2.11}$$

For an arbitrary state-space realization of $Z(s)$, M may not be positive real. If M is positive real, it is, as we shall see, very easy to synthesize a network N_r with impedance matrix M. So if a realization of $Z(s)$ can be found such that M is positive real, then it follows that a synthesis of $Z(s)$ is given by N, as shown in Fig. 8.2.5, where the impedance matrix of N_r exists, being that of (8.2.11), and is synthesizable.

In summary, *the problem of giving a nonreciprocal synthesis for a rational positive real $Z(s)$ via the reactance-extraction technique is essentially this: among the infinitely many realizations of $Z(s)$ with $Z(\infty)$ assumed to have finite entries, find one realization $\{F, G, H, J\}$ of $Z(s)$ such that M of (8.2.11) is positive real, or equivalently, such that (8.2.7) holds.*

Of course, if $Z(s)$ is not positive real, there is certainly no possibility of the existence of such a realization $\{F, G, H, J\}$ of $Z(s)$.

We shall now consider the problem of giving a reciprocal synthesis for a symmetric rational positive real $Z(s)$ via the reactance-extraction approach.

Reactance Extraction—Reciprocal Networks

When a synthesis of a *symmetric* rational positive real $Z(s)$ is required to be reciprocal (i.e., no gyrators may be used in the synthesis), the element replacements of Fig. 8.2.4 used for the nonreciprocal networks are not applicable, and we return to consider Fig. 8.2.3 as our proposed scheme for a reciprocal synthesis of $Z(s)$. Of course, N_r must be reciprocal if the network N synthesizing $Z(s)$ is to be reciprocal.

Suppose that N_r is described by an $(m + n) \times (m + n)$ *hybrid matrix* M in which the excitations at the first $m + n_1$ ports of N_r are currents and those at the remaining n_2 ports are voltages. Let M be partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{8.2.12}$$

where M_{11} is $m \times m$ and M_{22} is $n \times n$. Because of the lack of energy storage elements, M is constant. Because N_r is reciprocal, M satisfies

$$[I_m \dotplus I_{n_1} \dotplus (-1)I_{n_2}]M = \{[I_m \dotplus I_{n_1} \dotplus (-1)I_{n_2}]M\}' \tag{8.2.13}$$

and because N_r is passive, M satisfies

$$M + M' \geq 0 \tag{8.2.14}$$

Now noting that the hybrid matrix of N_l is simply sI_n with excitation variables of N_l appropriately defined, it is straightforward to see that the impedance observed at the input m ports of N_r when loaded in N_l is

$$Z(s) = M_{11} - M_{12}(sI_n + M_{22})^{-1}M_{21} \tag{8.2.15}$$

Examination of (8.2.15) reveals clearly that one state-space realization of $Z(s)$ is given by

$$\{F, G, H, J\} = \{-M_{22}, M_{21}, -M'_{12}, M_{11}\} \tag{8.2.16}$$

Conversely, a *possible* hybrid matrix M of N_r is

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} \tag{8.2.17}$$

where $\{F, G, H, J\}$ is *any* realization of $Z(s)$. Further, if a realization of $Z(s)$ can be found such that M is positive real (i.e., $M + M' \geq 0$) and $(I_m \dotplus \Sigma)M$ is symmetric with Σ an $n \times n$ diagonal matrix consisting of $+1$ and -1 diagonal entries only, then it follows, provided M can be synthesized to yield N_r, that a reciprocal synthesis of $Z(s)$ is given as shown in Fig. 8.2.3, in which the hybrid matrix for N_r is that of (8.2.17). As we shall see, the problem of synthesizing a constant hybrid matrix satisfying the passivity and reciprocity constraints is easy.

In summary, the problem of giving a reciprocal network synthesis for a symmetric positive real impedance matrix $Z(s)$ via the reactance-extraction technique is essentially equivalent to the problem of finding a realization $\{F, G, H, J\}$ among the infinitely many realizations of $Z(s)$, with $Z(\infty)$ assumed to have finite entries, such that M of (8.2.17) satisfies (8.2.13) and (8.2.14).

We can conclude from the resistance- and reactance-extraction problems posed in this section that the latter technique, that of reactance extraction, is the more natural approach to solving the synthesis problem from the state-space point of view. This is so because the constant impedance or hybrid matrix of the nondynamic network N_r (see Fig. 8.2.3) resulting from extraction of all reactive elements is closely associated through (8.2.11) or (8.2.17) with the four constant matrices F, G, H, and J, which constitute a state-space realization of the prescribed $Z(s)$. On the other hand, an appropriate description for the lossless network N_L resulting from extraction of all resistors (see Fig. 8.2.1) does not reveal any close association to F, G, H, and J in an obvious manner.

Two additional points are worth noting:

1. The idea of reactance extraction applies readily to the problem of hybrid-matrix synthesis. Problem 8.2.2 asks the reader to illustrate this point.
2. Another important property that the reactance-extraction approach possesses, but the resistance-extraction approach does not, is its ready application to the lossless network synthesis problem. From the reactance-extraction point of view, the lossless synthesis problem is essentially equivalent to the problem of synthesizing a constant hybrid (or immittance) matrix that satisfies the losslessness and (in the reciprocal-network case) the reciprocity constraints. This problem, as will be seen subsequently, is an extremely easy one. The resistance-extraction approach, on the other hand, essentially avoids consideration of lossless synthesis, assuming this can be carried out by classical procedures or by a state-space procedure such as a reactance-extraction synthesis.

Problem 8.2.1 Synthesize the lossless positive real impedance matrix sL, where L is nonnegative definite symmetric. Show that if syntheses of sL and $Z_0(s)$ are available, a synthesis of $sL + Z_0(s)$ is easily achieved. (*Hint:* If L is nonnegative definite symmetric, there exists a matrix T such that $L = T'T$.)

Problem 8.2.2 Suppose that $\mathcal{H}(s)$ is an $m \times m$ rational positive real hybrid matrix with a state-space realization $\{F, G, H, J\}$. Formulate the reactance-extraction synthesis problem for $\mathcal{H}(s)$ for nonreciprocal and reciprocal cases, and derive the conditions required of $\{F, G, H, J\}$ for synthesis.

Problem 8.2.3 Verify Eq. (8.2.15).

8.3 THE BASIC SCATTERING MATRIX SYNTHESIS PROBLEM

In this section we shall discuss the state-space scattering matrix synthesis problem in terms of the reactance-extraction method and the resistance-extraction method. As with the impedance synthesis problem, matrices appearing in the descriptions of a nondynamic coupling network arising in the reactance-extraction method exhibit close links with a state-space realization $\{F, G, H, J\}$ of the prescribed rational bounded real scattering matrix $S(s)$. The material dealing with the reactance-extraction method is based on [2], and that dealing with the resistance-extraction method is drawn from [4].

The Reactance-Extraction Problem

As before, we regard an m port N synthesizing a rational bounded real $S(s)$ as an interconnection of two subnetworks N_r and N_l, as shown in Fig. 8.2.3, except that we now have $S(s)$ as the scattering matrix observed at the left-hand m terminals of N_r, rather than $Z(s)$ as the impedance matrix. The subnetwork N_r is a nondynamic $(m + n)$ port, while the n-port subnetwork N_l consists of n_1 inductors and n_2 capacitors with $n_1 + n_2 = n$. If gyrators are allowed, then we may arrange for N_l to consist of n inductors, and Fig. 8.2.5 applies in the latter case.

Reactance Extraction—Reciprocal Networks. For the moment we wish to consider the reciprocal case in which N_r contains no gyrators. In contrast to the impedance synthesis problem, we shall allow arbitrary choice of the values of the inductors and capacitors of N_l. Assume that the n_1 inductors and n_2 capacitors are $L_1, L_2, \ldots, L_{n_1}$ henries and $C_1, C_2, \ldots, C_{n_2}$ farads. Suppose that N_r is described by a scattering matrix S_a; then because of the reciprocity, lack of energy storage elements, and passivity of N_r, S_a is symmetric and constant, and the bounded real condition becomes simply

$$I - S_a^2 \geq 0 \tag{8.3.1}$$

Of course, S_a is normalized to $+1$ at each of the m input ports (i.e., the left-hand m ports in Fig. 8.2.3) of N_r. The normalization numbers for the output (right-hand side) n ports have yet to be specified. (A different set of numbers results in a different describing matrix S_a for the same network N_r.) A useful choice we shall make is

$$r_{m+l} = \begin{cases} L_l & l = 1, 2, \ldots, n_1 \\ \dfrac{1}{C_l} & l = n_1 + 1, \ldots, n \end{cases} \tag{8.3.2}$$

We shall also write down the scattering matrix of N_l; the normalizing number of each port will be the same as the normalizing number at the corresponding port of N_r.

Now with normalization number L_l the reactance sL_l possesses a scattering coefficient of

$$\frac{sL_l - L_l}{sL_l + L_l} = \frac{s-1}{s+1}$$

for $l = 1, 2, \ldots, n_1$, while with normalization number $1/C_l$ the reactance $1/sC_l$ possesses a scattering coefficient of

$$\frac{(1/sC_l) - (1/C_l)}{(1/sC_l) + (1/C_l)} = \frac{1-s}{1+s}$$

$l = n_1 + 1, \ldots, n$. Evidently the scattering matrix of N_l is

$$B(s) = \frac{s-1}{s+1}\Sigma \tag{8.3.3}$$

where

$$\Sigma = I_{n_1} \dotplus (-1)I_{n_2} \tag{8.3.4}$$

If the resulting S_a having the above normalization numbers is partitioned like the ports of N_r as

$$S_a = \begin{bmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{bmatrix} \tag{8.3.5}$$

where $S_{11} = S'_{11}$ is $m \times m$ and $S_{22} = S'_{22}$ is $n \times n$, then the termination of N_r in N_l results in N with the prescribed matrix $S(s)$.

The calculation of $S(s)$ from S_a and $B(s)$ has been done in an earlier chapter. The result is

$$\begin{aligned} S(s) &= S_{11} + S_{12}[B^{-1}(s) - S_{22}]^{-1}S'_{12} \\ &= S_{11} + S_{12}\left(\frac{s+1}{s-1}\Sigma - S_{22}\right)^{-1}S'_{12} \\ &= S_{11} + S_{12}(pI - \Sigma S_{22})^{-1}\Sigma S'_{12} \end{aligned} \tag{8.3.6}$$

with

$$p = \frac{s+1}{s-1} \quad \text{or} \quad s = \frac{p+1}{p-1} \tag{8.3.7}$$

Therefore, we can view (8.3.7) as a change of complex variables, so that the function $S(\cdot)$ of the variable s may naturally be regarded as a function $W(\cdot)$ of the variable p, via

$$S(s) = S\left(\frac{p+1}{p-1}\right) = W(p) \tag{8.3.8}$$

It follows then that

$$W(p) = S_{11} + S_{12}(pI - \Sigma S_{22})^{-1}\Sigma S'_{12} \tag{8.3.9}$$

In summary, if we have available a network N synthesizing $S(s)$, this determines a state-space realization of

$$W(p) = S\left(\frac{p+1}{p-1}\right)$$

via (8.3.9), where the S_{ij} are defined [see (8.3.5)] by partitioning of the scattering matrix of the nondynamic part of N; further, S_{11} and S_{22} are symmetric and Eq. (8.3.1) holds.

Conversely, we may state the reciprocal passive synthesis problem as follows. *First, from $S(s)$ form $W(p)$ according to the change of variables (8.3.7), and construct a state-space realization of $W(p)$ such that quantities S_{11}, S_{12}, and S_{22} are defined via (8.3.9) by this realization, and S_a in (8.3.5) is symmetric and satisfies the passivity condition (8.3.1). Second, synthesize S_a with a nondynamic reciprocal network.* (This step turns out to be straightforward.)

Let us now rephrase this problem slightly. On examination of (8.3.9), we can see that *given* N and S_a, one possible state-space realization for $W(p)$ is

$$\{F_w, G_w, H_w, J_w\} = \{\Sigma S_{22}, \Sigma S'_{12}, S'_{12}, S_{11}\}$$

Therefore

$$S_a = [I_m + \Sigma]M \tag{8.3.10}$$

where

$$M = \begin{bmatrix} J_w & H'_w \\ G_w & F_w \end{bmatrix} \tag{8.3.11}$$

Note that the quadruple $\{F_w, G_w, H_w, J_w\}$ obtained from the scattering matrix S_a of N, via (8.3.10), although a state-space realization of $W(p)$, is not (except in a chance situation) a state-space realization $\{F, G, H, J\}$ of the prescribed $S(s)$. However, one quadruple may always be obtained from the other via a set of invertible relations; we shall state explicitly these relations in a later chapter.

Considering again the synthesis problem, we see that we can state the symmetry condition and the passivity condition (8.3.1) on S_a directly in terms of M as follows:

$$(I_m + \Sigma)M = M'(I_m + \Sigma) \tag{8.3.12}$$

and

$$I - M'M \geq 0 \tag{8.3.13}$$

This allows us to restate the passive reciprocal scattering matrix synthesis problem.

1. *Form a real rational $W(p)$ from the prescribed rational bounded real $S(s)$ via a change of variable defined in (8.3.7), and*
2. *Derive a state-space realization $\{F_w, G_w, H_w, J_w\}$ of $W(p)$ such that with the definition (8.3.11), both (8.3.12) and (8.3.13) are fulfilled.*

As we shall show later, once 1 and 2 have been done, synthesis is easy.

Reactance Extraction—Nonreciprocal Networks. We now turn our attention to the nonreciprocal synthesis case in which gyrators are acceptable as circuit elements. Suppose that a synthesis is available in this case; all reactances may be assumed inductive, since any capacitor may be replaced by a gyrator terminated in an inductor, as shown in Fig. 8.2.4, and the gyrator part may be then regarded as part of the nondynamic network N_r. By choosing the normalization numbers for the n inductively terminated ports of N_r according to (8.3.2), this then determines a real constant scattering matrix S_a for N_r, which is a nonsymmetric matrix and satisfies the passivity condition (8.3.1). If we partition S_a similarly to the ports of N_r as

$$S_a = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{8.3.14}$$

and proceed as for the reciprocal case, we obtain, instead of (8.3.9),

$$W(p) = S_{11} + S_{12}(pI - S_{22})^{-1}S_{21} \tag{8.3.15}$$

Thus, knowing N and S_a, we can write down one state-space realization for $W(p)$ as

$$\{F_w, G_w, H_w, J_w\} = \{S_{22}, S_{21}, S'_{12}, S_{11}\}$$

This may be rewritten as

$$S_a = M = \begin{bmatrix} J_w & H'_w \\ G_w & F_w \end{bmatrix} \tag{8.3.16}$$

and the passivity condition of (8.3.1) in terms of M is therefore

$$I - M'M \geq 0 \tag{8.3.13}$$

The argument is reversible as for the reciprocal case. Hence we conclude that to solve the nonreciprocal scattering matrix synthesis problem via reactance extraction, we need to do the following things:

1. *Form a real rational $W(p)$ from the given rational bounded real $S(s)$ via a change of variable (8.3.7), and*
2. *Find a state-space realization $\{F_w, G_w, H_w, J_w\}$ for $W(p)$ such that with the definition of (8.3.16), the passivity condition (8.3.13) is satisfied.*

Once S_a is known, synthesis is, as we shall see, very straightforward. Therefore, we do not bother to list a further step demanding passage from S_a to a network synthesizing it.

An important application of the reactance-extraction technique is to the problem of *lossless* scattering matrix synthesis. The nondynamic coupling network N_r is lossless, and accordingly its scattering matrix is constant and orthogonal. The lossless synthesis problem is therefore essentially equivalent to one of synthesizing a constant orthogonal scattering matrix that is also symmetric in the reciprocal-network case. This problem turns out to be particularly straightforward.

The Resistance-Extraction Problem

Consider an arrangement of an m port N synthesizing a prescribed rational bounded real $S(s)$, as shown in Fig. 8.2.1, save that the scattering matrix $S(s)$ [rather than an impedance matrix $Z(s)$] is observed at the left-hand ports of N_L. Here a p-port subnetwork N_l consisting of p uncoupled unit resistors loads a lossless $(m + p)$-port coupling network N_L to yield N. Clearly, the total number of resistors employed in the synthesis of $S(s)$ is p. Let $S_L(s)$ be the scattering matrix of N_L, partitioned as the ports

$$S_L(s) = \begin{bmatrix} S_{11}(s) & S_{12}(s) \\ S_{21}(s) & S_{22}(s) \end{bmatrix} \tag{8.3.17}$$

where $S_{11}(s)$ is $m \times m$ and $S_{22}(s)$ is $p \times p$. The network N_l is described by a constant scattering matrix $S_l = 0_p$, and it is known that the cascade-load connection of N_l and N_L yields a scattering matrix $S(s)$ viewed from the unterminated m ports that is given by

$$S(s) = S_{11}(s) \tag{8.3.18}$$

Therefore, the top left $m \times m$ submatrix of $S_L(s)$ of N_L must necessarily be $S(s)$. Since N_L is lossless, $S_L(s)$ necessarily satisfies the lossless bounded real requirement $S'_L(-s)S_L(s) = I$.

We shall digress from the main problem temporarily to note an important result that specifies the minimum number of resistors necessary to physically construct a finite passive m port. It is easy to see that the requirement $S'_L(-s)S_L(s) = I$ forces, among others, the equation

$$I - S'_{11}(-s)S_{11}(s) = I - S'(-s)S(s) = S'_{21}(-s)S_{21}(s) \tag{8.3.19}$$

Since $S_{21}(s)$ is an $m \times p$ matrix, $S'_{21}(-s)S_{21}(s)$ has normal rank no larger than p. It follows therefore that the rank ρ of the matrix $I - S'(-s)S(s)$, called the *resistivity matrix* and designated by $\mathcal{R}(s)$, is at most equal to the number of resistors in a synthesis of $S(s)$. Thus we may state the above result in the following theorem:

> **Theorem 8.3.1.** [4]: The number of resistors p in a finite passive synthesis of $S(s)$ is bounded below by the normal rank ρ of the resistivity matrix $\mathcal{R}(s) = I - S'(-s)S(s)$; i.e., $p \geq \rho$.

As we shall see in a later chapter, we can synthesize $S(s)$ with exactly ρ resistors.

The result of Theorem 8.3.1 can also be stated in terms of the impedance matrix when it exists, as follows (a proof is requested in the problems):

> **Corollary 8.3.1.** The number of resistors in a passive network synthesis of a prescribed impedance matrix $Z(s)$ is bounded below by the normal rank ρ of the matrix $Z(s) + Z'(-s)$.

Let us now return to the mainstream of the argument, but now considering the synthesis problem. The idea is to split the synthesis problem into two parts: first, to obtain from a prescribed $S(s)$ (by augmentation of it with p rows and columns) a lossless scattering matrix $S_L(s)$—here, p is no less than the rank ρ of the resistivity matrix; second, to obtain a network N_L synthesizing $S_L(s)$. If these two steps are executed, it follows that termination of the appropriate p ports of N_L will lead to $S(s)$ being observed at the remaining m ports.

Since the second step turns out to be easy, we shall now confine discussion to the first step, and translate the idea of the first step into state-space terms. Suppose that the lossless $(m + p) \times (m + p)$ $S_L(s)$ has a minimal state-space realization $\{F_L, G_L, H_L, J_L\}$, where F_L is $n \times n$. (Note that the dimensions of the other matrices are automatically fixed.)

Let us partition G_L, H_L, and J_L as

$$G_L = [G_{L1} \quad G_{L2}] \qquad H_L = [H_{L1} \quad H_{L2}] \qquad J_L = \begin{bmatrix} J_{L1} & J_{L2} \\ J_{L3} & J_{L4} \end{bmatrix} \tag{8.3.20}$$

where G_{L1} and H_{L1} have m columns, and J_{L1} has m columns and rows. Straightforward calculations show that

$$\begin{aligned} S_L(s) &= J_L + H'_L(sI - F_L)^{-1}G_L \\ &= \begin{bmatrix} J_{L1} + H'_{L1}(sI - F_L)^{-1}G_{L1} & J_{L2} + H'_{L1}(sI - F_L)^{-1}G_{L2} \\ J_{L3} + H'_{L2}(sI - F_L)^{-1}G_{L1} & J_{L4} + H'_{L2}(sI - F_L)^{-1}G_{L2} \end{bmatrix} \end{aligned}$$

It follows therefore from (8.3.18) that $\{F_L, G_{L1}, H_{L1}, J_{L1}\}$ constitutes a state-space realization (not necessarily minimal) of $S(s)$. Also, the lossless property of N_L requires the quadruple $\{F_L, G_L, H_L, J_L\}$ to satisfy the following equations of the lossless bounded real lemma:

$$\begin{aligned} PF_L + F'_L P &= -H_L H'_L \\ -PG_L &= H_L J_L \\ J'_L J_L &= I \end{aligned} \tag{8.3.21}$$

where P is an $n \times n$ positive definite symmetric matrix.

Further, if a reciprocal synthesis N of a prescribed $S(s)$ is required [of course, one must have $S(s) = S'(s)$], N_L must also be reciprocal, and therefore it is necessary that $S_L(s) = S'_L(s)$, or that

$$J_L = J'_L \qquad H'_L(sI - F_L)^{-1}G_L = G'_L(sI - F'_L)^{-1}H_L \tag{8.3.22}$$

Leaving aside the problem of lossless scattering matrix synthesis, we conclude that *the scattering matrix synthesis problem via resistance extraction is equivalent to the problem of finding constant matrices F_L, G_L, H_L, and J_L such that*

1. *For nonreciprocal networks,*
   (a) *the conditions of the lossless bounded real lemma (8.3.21) are fulfilled, and*
   (b) *with the partitioning as in (8.3.20), $\{F_L, G_{L1}, H_{L1}, J_{L1}\}$ is a realization of the prescribed $S(s)$.*
2. *For reciprocal networks,*
   (a) *the conditions of the lossless bounded real lemma (8.3.21) and the conditions of the reciprocity property (8.3.22) are fulfilled, and*
   (b) *with the partitioning as in (8.3.20), $\{F_L, G_{L1}, H_{L1}, J_{L1}\}$ is a realization of the prescribed matrix $S(s)$.*

As with the impedance synthesis problem, once a desired lossless scattering matrix S_L has been found, the second step requiring a synthesis of S_L is straightforward and readily solved through an application of the reactance-extraction technique considered previously. This will be considered in detail later.

Problem 8.3.1 Comment on the significance of the change of variables

$$p = \frac{s+1}{s-1} \qquad s = \frac{p+1}{p-1}$$

and then give an interpretation of the (frequency-domain) bounded real properties of $S(s)$ in terms of the real rational

$$W(p) = S\left(\frac{p+1}{p-1}\right)$$

Does there exist a one-to-one transformation between a state-space realization of $S(s)$ and that of $W(p)$?

Problem 8.3.2 Give a proof of Corollary 8.3.1 using the result of Theorem 8.3.1.

8.4 PRELIMINARY SIMPLIFICATION BY LOSSLESS EXTRACTIONS

Before we proceed to consider synthesis procedures that answer the state-space synthesis problems for finite passive networks posed in the previous sections, we shall indicate simplifications to the synthesis task. These simplifications involve elementary operations on the matrix to be synthesized, which amount to extracting, in a way to be made precise, lossless sections composed of inductors, capacitors, transformers, and perhaps gyrators. The operations may be applied to a prescribed positive real or bounded real matrix prior to carrying out various synthesis methods yet to be discussed and, after their application, lead to a simpler synthesis problem than the original. (For those familiar with classical network synthesis, the procedure may be viewed as a partial parallel of the classical Foster preamble performed on a positive real function preceding such synthesis methods as the Brune synthesis and the Bott–Duffin synthesis [5]. Essentially, in the Foster preamble a series of reactance extractions is successively carried out until the residual positive real function is minimum reactive and minimum susceptive; i.e., the function has a strictly Hurwitz numerator and a strictly Hurwitz denominator.)

The reasons for using these preliminary simplification procedures are actually several. All are basically associated with reducing computational burdens.

First, with each lossless section extracted, the degree of a certain residual matrix is lowered by exactly the number of reactive elements extracted. This, in turn, results in any minimal state-space realization of the residual matrix possessing lower dimension than that of any minimal state-space realization of the original matrix. Now, after extraction, it is the residual matrix that has to be synthesized. Clearly, the computations to be performed on the constant matrices of a minimal realization of the residual matrix in a synthesis of this matrix will generally be less than in the case of the original matrix, though this reduction is, of course, at the expense of calculations necessary for synthesis of the lossless sections extracted. Note though that these calculations are generally particularly straightforward.

Second, as we shall subsequently see, in solving the state-space synthesis problems formulated in the previous two sections, it will be necessary to compute solutions of the positive real or the bounded real lemma equations. Various techniques for computation may be found earlier; recall that one method depends on solving an algebraic quadratic matrix inequality and another one on solving a Riccati differential equation. There is however a restriction that must be imposed in applying these methods: the matrix $Z(\infty) + Z'(\infty)$ or $I - S'(\infty)S(\infty)$ must be nonsingular. Now it is the lossless extraction process that enables this restriction to be met after the extraction process is carried out, even if the restriction is not met initially. Thus the extraction procedure enables the synthesis procedures yet to be given to be purely algebraic if so desired.

Third, it turns out that a version of the equivalent network problem, the problem of deriving *all* those networks with a minimal number of reactive elements that synthesize a prescribed positive real impedance or bounded real scattering matrix, is equivalent to deriving all triples P, L, and W_0 satisfying the equations of the positive real lemma or the bounded real lemma. The solution to the latter problem (and hence the equivalent network problem) hinges, as we have seen, on finding all solutions of a quadratic matrix inequality,* given the restriction that the matrix $Z(\infty) + Z'(\infty)$ or $I - S'(\infty)S(\infty)$ is nonsingular. So, essentially, the lossless extraction operations play a vital part in solving the minimal reactive element (nonreciprocal) equivalence problem.

There are also other less important reasons, which we shall try to point out as the occasion arises. But notice that none of the above reasons is so strong as to demand the execution of the procedure as a necessary preliminary to synthesis. *Indeed, and this is most important, save for an extraction operation corresponding to the removal of a pole at infinity of entries of a prescribed positive real matrix (impedance, admittance, or hybrid) that is to be synthesized, the procedures to be described are generally not necessary to, but merely beneficial for, synthesis.*

Simplifications for the Immittance Synthesis Problem

The major specific aim now is to show how we can reduce, by a sequence of lossless extractions, the problem of synthesizing an arbitrary prescribed positive real $Z(s)$, with $Z(\infty) < \infty$, to one of synthesizing a second positive real matrix, $\hat{Z}(s)$ say, for which $\hat{Z}(\infty) + \hat{Z}'(\infty)$ is nonsingular. We shall also show how $\hat{Z}(s)$ can sometimes be made to possess certain additional properties.

*The discussion of the quadratic matrix inequality appeared in Chapter 6, which may have been omitted at a first reading.

In particular, we shall see how the sequence of simple operations can reduce the problem of synthesizing $Z(s)$ to one of synthesizing $\hat{Z}(s)$ such that

1. $\hat{Z}(\infty) + \hat{Z}'(\infty)$ is nonsingular.
2. $\hat{Z}(\infty)$ is nonsingular.
3. A reciprocal synthesis of $Z(s)$ follows from a reciprocal synthesis of $\hat{Z}(s)$.
4. A synthesis of $Z(s)$ with the minimum possible number of resistors follows from one of $\hat{Z}(s)$ with a minimum possible number of resistors.
5. A synthesis of $Z(s)$ with the minimum possible number of reactive elements follows from one of $\hat{Z}(s)$ with a minimum possible number of reactive elements.

Some remarks are necessary here. We shall note that although the sequence of operations to be described is *potentially* capable of meeting any of the listed conditions, there are certainly cases in which one or more of the conditions represents a pointless goal. For example, it is clear that 3 is pointless when the prescribed $Z(s)$ is not symmetric to start with. Also, it is a technical fact established in [1] that in general there exists *no* reciprocal synthesis that has both the minimum possible number of resistors of any synthesis (reciprocal or nonreciprocal), and simultaneously the minimum possible number of reactive elements of any synthesis (again, reciprocal or nonreciprocal). Thus it is not generally sensible to adopt 3, 4, and 5 simultaneously as goals, though the adoption of any two turns out to be satisfactory.

Since reciprocity of a network implies symmetry of its impedance or admittance matrix, property 3 is equivalent to saying that

3a. $\hat{Z}(s)$ is symmetric if $Z(s)$ is symmetric.

[There is an appropriate adjustment in case $Z(s)$ is a hybrid matrix.] Further, in order to establish 4, we need only to show that

4a. Normal rank $[Z(s) + Z'(-s)]$ = normal rank $[\hat{Z}(s) + \hat{Z}'(-s)]$.

The full justification for this statement will be given subsequently, when we show that there exist syntheses of $Z(s)$ using a number of resistors equal to p = normal rank $[Z(s) + Z'(-s)]$. Since we showed in the last section that the number of resistors in a synthesis cannot be less than p, it follows that a synthesis using p resistors is one using the minimum possible number of resistors.

In the sequel we shall consider four classes of operations. For convenience, we shall consider the application of these only to impedance or admittance matrices and leave the hybrid-matrix case to the reader. For ease of notation, $Z(s)$, perhaps with a subscript, will denote both an impedance and admittance matrix. Following discussion of the four classes of operation, we shall be able to note quickly how the restrictions 1–5 may be obtained. The four classes of operations are

1. Reduction of the problem of synthesizing $Z(s)$, singular throughout the s plane, to the problem of synthesizing $Z_1(s)$, nonsingular almost everywhere.
2. Reduction of the problem of synthesizing $Z(s)$ with $Z(\infty)$ singular to the problem of synthesizing $Z_1(s)$ with some elements possessing a pole at infinity.
3. Reduction of the problem of synthesizing $Z(s)$ with some elements possessing a pole at infinity to the problem of synthesizing $Z_1(s)$ with $\delta[Z_1(s)] < \delta[Z(s)]$ and with no element of $Z_1(s)$ possessing a pole at infinity.
4. Reduction of the problem of synthesizing $Z(s)$ with $Z(\infty)$ nonsymmetric to the problem of synthesizing $Z_1(s)$ with $Z_1(\infty)$ symmetric.

1. Singular $Z(s)$. Suppose that $Z(s)$ is positive real and singular. We shall suppose that $Z(\infty) < \infty$; if not, operation 3 can be carried out. Then the problem of synthesizing $Z(s)$ can be modified by using the following theorem:

> **Theorem 8.4.1.** Let $Z(s)$ be an $m \times m$ positive real matrix with $Z(\infty) < \infty$ and with minimal realization $\{F, G, H, J\}$. Suppose that $Z(s)$ is singular throughout the s plane, possessing rank $m' < m$. Then there exist a constant nonsingular matrix T and an $m' \times m'$ positive real matrix $Z_1(s)$, nonsingular almost everywhere, such that
>
> $$Z(s) = T'[Z_1(s) \dotplus 0_{m-m',m-m'}]T \tag{8.4.1}$$
>
> The matrix $Z_1(s)$ has a minimal realization $\{F_1, G_1, H_1, J_1\}$, where
>
> $$\begin{aligned} F_1 &= F \qquad G_1 = GT^{-1}[I_{m'} \quad 0_{m',m-m'}]' \\ H_1 &= HT^{-1}[I_{m'} \quad 0_{m',m-m'}]' \\ J_1 &= [I_{m'} \quad 0_{m',m-m'}](T')^{-1}JT^{-1}[I_{m'} \quad 0_{m',m-m'}]' \end{aligned} \tag{8.4.2}$$
>
> Moreover, $\delta[Z(s)] = \delta[Z_1(s)]$, $Z_1(s)$ is symmetric if and only if $Z(s)$ is symmetric, and normal rank $[Z(s) + Z'(-s)] =$ normal rank $[Z_1(s) + Z_1'(-s)]$.

For the proof of this theorem up to and including Eq. (8.4.2), see Problem 8.4.1. (The result is actually a standard one of network theory and is derived in [1] by frequency-domain arguments.) The claims of the theorem following Eq. (8.4.2) are immediate consequences of Eq. (8.4.1).

The significance of this theorem lies in the fact that if a synthesis of $Z_1(s)$ is on hand, a synthesis of $Z(s)$ follows by appropriate use of a multiport trans-

former associated with the synthesis of $Z_1(s)$. If $Z_1(s)$ and $Z(s)$ are impedances, one may take a transformer of turns-ratio matrix T and terminate its first m' secondary ports in a network N_1 synthesizing $Z_1(s)$ and its last $m - m'$ ports in short circuits. The impedance $Z(s)$ will be observed at the input ports. Alternatively, if $\tilde{T}$ denotes T with its last $m - m'$ rows deleted, a transformer of turns-ratio matrix $\tilde{T}$ terminated at its m' secondary ports in a network N_1 synthesizing $Z_1(s)$ will provide the same effect (see Fig. 8.4.1). For admittances, the situation is almost the same—one merely interchanges the roles of the primary and secondary ports of the transformer.

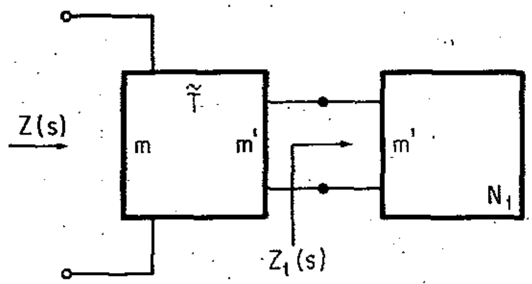

FIGURE 8.4.1. Conversion of a Singular $Z(s)$ to a Nonsingular $Z_1(s)$.

2. Singular $Z(\infty)$, *Nonsingular* $Z(s)$. We now suppose that $Z(s)$ is nonsingular almost everywhere, but singular at $s = \infty$. [Again, operation 3 can be carried out if necessary to ensure that $Z(\infty) < \infty$.] We shall make use of the following theorem.

Theorem 8.4.2. Let $Z(s)$ be an $m \times m$ positive real matrix with $Z(\infty) < \infty$, $Z(s)$ nonsingular almost everywhere, and $Z(\infty)$ singular. Then $Z_1(s) = [Z(s)]^{-1}$ is positive real and has elements possessing a pole at infinity. Moreover, $\delta[Z(s)] = \delta[Z_1(s)]$, $Z_1(s)$ is symmetric if and only if $Z(s)$ is symmetric, and normal rank $[Z(s) + Z'(-s)] =$ normal rank $[Z_1(s) + Z_1'(-s)]$.

Proof. That $Z_1(s)$ is positive real because $Z(s)$ is positive real is a standard result, which may be proved in outline form as follows. Let $S(s) = [Z(s) - I][Z(s) + I]^{-1}$; then $S(s)$ is bounded real. It follows that $S_1(s) = -S(s)$ is bounded real, and thus that $Z_1(s) = [I + S_1(s)][I - S_1(s)]^{-1}$ is positive real. Expansion of this argument is requested in Problem 8.4.2.

The definition of $Z_1(s)$ implies that $Z_1(s)Z(s) = I$; setting $s = \infty$ and using the singularity of $Z(\infty)$ yields the fact that some elements of $Z_1(s)$ possess a pole at infinity.

The degree property is standard (see Chapter 3) and the symmetry property is obvious. The normal-rank property follows from the easily established relation

$$Z_1(s) + Z_1'(-s) = Z^{-1}(s)[Z(s) + Z'(-s)][Z'(-s)]^{-1} \qquad \nabla \nabla \nabla$$

This theorem really only becomes significant because it may provide the conditions necessary for the application of operation 3. Notice too that the theorem is really a theorem concerning admittance and impedance matrices that are inverses of one another. It suggests that *the problem of synthesizing an impedance $Z(s)$ with $Z(\infty) < \infty$ and $Z(\infty)$ singular may be viewed as the problem of synthesizing an admittance $Z_1(s)$ with $Z_1(\infty) < \infty$ failing, and with various other relations between $Z(s)$ and $Z_1(s)$.*

3. Removal of Pole at Infinity. Suppose that $Z(s)$ is positive real, with some elements possessing a pole at infinity. A decomposition of $Z(s)$ is possible, as described in the following theorem.

Theorem 8.4.3. Let $Z(s)$ be an $m \times m$ positive real matrix, with $Z(\infty) < \infty$ failing. Then there exists a nonnegative definite symmetric L and a positive real $Z_1(s)$, with $Z_1(\infty) < \infty$, such that

$$Z(s) = sL + Z_1(s) \tag{8.4.3}$$

Moreover, $\delta[Z(s)] = \text{rank } L + \delta[Z_1(s)]$, $Z_1(s)$ is symmetric if and only if $Z(s)$ is symmetric, and normal rank $[Z(s) + Z'(-s)] =$ normal rank $[Z_1(s) + Z_1'(-s)]$.

We have established the decomposition (8.4.3) in Chapter 5. The remaining claims of the theorem are straightforward.

The importance of this theorem in synthesis is as follows. First, *observe that sL is simply synthesizable*, for the nonnegativity of L guarantees that a matrix T_l' exists with m rows and $l = \text{rank } L$ columns, such that

$$L = T_l' T_l$$

If sL is an impedance, the network of Fig. 8.4.2, showing unit inductors terminating the secondary ports of a transformer of turns-ratio matrix T_l, yields a synthesis of sL. If sL is an admittance, the transformer is reversed and terminated in unit capacitors, as shown in Fig. 8.4.3.

With a synthesis of an admittance or impedance sL achievable so easily, it follows that *a synthesis of $Z(s)$ can be achieved from a synthesis of $Z_1(s)$ by, in the impedance case, series connecting the synthesis of $Z_1(s)$ and the synthesis of sL, and, in the admittance case, paralleling two such syntheses.*

Since $\delta[sL] = \text{rank } L$, the syntheses we have given of sL, both impedance

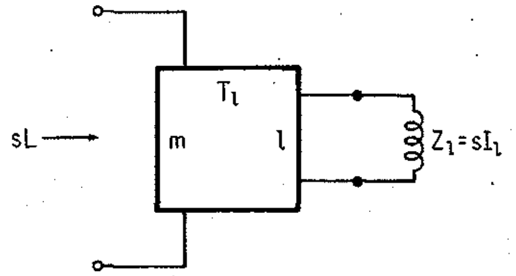

FIGURE 8.4.2. Synthesis of Impedance sL.

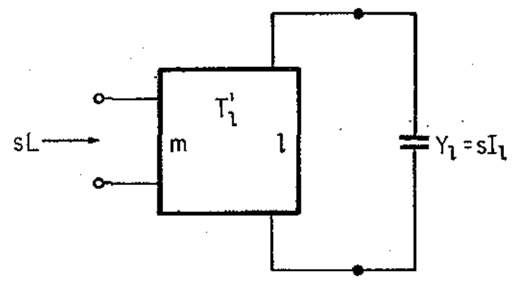

FIGURE 8.4.3. Synthesis of Admittance sL.

and admittance, use $\delta[sL]$ reactive elements—actually the minimum number possible. *Hence if $Z_1(s)$ is synthesized using the minimal number of reactive elements, the same is true of $Z(s)$.*

4. $Z(\infty)$ Nonsymmetric. The synthesis reduction depends on the following result.

> **Theorem 8.4.4.** Let $Z(s)$ be positive real, with $Z(\infty) < \infty$ and nonsymmetric. Then the matrix $Z_1(s)$ defined by
>
> $$Z(s) = Z_1(s) + \tfrac{1}{2}[Z(\infty) - Z'(\infty)] \tag{8.4.4}$$
>
> is positive real, has $Z_1(\infty) < \infty$, and $Z_1(\infty) = Z_1'(\infty)$. Moreover, $\delta[Z(s)] = \delta[Z_1(s)]$ and normal rank $[Z(s) + Z'(-s)] =$ normal rank $[Z_1(s) + Z_1'(-s)]$.

The straightforward proof of this theorem is called for in the problems.

The significance for synthesis is that a synthesis of $Z(s)$ follows from a series connection in the impedance case, or parallel connection in the admittance case, of syntheses of $Z_1(s)$ and of $\frac{1}{2}[Z(\infty) - Z'(\infty)]$. The latter is easily synthesized, as we now indicate. The matrix $\frac{1}{2}[Z(\infty) - Z'(\infty)]$ is skew symmetric; if $Z(\infty)$ is $m \times m$ and $\frac{1}{2}[Z(\infty) - Z'(\infty)]$ has rank $2g$, evenness of the rank being guaranteed by the skew symmetry [6], it follows that there exists a real $2g \times m$ matrix T_g, readily computable, such that

$$\tfrac{1}{2}[Z(\infty) - Z'(\infty)] = T'_g[E \dotplus E \dotplus \cdots \dotplus E]T_g \tag{8.4.5}$$

where

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

in the case when $Z(s)$ is an impedance, and

$$E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

in the case when $Z(s)$ is an admittance, and there are g summands in the direct sum on the right side of (8.4.5). The first E is the impedance matrix and the second E the admittance matrix of the gyrator with gyration resistance of 1 Ω. Therefore, appropriate use of a transformer of turns ratio T_g in the impedance case and T'_g in the admittance case, terminated in g unit gyrators, will lead to a synthesis of $\frac{1}{2}[Z(\infty) - Z'(\infty)]$. See Fig. 8.4.4 for an illustration of the impedance case.

Our discussion of the four operations considered separately is complete. Observe that each operation consists of mathematical operations on the immittance matrices, coupled with the physical operations of lossless element extraction; all classes of lossless elements, transformers, inductors, capacitors, and gyrators arise.

We shall now observe the effect of combining the four operations. Figure 8.4.5 should be studied. It shows in flow-diagram form the sequence in which the operations are performed. Notice that there is one major loop. By Theorem 8.4.3, each time this loop is traversed, degree reduction in the residual immittance to be synthesized occurs. Since the degree of the initially given immittance is finite and any degree is positive, looping has to eventually end. From Fig. 8.4.5 it is clear that the immittance finally obtained, call it $\hat{Z}(s)$, will have $\hat{Z}(\infty)$ finite, $\hat{Z}(\infty) = \hat{Z}'(\infty)$, and $\hat{Z}(\infty)$ nonsingular. As a consequence, $\hat{Z}(\infty) + \hat{Z}'(\infty)$ is nonsingular. Notice too, using Theorems 8.4.1 through 8.4.4, that $\hat{Z}(s)$ will be symmetric if $Z(s)$ is symmetric, and normal rank $[Z(s) + Z'(-s)] =$ normal rank $[\hat{Z}(s) + \hat{Z}'(-s)]$. Finally, the only step at which degree reduction of the residual immittances occurs is in the looping procedure. By the remarks following Theorem 8.4.3, it follows that

$$\begin{aligned} \delta[Z(s)] &= \delta[\hat{Z}(s)] + \textstyle\sum \text{(degree of inductor or capacitor element extractions)} \\ &= \delta[\hat{Z}(s)] + \text{(number of reactive elements extracted)} \end{aligned}$$

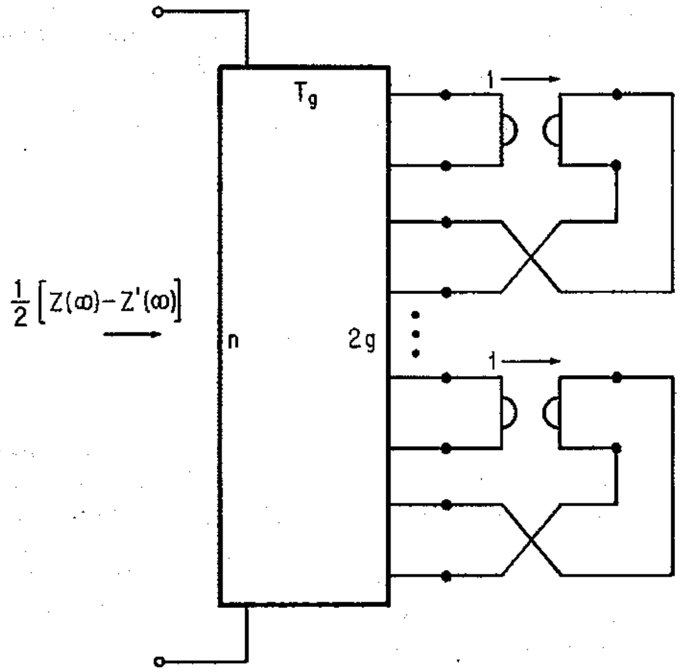

FIGURE 8.4.4. Synthesis for Impedance $\frac{1}{2}[Z(\infty) - Z'(\infty)]$.

Since, as we remarked in Section 8.1, the minimum number of reactive elements in any synthesis of $Z(s)$ is $\delta[Z(s)]$, it follows that a synthesis of $Z(s)$ using the minimal number of reactive elements will follow from a minimal reactive element synthesis of $\hat{Z}(s)$.

Example 8.4.1 Consider the impedance matrix

$$Z(s) = \begin{bmatrix} s + \frac{1}{s+1} & \frac{1}{s+1} + 1 \\ -1 + \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Adopting the procedure of the flow chart, we would first write

$$Z(s) = s\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + Z_1(s)$$

where

$$Z_1(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} + 1 \\ -1 + \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

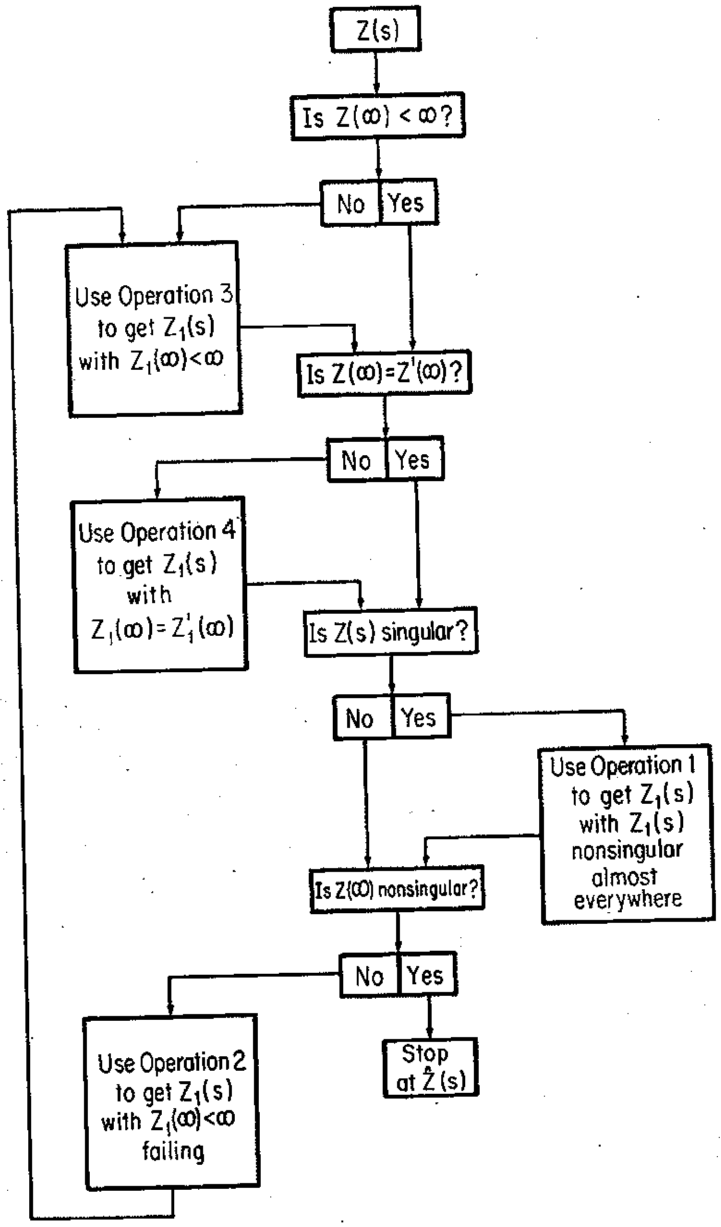

FIGURE 8.4.5. Flow Diagram Depicting Preliminary Lossless Extractions for Immittance Matrix Problem.

Next, we observe that $Z_1(\infty) \neq Z_1'(\infty)$, and so we write

$$Z_1(s) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + Z_2(s)$$

where

$$Z_2(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Now we observe that $Z_2(s)$ is singular. We write it as

$$Z_2(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{s+1} [1 \quad 1]$$

and take

$$Z_3(s) = \frac{1}{s+1}$$

We see that $Z_3(\infty)$ is singular, and thus we form

$$Z_4(s) = [Z_3]^{-1} = s + 1$$

Finally, we take $Z_5(s) = 1$, and at this point the procedure stops. In sequence, we have extracted a series inductor, extracted a series gyrator, represented the remaining impedance by a transformer terminated in a one-port network with a certain impedance, converted this impedance to an admittance, and represented the admittance as a parallel combination of a capacitor and resistor. Although in this case application of the "preliminary" simplifications led to a complete synthesis, this is not the case in general.

There is one additional point we shall make here. Suppose that the positive real matrix $\hat{Z}(s)$ deduced from $Z(s)$ possesses one or more elements with pure imaginary poles. If desired, these poles may be removed, thereby simplifying $\hat{Z}(s)$ and resulting in another positive real impedance, $\tilde{Z}(s)$ say, the elements of which are free of imaginary poles. More precisely, as indicated in Section 5.1, $\hat{Z}(s)$ is expressed, using a partial fraction decomposition, as a sum of two positive real impedances, one with elements possessing purely imaginary poles—this being $Z_L(s)$, and one with elements possessing no purely imaginary pole together with $\hat{Z}(\infty)$—this being $\tilde{Z}(s)$. Thus $\hat{Z}(s) = Z_L(s) + \tilde{Z}(s)$. The matrix $Z_L(s)$ is, of course, lossless positive real. Synthesis procedures using a minimal number, $\delta[Z_L(s)]$, of reactive elements are available (see [1] for classical approaches and later sections in this book for a state-space approach).

Because $\hat{Z}(\infty)$ is symmetric and nonsingular, $\tilde{Z}(\infty) = \hat{Z}(\infty)$ is also. Moreover, because entries of $Z_L(\cdot)$ and $\tilde{Z}(\cdot)$ can have no common poles,

$$\delta[\hat{Z}(s)] = \delta[Z_L(s)] + \delta[\tilde{Z}(s)]$$

which implies that a minimal reactive element synthesis of $\hat{Z}(s)$ follows from a series connection of minimal reactive element syntheses of $Z_L(s)$ and $\tilde{Z}(s)$. Since a minimal reactive element synthesis of $Z_L(s)$ always exists, as remarked above, it follows therefore that the problem of giving a minimal reactive synthesis for $\hat{Z}(s)$ is equivalent to that of giving one for $\tilde{Z}(s)$ also with a minimal number of reactive elements. Note also that $Z_L(s) + Z'_L(-s) = 0$ for all s, and therefore $\hat{Z}(s) + \hat{Z}'(-s) = \tilde{Z}(s) + \tilde{Z}'(-s)$ and has normal rank equal to the normal rank of $Z(s) + Z'(-s)$. Finally, symmetry of $\hat{Z}(s)$ is readily found to guarantee symmetry of $Z_L(s)$ and $\tilde{Z}(s)$.

Therefore, assuming synthesis of lossless $Z_L(s)$ is possible, we conclude that the problem of synthesizing an arbitrary positive real $Z(s)$ may even be reduced to the problem of synthesizing another positive real $\tilde{Z}(s)$, with $\tilde{Z}(s)$ satisfying conditions 1 to 5 on page 349 [with $\hat{Z}(s)$ replaced by $\tilde{Z}(s)$], and also

6. No element of $\tilde{Z}(s)$ possesses a purely imaginary pole.

Simplification for the Scattering Matrix Synthesis Problem

In this subsection we shall use the sort of lossless extractions just considered to show how the problem of synthesizing an arbitrary prescribed bounded real $S(s)$ may be reduced to one of synthesizing another bounded real $\hat{S}(s)$ such that

1. $I - \hat{S}'(\infty)\hat{S}(\infty)$ is nonsingular.
2. A reciprocal synthesis of $S(s)$ follows from a reciprocal synthesis of $\hat{S}(s)$.
3. A synthesis of $S(s)$ with the minimum possible number of resistors follows from a synthesis of $\hat{S}(s)$ with a minimum possible number of resistors.
4. A synthesis of $S(s)$ with the minimum possible number of reactive elements follows from one of $\hat{S}(s)$ with a minimum possible number of reactive elements.

Since the necessary and sufficient condition for a network to be reciprocal is that its scattering matrix be symmetric, and since the minimum possible number of resistors necessary to give a synthesis of a bounded real $S(s)$ is given by normal rank $[I - S'(-s)S(s)]$ (which, as we shall see, is actually achievable in practice), it is clear that 2 and 3 are in fact equivalent to

2a. $\hat{S}(s)$ is symmetric if $S(s)$ is symmetric.

3a. Normal rank $[I - \hat{S}'(-s)\hat{S}(s)] =$ normal rank $[I - S'(-s)S(s)]$.

Similarly, we note that the same sort of remarks as those made in the immittance-matrix case may be made; i.e., although the sequence of operations to be specified is capable of fulfilling any of the conditions listed above, one or more of them may become a pointless objective in a particular case.

Below, three classes of operations will be introduced. Then we shall be able to note quickly how the listed restrictions 1–4 may be obtained using these three classes in conjunction with the other four considered previously for the immittance matrices.

The three classes of operations are

5. Reduction of the problem of synthesizing $S(s)$ with $S(\infty)$ nonsymmetric to the problem of synthesizing $S_1(s)$ with $S_1(\infty)$ symmetric.
6. Reduction of the problem of synthesizing $S(s)$ with $I - S(s)$ singular throughout the s plane to the problem of synthesizing a matrix $S_1(s)$ of smaller dimensions with $I - S_1(s)$ nonsingular almost everywhere.
7. Reduction of the problem of synthesizing $S(s)$ with $I - S(s)$ nonsingular almost everywhere to the problem of synthesizing a positive real impedance $Z(s)$, and the reverse process of reduction of the problem of synthesizing a positive real immittance $Z(s)$ to the problem of synthesizing a bounded real $S(s)$.

5. $S(\infty)$ Nonsymmetric. The synthesis problem may be modified by using the following result:

Theorem 8.4.5. Let $S(s)$ be an $m \times m$ rational bounded real matrix with minimal realization $\{F, G, H, J\}$. Suppose that J is nonsymmetric. Then there exists an $m \times m$ real orthogonal matrix U and an $m \times m$ bounded real matrix $S_1(s)$ with $J_1 = S_1(\infty)$ symmetric, such that

$$S(s) = US_1(s) \tag{8.4.6}$$

The matrix $S_1(s)$ has a minimal realization $\{F_1, G_1, H_1, J_1\}$, where

$$F_1 = F \qquad G_1 = G \qquad H_1 = HU \qquad J_1 = U'J \tag{8.4.7}$$

Moreover, $\delta[S(s)] = \delta[S_1(s)]$, and normal rank $[I - S'(-s)S(s)]$ $=$ normal rank $[I - S_1'(-s)S_1(s)]$.

The proof of this theorem (see Problem 8.4.5) is straightforward and simply depends on a well-established result that for a given real constant matrix J, there exist a real orthogonal U and a real constant symmetric matrix J_1 such that $J = UJ_1$.

The significance of the above result for synthesis is that a synthesis of $S(s)$ follows from syntheses of U and $S_1(s)$ via a coupling network N_c of m uncoupled gyrators, as shown in Fig. 8.4.6. The scattering matrix U, being a real orthogonal matrix, is bounded real and easily synthesizable using transformer-coupled gyrators, open circuits, and short circuits. Full details justifying the scheme of Fig. 8.4.6 and the synthesis of U are requested in Problems 8.4.6 and 8.4.7.

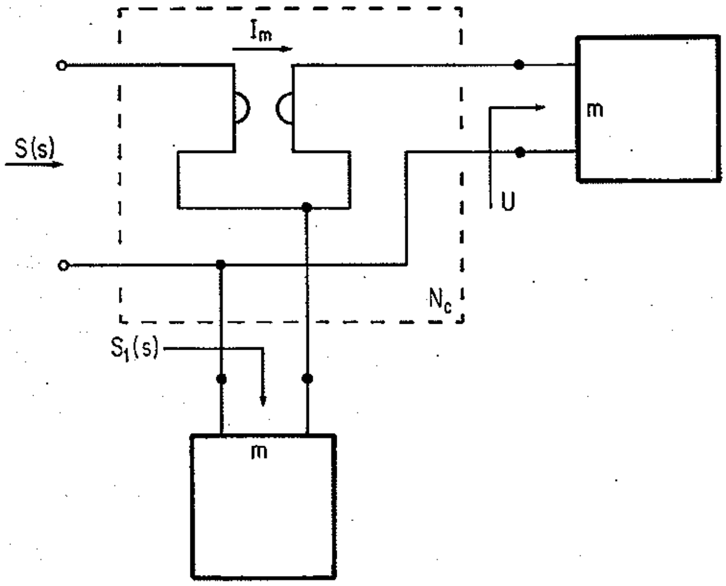

FIGURE 8.4.6. Connection for Scattering-Matrix Multiplication.

6. Singular $I - S(s)$. Suppose that $S(s)$ is bounded real and $I - S(s)$ singular throughout the s plane. Then the synthesis simplification depends on the following theorem:

Theorem 8.4.6. Let $S(s)$ be an $m \times m$ bounded real matrix with minimal realization $\{F, G, H, J\}$. Suppose that $I - S(s)$ is singular throughout the s plane and has rank $m' < m$. Then there exist a real *orthogonal* matrix T and an $m' \times m'$ bounded real matrix $S_1(s)$ with $I - S_1(s)$ nonsingular almost everywhere, such that

$$S(s) \mp T[S_1(s) + I_{m-m'}]T' \qquad (8.4.8)$$

The matrix $S_1(s)$ has a minimal realization $\{F_1, G_1, H_1, J_1\}$, where

$$F_1 = F \qquad G_1 = GT[I_{m'} \quad 0_{m',m-m'}]'$$
$$H_1 = HT[I_{m'} \quad 0_{m',m-m'}]' \qquad J_1 = [I_{m'} \quad 0_{m',m-m'}]T'JT[I_{m'} \quad 0_{m',m-m'}]' \tag{8.4.9}$$

Furthermore, $\delta[S(s)] = \delta[S_1(s)]$, $S_1(s)$ is symmetric if and only if $S(s)$ is symmetric, $S_1(\infty)$ is symmetric if and only if $S(\infty)$ is symmetric, and normal rank $[I - S'(-s)S(s)] =$ normal rank $[I - S_1'(-s)S_1(s)]$.

The result of the theorem is actually a standard one of network theory and is established in [1] by frequency-domain arguments. The theorem may also be proved using the arguments used to establish Theorem 8.4.1 (see Problem 8.4.1).

The significance of this theorem for synthesis lies in the fact that if a synthesis of $S_1(s)$ is on hand, a synthesis of $S(s)$ follows by appropriate use of an orthogonal transformer of turns-ratio matrix T. Now a scattering matrix $I_{m-m'}$ represents $m - m'$ open circuits, so one may take a transformer with the turns-ratio matrix $\bar{T}$ resulting from T with its last $m - m'$ columns deleted and terminate its m' primary ports in a network N_1 synthesizing $S_1(s)$, to yield a synthesis of $S(s)$ at the unterminated ports (see Fig. 8.4.7).

7. Conversion of $S(s)$ to $Z(s)$ and Conversely. Suppose that $S(s)$ is bounded real, with $I - S(s)$ nonsingular almost everywhere. Then the problem of synthesizing $S(s)$ can be modified to an equivalent problem of synthesizing a positive real impedance matrix $Z(s)$, as contained in the following result:

Theorem 8.4.7. Let $S(s)$ be an $m \times m$ bounded real matrix, with $I - S(s)$ nonsingular almost everywhere. Then

$$Z(s) = [I + S(s)][I - S(s)]^{-1} \tag{8.4.10}$$

is positive real. Conversely, if $Z(s)$ is positive real, then

$$S(s) = [Z(s) + I]^{-1}[Z(s) - I] \tag{8.4.11}$$

always exists and is bounded real. Moreover, $\delta[S(s)] = \delta[Z(s)]$, $Z(s)$ is symmetric if and only if $S(s)$ is symmetric, $Z(\infty)$ is symmetric if and only if $S(\infty)$ is symmetric, and normal rank $[I - S'(-s)S(s)] =$ normal rank $[Z(s) + Z'(-s)]$.

The results of the theorem have all been demonstrated earlier.

The main application of this theorem is as follows. As illustrated in the immittance case, the operations arising in the preliminary synthesis steps are

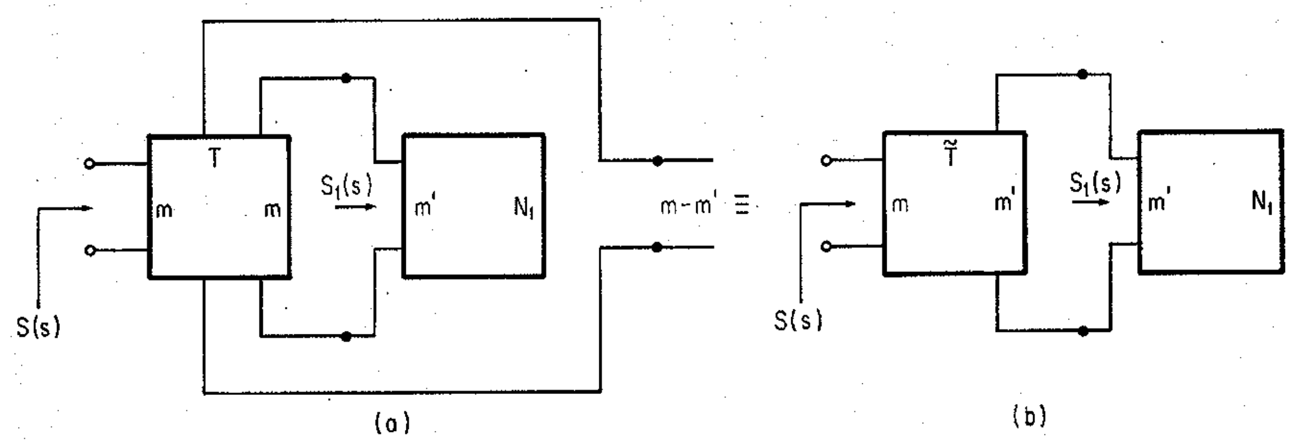

FIGURE 8.4.7. Conversion of $S(s)$ to $S_1(s)$.

likely in some cases to involve extractions of transformer-coupled inductors and transformer-coupled capacitors (or even transformer-coupled tuned circuits if desired). These operations are simpler by far to define in terms of immittance matrices.

Observe that operations 5 and 6 consist of mathematical operations on the scattering matrices which are associated with lossless element extractions involving transformers and gyrators only. However, through operation 7 coupled with those operations considered in the immittance-matrix case, all classes of lossless elements can be involved in simplifications to the scattering matrix synthesis problem.

We shall now observe the result of combining all the operations considered in this section. Figure 8.4.8 together with Fig. 8.4.5 is to be used for this purpose, and it shows in flow-diagram form the sequence in which the operations are performed. Notice that there is still only one major loop in the sequence. By Theorems 8.4.3 and 8.4.7, each time this loop is traversed, degree reduction in the residual matrix to be synthesized occurs and, as explained previously, the sequence of operations has to eventually end. Now at the end of the looping (if looping is necessary) the immittance finally obtained, call it $\hat{Z}(s)$, will have $\hat{Z}(\infty)$ finite and nonsingular, and $\hat{Z}(\infty) = \hat{Z}'(\infty)$. Therefore, the corresponding bounded real matrix, $\hat{S}(s)$ say, will have $\hat{S}(\infty) = \hat{S}'(\infty)$, $I - S(\infty)$ and $I + S(\infty)$ nonsingular, *and thus* $I - \hat{S}'(\infty)\hat{S}(\infty)$ *nonsingular*. Notice then (using Theorems 8.4.1 through 8.4.7 and the line of argument as applied to the immittance-matrix case) that $S(s)$ will satisfy conditions 2 through 4 on page 358.

Application to Lossless Synthesis

Before we conclude the section we wish to make the following point. The preliminary simplification procedures that have just been described, though basically intended to help in the synthesis of an arbitrary positive real or bounded real matrix, provide a technique for the complete synthesis of a lossless positive real or bounded real matrix. To see why, consider in more detail the case of a lossless positive real matrix. At no stage in the application of the preliminary procedures can a positive real matrix arise that is not lossless, and therefore at no stage will a matrix $\hat{Z}(s)$ be arrived at for which $\hat{Z}(\infty)$ is symmetric and nonsingular. But as reference to Fig. 8.4.5 shows, the preliminary extraction procedure will not terminate unless $\hat{Z}(\infty)$ is symmetric and nonsingular, unless of course $\hat{Z}(s) = 0$, i.e., unless a complete synthesis is achieved. In the case of a lossless immittance, the resulting synthesis resembles that obtained via the *first Cauer synthesis* (see [1] for a detailed discussion).

Example 8.4.2 Consider the lossless positive real impedance matrix

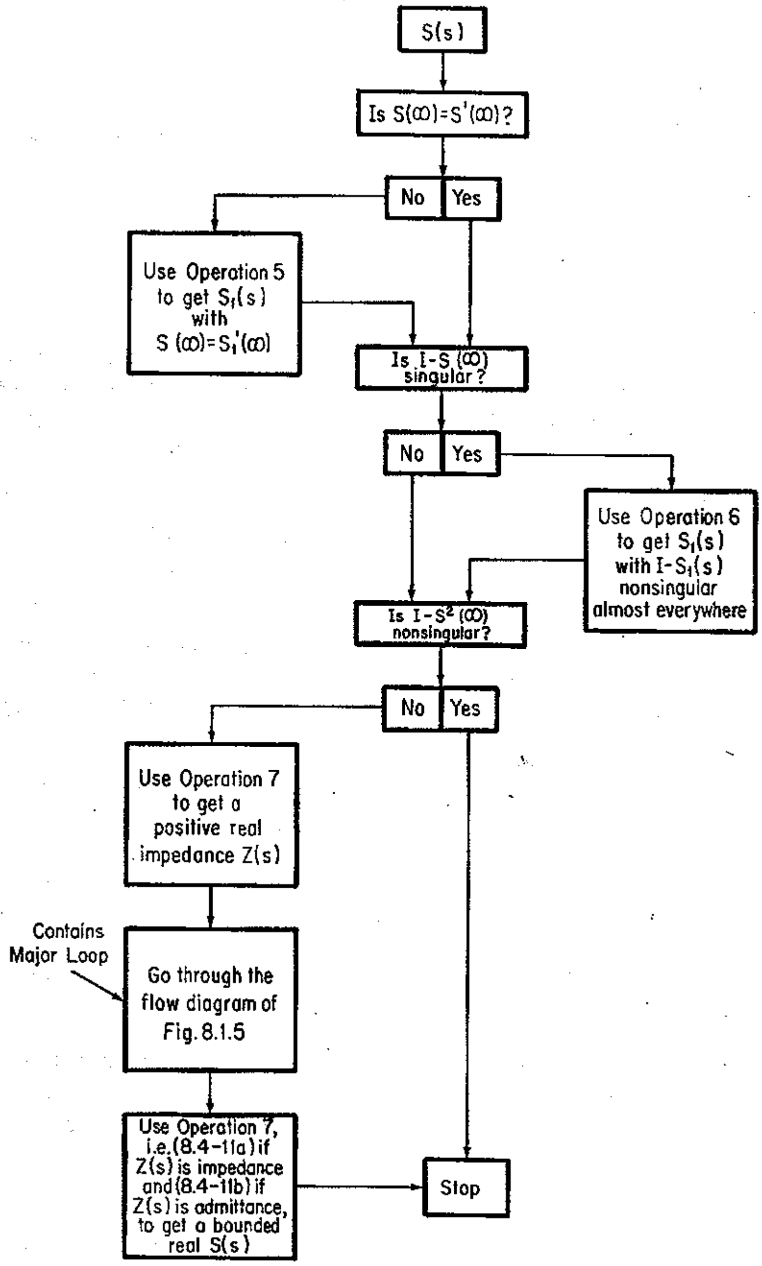

FIGURE 8.4.8. Flow Diagram Depicting Preliminary Lossless Extractions for Scattering Matrix Problem.

$$Z(s) = \begin{bmatrix} \dfrac{2s^2+2}{s} & \dfrac{s+1}{s} \\ \dfrac{-s+1}{s} & \dfrac{1}{s} \end{bmatrix}$$

First we separate out those elements of $Z(s)$ that possess a pole at infinity (operation 3). Thus

$$Z(s) = \begin{bmatrix} 2s & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \dfrac{2}{s} & \dfrac{s+1}{s} \\ \dfrac{-s+1}{s} & \dfrac{1}{s} \end{bmatrix}$$

The first term is readily synthesized (merely an inductor for port 1 and a short circuit for port 2). Next, the remaining positive real impedance $Z_1(s)$, with $Z_1(\infty) < \infty$, is examined to see if $Z_1(\infty) = Z_1'(\infty)$. The answer is no, so we apply operation 4 and write

$$Z_1(s) = \begin{bmatrix} \dfrac{2}{s} & \dfrac{1}{s} \\ \dfrac{1}{s} & \dfrac{1}{s} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

for which a gyrator may be extracted synthesizing the second term. Now the remaining positive real matrix $Z_2(s)$ is nonsingular almost everywhere, but $Z_2(\infty)$ is singular. We then invert $Z_2(s)$ (operation 2) to give a positive real admittance matrix

$$Y_2(s) = \begin{bmatrix} s & -s \\ -s & 2s \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

which evidently has elements with a pole at ∞, and these may be synthesized (operation 3) using capacitors terminating a multiport transformer. Since $Y_2(s)$ consists of only elements with a pole at ∞, the remaining admittance is simply a zero matrix; hence the synthesis of $Z(s)$ is completed and is depicted in Fig. 8.4.9.

Later we shall be considering state-space syntheses of lossless positive real and lossless bounded real matrices that are simple to carry out and are able to avoid the matrix inversion operations contained in the preceding procedures.

Problem 8.4.1 Prove Theorem 8.4.1. First write down the positive real lemma equations. Show that if w is a constant vector in the nullspace of $Z(s)$, then $Gw = 0$ and $Jw = 0$. Conclude that $Hw = 0$. Then take T as a matrix whose last m' columns span the nullspace of $Z(s)$. In like manner, prove Theorem 8.4.6.

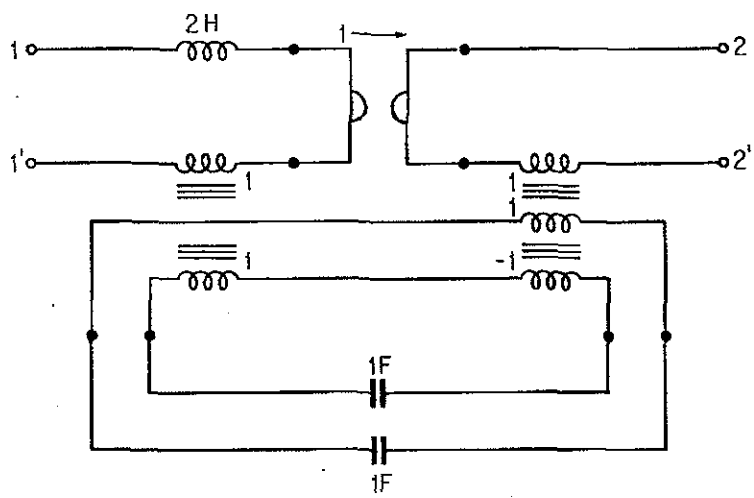

FIGURE 8.4.9. A Synthesis of $Z(s)$.

Problem 8.4.2 Prove that part of the statement of Theorem 8.4.2 which claims that $[Z(s)]^{-1}$ is positive real if $Z(s)$ is positive real.

Problem 8.4.3 Prove Theorem 8.4.4.

Problem 8.4.4 Consider a positive real impedance matrix

$$Z(s) = \begin{bmatrix} \dfrac{s+1}{s^2+2s+2} & \dfrac{\sqrt{2}(s+1)}{s^2+2s+2} \\ \dfrac{\sqrt{2}(s+1)}{s^2+2s+2} & \dfrac{2(s+1)}{s^2+2s+2} \end{bmatrix}$$

Show how to reduce $Z(s)$ to another positive real matrix $\hat{Z}(s)$ such that $\hat{Z}(\infty)$ is nonsingular. Then verify that the normal-rank property and the degree property as stated in the text hold for $Z(s)$ above.

Problem 8.4.5 Prove Theorem 8.4.5.

Problem 8.4.6 Consider the scheme of Fig. 8.4.10, which depicts an interconnection of m gyrators with an m-port network N_2 of scattering matrix S_2. Show that the resultant $2m$ port has scattering matrix

$$\begin{bmatrix} 0 & I \\ S_2 & 0 \end{bmatrix}$$

using the fact that the scattering matrix of the unit gyrator is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

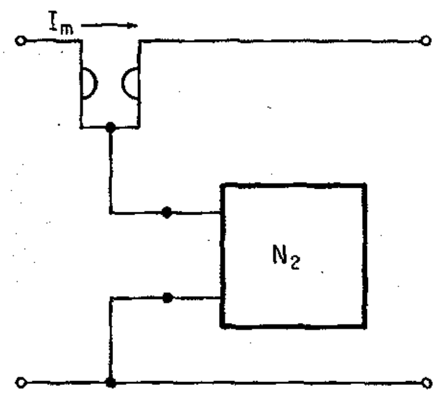

FIGURE 8.4.10. Building Block for the Network of Figure 8.4-11.

Conclude that the scattering matrix of the scheme of Fig. 8.4.11 is S_1S_2, where N_1 has scattering matrix S_l. (The cascade load formula may be helpful here.)

Problem 8.4.7 In this problem you will need to recall the fact that an arbitrary orthogonal matrix U can be written as

$$U = T'VT$$

where T is orthogonal and V is a direct sum of blocks of the form [1], [−1], and $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $\theta \neq 0$. Show that the scattering matrix of a gyrator with gyrator resistance γ ohms is

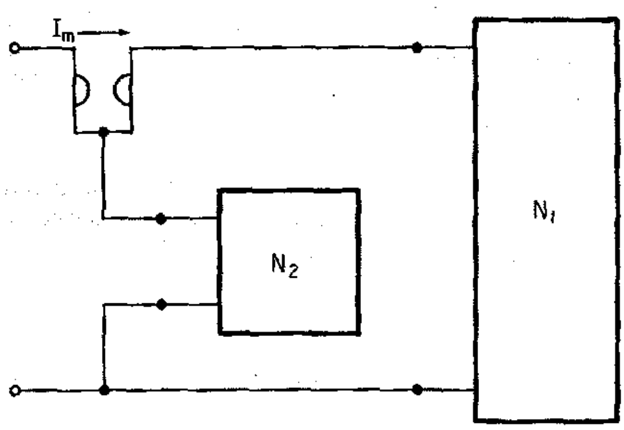

FIGURE 8.4.11. Network for Problem 8.4.6.

$$\begin{bmatrix} \frac{\gamma^2 - 1}{\gamma^2 + 1} & \frac{2\gamma}{\gamma^2 + 1} \\ \frac{-2\gamma}{\gamma^2 + 1} & \frac{\gamma^2 - 1}{\gamma^2 + 1} \end{bmatrix}$$

and then show that a scattering matrix that is a real constant orthogonal matrix can be synthesized by appropriately terminating a transformer in gyrators.

Problem 8.4.8 Synthesize the lossless bounded real matrix

$$S(s) = \frac{1}{10s + 7}\begin{bmatrix} -1 & -10s + 4\sqrt{3} \\ 10s + 4\sqrt{3} & 1 \end{bmatrix}$$

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9

Impedance Synthesis

9.1 INTRODUCTION

Earlier we formulated two separate approaches to the problem of synthesizing a positive real $m \times m$ impedance matrix $Z(s)$; one is based on the reactance-extraction concept; the other is based on the resistance-extraction concept. We shall devote most of this chapter to giving synthesis procedures using both approaches yielding *nonreciprocal networks*, in which gyrators are included as circuit elements.

Although synthesis procedures that yield a nonreciprocal network with a prescribed *nonsymmetric* positive real impedance matrix can naturally be applied to obtain a synthesis of a *symmetric* positive real impedance matrix (since the latter can be regarded as a special case of the former), one objection is that gyrators generally arise if a synthesis method for a nonsymmetric $Z(s)$ is applied to a symmetric $Z(s)$—despite the fact that reciprocal (gyratorless) networks synthesizing a prescribed symmetric positive real $Z(s)$ can (and will) be shown to exist. The nonreciprocal synthesis procedures are therefore often unsuitable for the synthesis of a symmetric positive real $Z(s)$, and other methods are desired which always lead to a reciprocal synthesis. Section 9.4 and much of Chapter 10 will provide some reciprocal syntheses of symmetric positive real impedance matrices.

We now ask the reader to note the following points concerning the material in this chapter. First, we shall assume throughout that $Z(s)$ is such that $Z(\infty) < \infty$. As noted earlier, this restriction can always be satisfied with no loss of generality; for if a prescribed $Z_0(s)$ does not possess this property, one can always remove the pole at infinity by a series extraction of transformer-coupled inductors, and the problem of synthesizing the original $Z_0(s)$ is then equivalent to the problem of synthesizing the remaining positive real $Z(s)$ satisfying the restriction $Z(\infty) < \infty$.

Second, recall that the formulations of various approaches to synthesis were posed in terms of a state-space realization $\{F, G, H, J\}$ of $Z(s)$, sometimes with no specific reference to the size of the associated state vector. The synthesis procedures to be considered will, however, be stated *in terms of a particular minimal realization of* $Z(s)$. The reason becomes apparent if one observes that the algebraic characterization of the positive real property of $Z(s)$ in state-space terms, as contained in the positive real lemma, applies only to minimal realizations of $Z(s)$. That is, the existence is guaranteed of a positive definite symmetric P and real matrices L and W_0 that satisfy the positive real lemma equations only if $\{F, G, H, J\}$ constitutes a *minimal* realization of the positive real $Z(s)$. Since the synthesis of $Z(s)$ in state-space terms depends on the algebraic characterization of its positive real property, the above restriction naturally follows. This in one sense provides an advantage over some classical synthesis methods in that a synthesis of $Z(s)$ is always assured that uses the minimum number of reactive elements, this number being precisely $\delta[Z]$ or the size of the F matrix in a minimal realization.

Third, we shall even assume knowledge of a minimal realization $\{F, G, H, J\}$ of $Z(s)$ such that

$$\begin{aligned} F + F' &= -LL' \\ G &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{9.1.1}$$

for some real matrices L and W_0. The existence of the minimal realization $\{F, G, H, J\}$ and matrices L and W_0 satisfying (9.1.1) is guaranteed by the following theorem—a development of the positive real lemma:

Theorem 9.1.1. Let $Z(s)$ be a rational positive real matrix with $Z(\infty) < \infty$. Let $\{F_0, G_0, H_0, J\}$ be a minimal realization and $\{P, L_0, W_0\}$ be a triple that satisfies the positive real lemma equations for the above minimal realization. With T any matrix ranging over the set of matrices for which

$$T'T = P \tag{9.1.2}$$

then the minimal realization $\{F, G, H, J\}$, with $F = TF_0T^{-1}$, $G = TG_0$, and $H = (T^{-1})'H_0$, satisfies Eq. (9.1.1), where $L = (T^{-1})'L_0$.

The proof of the above result follows immediately on invoking the positive real lemma equations and on performing a few simple calculations. Note that F, G, H, L, and W_0 are certainly not unique: if $\{F_0, G_0, H_0, J\}$ is an arbitrary minimal realization of $Z(s)$, we know from Chapter 6* that there exists an infinity of matrices P satisfying the positive real lemma equations, and for any one P an infinity of L and W_0 satisfying the same equations. Further, for any one P there exists an infinity of matrices T satisfying (9.1.2). So the nonuniqueness arises on several counts. Even the dimensions of L and W_0 are not unique, for, as noted earlier, the number of rows of L'_0 and W_0 (and thus of L' and W_0) is bounded below by $\rho =$ normal rank $[Z(s) + Z'(-s)]$, but is otherwise arbitrary (see also Problem 9.1.1).

In the following synthesis procedures, except for the lossless synthesis, we shall take (9.1.1) as the starting point. In the case of the synthesis of a lossless impedance, we know from previous considerations that the matrices L and W_0 are zero and (9.1.1) therefore is replaced by

$$\begin{aligned} F + F' &= 0 \\ G &= H \\ J + J' &= 0 \end{aligned} \tag{9.1.3}$$

As we know from our discussions on the computation of solutions to the positive real lemma equations, a minimal realization of a lossless positive real $Z(s)$ satisfying (9.1.3) is much easier to derive from an arbitrary minimal realization than is a minimal realization of a nonlossless or lossy positive real $Z(s)$ satisfying (9.1.1).

An outline of this chapter is as follows.

In Section 9.2 we consider the synthesis of nonreciprocal networks using the reactance-extraction approach. The procedure includes as a special case the synthesis of lossless nonreciprocal networks. Since the latter can be achieved with much less computational effort than the synthesis of lossy nonreciprocal networks, and since lossless networks play a major role in network theory and applications, the lossless synthesis is therefore considered in a separate subsection. The material in Section 9.2 is drawn from [1] and [2].

In Section 9.3 a synthesis procedure for nonreciprocal networks using the resistance-extraction approach is discussed. This material is drawn from [3]. Finally, Section 9.4 is devoted to considering a straightforward synthesis procedure for a prescribed symmetric positive real $Z(s)$ that results in a reciprocal network. The material of this section is drawn from [4] and [5].

*This chapter may have been omitted at a first reading.

Problem 9.1.1 Show that if F, G, H, J are such that (9.1.1) holds for some pair of matrices L and W_0, then it holds for L and W_0 replaced by LV and $V'W_0$ for any V, not necessarily square, with $VV' = I$. Show also that the minimum number of rows in L' and W_0 is ρ = normal rank $[Z(s) + Z'(-s)]$, and that, given the existence of L' and W_0 with ρ rows, there exist other L' and W_0 with any number of rows greater than ρ. (Existence of L' and W_0 with ρ rows is established in Chapters 5 and 6.)

9.2 REACTANCE-EXTRACTION SYNTHESIS

We recall that the problem of finding a passive structure synthesizing a prescribed $m \times m$ positive real $Z(s)$ via reactance extraction essentially rests on finding a realization $\{F_0, G_0, H_0, J\}$ for $Z(s)$, such that the real constant $(m + n) \times (m + n)$ matrix

$$M = \begin{bmatrix} J & -H_0' \\ G_0 & -F_0 \end{bmatrix} \tag{9.2.1}$$

has its symmetric part nonnegative definite:

$$M + M' \geq 0 \tag{9.2.2}$$

In other words, M is positive real. With n denoting the dimension of F, the constant positive real matrix M is the impedance of a nondynamic $(m + n)$-port coupling network N_c. On terminating N_c at its last n ports in 1-H inductors, a network N synthesizing the prescribed $Z(s)$ is obtained.

As is guaranteed by Theorem 9.1.1, we can find a minimal realization $\{F, G, H, J\}$ of $Z(s)$ and real matrices L and W_0, such that

$$\begin{aligned} F + F' &= -LL' \\ G &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{9.2.3}$$

The dimensions of the various matrices are automatically fixed except for the number of rows of both L' and W_0. These are arbitrary, save that the number must be at least ρ, the normal rank of $[Z(s) + Z'(-s)]$. As we have already noted, there are infinitely many minimal realizations $\{F, G, H, J\}$, which, with some L and W_0, satisfy (9.2.3). We shall show that any one of these provides a synthesizable M, i.e., a matrix M satisfying (9.2.2).

By direct calculation, we have

$$M + M' = \begin{bmatrix} J + J' & (G - H)' \\ G - H & -F - F' \end{bmatrix}$$

$$= \begin{bmatrix} W_0'W_0 & -W_0'L' \\ -LW_0 & LL' \end{bmatrix}$$

$$= \begin{bmatrix} W_0' \\ -L \end{bmatrix} [W_0 \quad -L'] \tag{9.2.4}$$

The first equality follows from the definition of M, the second from (9.2.3), and the last from an obvious factorization. Thus, *we conclude that*

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} \tag{9.2.5}$$

[*where* $\{F, G, H, J\}$ *is any one minimal realization of* $Z(s)$ *satisfying* (9.2.3)] *has the property that* M *is positive real, or equivalently that the symmetric part of* M *is nonnegative definite:*

$$M + M' \geq 0 \tag{9.2.2}$$

We have claimed earlier that if M satisfies this constraint, it is readily synthesizable. We shall now justify this claim. Note that a synthesis of $Z(s)$ follows immediately once a synthesis of M is known.

Synthesis of a Constant PR Impedance Matrix M

To realize a constant positive real impedance M, we make use of the simple identity

$$M = M_{sy} + M_{sk} \tag{9.2.6}$$

where M_{sy} denotes the symmetric part of M, i.e., $\frac{1}{2}(M + M')$, and M_{sk} denotes the skew-symmetric part of M, i.e., $\frac{1}{2}(M - M')$; then we use the immediately verifiable fact that M_{sy} and M_{sk} are *both positive* real impedances. As may also be easily checked, M_{sk} is even lossless. A synthesis of M is then obtained by a series connection of a synthesis N_1 of M_{sy} and a synthesis N_2 of M_{sk}, both with $m + n$ ports, as illustrated in Fig. 9.2.1. A synthesis of the skew impedance matrix M_{sk} is readily obtained, as seen earlier, on noting that M_{sk} may be expressed as

$$M_{sk} = T_g'[E \dotplus E \dotplus \cdots \dotplus E]T_g \tag{9.2.7}$$

where $E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and there are g summands in (9.2.7) with $2g =$ rank

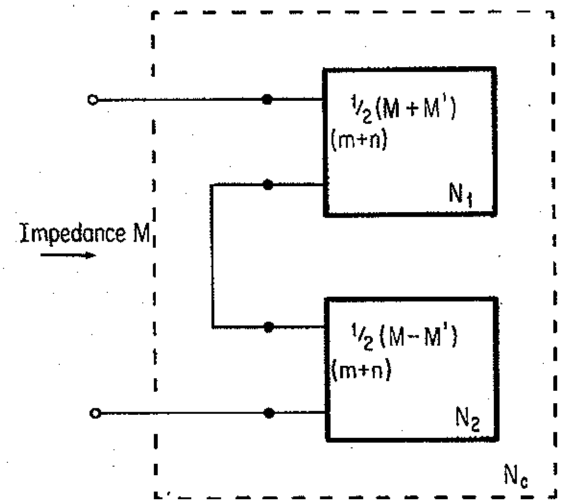


FIGURE 9.2.1. Series Connection of Impedances $\frac{1}{2}(M + M')$ and $\frac{1}{2}(M - M')$.

M_{sk}. Thus M_{sk} may be synthesized by a multiport transformer of turns-ratio matrix T_g terminated by g unit gyrators. Note that the synthesis of M_{sk} does not use any resistors, as predicted by the lossless (or skew) character of M_{sk}. To synthesize M_{sy}, we note from (9.2.4) that

$$M_{sy} = \tfrac{1}{2}(M + M') = \tfrac{1}{2}\begin{bmatrix} W_0' \\ -L \end{bmatrix} I_r[W_0 \quad -L'] \tag{9.2.8}$$

where r is the number of rows of L' and W_0. This equation implies that M_{sy} may be synthesized by terminating a multiport transformer of turns ratio $1/\sqrt{2}[W_0 \quad -L']$ in r unit resistors. Since the synthesis of M_{sk} does not use any resistors, it follows that the total number of resistors used in synthesizing N is equal to the number of rows r of the matrices L' and W_0 in (9.2.3).

We now wish to note a number of interesting properties of the synthesis procedure given above.

1. Since the number of reactive elements, in this case inductors, needed in the synthesis of $Z(s)$ is n, the dimension of F, and since F is part of a minimal state-space realization of $Z(s)$, the number of reactive elements used is $\delta[Z(\cdot)]$. We conclude that *the synthesis of $Z(s)$ achieved by this method always uses a minimal number of reactive elements.*
2. As the total number of resistors required in the synthesis of $Z(s)$ is the same as the number of rows of the matrices L' and W_0 in (9.2.3), it follows that the smallest number of resistors that can synthesize $Z(s)$

via the above method is equal to the lower bound on the number of rows of L' and W_0, and this number is, as we have noted, $\rho =$ normal rank $[Z(s) + Z'(-s)]$. As we have also noted in Chapter 8, no synthesis of $Z(s)$ is possible using fewer than ρ resistors. Hence we can always achieve a synthesis of $Z(s)$ using the minimum number ρ of resistors and *simultaneously* the minimum number of reactive elements, since *it is always possible to compute matrices L' and W_0 with ρ rows that* satisfy (9.2.3), as shown in Chapters 5 and 6.*

3. The synthesis procedure described requires only that $Z(s)$ be positive real; hence it always yields a passive synthesis for any positive real $Z(s)$, either symmetric or nonsymmetric. However, gyrators are almost always required even when $Z(s)$ is symmetric, and so *the method is generally unsuitable for synthesizing a symmetric positive real $Z(s)$ when a synthesis is required that uses no gyrators.*
4. Problem 9.2.4 requests a proof of the easily established fact that every (nonreciprocal) minimal reactive synthesis defines a particular minimal realization $\{F, G, H, J\}$ of $Z(s)$ together with real matrices L and W_0 that satisfy (9.2.3). Also, as we have just proved, each minimal realization $\{F, G, H, J\}$ with associated real matrices L and W_0, such that (9.2.3) hold, yields a minimal reactive synthesis. It follows that all possible minimal reactive element syntheses of $Z(s)$ are obtained, i.e., the minimal reactive element equivalence problem for nonreciprocal networks is solved, if all minimal realizations can be found that satisfy (9.2.3) together with real matrices L and W_0. How may this be done? Suppose that $\{F_1, G_1, H_1, J_1\}$ is any one minimal realization of $Z(s)$. From Theorem 9.1.1, it is clear that from each solution triple P, L_1, and W_0 of the positive real lemma equations associated with $\{F_1, G_1, H_1, J\}$, there arises a set of minimal realizations satisfying (9.2.3) for some real matrices L and W_0. Further, we know from Chapter 6* how to compute essentially *all* solution triples P, L_1, and W_0 for an arbitrary *initial minimal realization* $\{F_1, G_1, H_1, J\}$. It follows that we can derive all possible minimal realizations satisfying (9.2.3) from a particular $\{F_1, G_1, H_1, J\}$ by obtaining all matrices P, L_1, and W_0 and then all T such that $T'T = P$. This is also the set of all possible minimal realizations with the properties given in (9.2.3) that may be derived from any other initial minimal realization, and so from *all* arbitrary minimal realizations. To see this, suppose that $\{F_1, G_1, H_1, J\}$ and $\{F_2, G_2, H_2, J\}$ are two minimal realizations of the same $Z(s)$, with $\hat{T}$ a nonsingular matrix such that $\hat{T}F_2\hat{T}^{-1} = F_1$, etc.; if P_1, L_1, and W_0 constitute a solution of the positive real lemma corresponding to $\{F_1, G_1, H_1, J\}$, then $P_2 = \hat{T}'P_1\hat{T}$, $L_2 = \hat{T}'L_1$, and W_0 constitute a solution corresponding

*This material may have been omitted at a first reading.

to $\{F_2, G_2, H_2, J\}$. The factorization $P_1 = T'T$ leads to the same $\{F, G, H, J\}$ from $\{F_1, G_1, H_1, J\}$ as does the factorization $P_2 = (T\hat{T})'T\hat{T}$ from $\{F_2, G_2, H_2, J\}$.

Lossless-Impedance Synthesis

Suppose that we are given a lossless $m \times m$ impedance $Z(s)$. Recall that

$$Z(s) + Z'(-s) = 0 \tag{9.2.9}$$

By removing the pole at infinity of all elements of $Z(s)$, if any, we may consider a lossless $Z(s)$ with $Z(\infty) < \infty$. Note, however, and this is important, that *the simplification procedure involving lossless extraction considered in Chapter 8 should not be applied,* save for making $Z(\infty) < \infty$, since this would eventually completely synthesize a lossless $Z(s)$.

Let $\{F, G, H, J\}$ be a minimal state-space realization of $Z(s)$ with F an $n \times n$ matrix, easily derivable from an arbitrary minimal realization of $Z(s)$, such that

$$\begin{aligned} F + F' &= 0 \\ G &= H \\ J + J' &= 0 \end{aligned} \tag{9.2.10}$$

Of course, the lossless positive real lemma guarantees that such a quadruple $\{F, G, H, J\}$ always exists. Now the matrix

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} \tag{9.2.5}$$

is obviously *skew*, because of (9.2.10). As shown earlier, the real constant matrix M is the impedance, *lossless* in this case, of an $(m + n)$-port lossless nondynamic network N_c synthesizable with transformer-coupled gyrators. Termination of the last n ports of N_c with uncoupled 1-H inductors yields a lossless synthesis of $Z(s)$ that uses a minimum number n of reactive elements.

Once any minimal realization of a lossless $Z(s)$ has been found, synthesis is easy. We have described in Chapter 6* the straightforward calculations required to generate, from an arbitrary minimal realization, a minimal realization satisfying (9.2.10). As we have just seen, synthesis, given (9.2.10), is easy.

Classical synthesis procedures for lossless impedances, though not computationally difficult, may not be quite so straightforward as the state-space

*This chapter may have been omitted at a first reading.

procedure. For example, there is a partial fraction synthesis that generalizes the famous Foster reactance-function synthesis [6]. The method rests on the decomposition of $Z(s)$, as a consequence of its lossless positive real property, in the form

$$Z(s) = J + sL + s^{-1}C + \sum_i \frac{sA_i + B_i}{s^2 + \omega_i^2} \tag{9.2.11}$$

where each term is individually lossless positive real; the matrices L, C, and A_i are real, constant, symmetric, and nonnegative definite, while J and B_i are real, constant, and skew symmetric. A synthesis of $Z(s)$ follows by series connection of syntheses of the individual summands of (9.2.11). The terms J, sL, and $s^{-1}C$ are easily synthesized, as is $sA_i/(s^2 + \omega_i^2)$ if $B_i = 0$. [*Note:* If $Z(s) = Z'(s)$, $B_i = 0$ for all i.] However, a synthesis of the terms $(sA_i + B_i)/(s^2 + \omega_i^2)$ is not quite so easily computed. It could be argued that the

Table 9.2.1
SUMMARY OF SYNTHESIS PROCEDURE

Step	Operation: Lossless-impedance case	Operation: Lossy-impedance case
1. Extract	Pole at infinity of all elements of $Z(s)$, to write $Z(s)$ as $sL + Z_0(s)$, with $Z_0(\infty) < \infty$	
2. Synthesize	sL via transformer-coupled inductors, so that a synthesis of $Z(s)$ will follow by series connection with a synthesis of $Z_0(s)$ (also if desired, carry out the other preliminary extraction procedures for the lossy-impedance case)	
3. Find	A minimal realization of the impedance to be synthesized	
4. Solve	The positive real lemma equations	
5. Find	A minimal realization $\{F, G, H, J\}$ for $Z(s)$ such that (9.2.10) holds	A minimal realization $\{F, G, H, J\}$ for $Z(s)$ together with real matrices L and W_0 such that (9.2.3) holds
6. Form	The impedance matrix $$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} \tag{9.2.5}$$	
7. Synthesize	M as a lossless nondynamic network N_c consisting of transformer-coupled gyrators; terminate the last $\delta[Z]$ ports in unit inductors to yield N	M as a nondynamic network N_c comprised of a series connection of a transformer-resistor network of impedance matrix $\frac{1}{2}(M + M')$ and a transformer-gyrator network of impedance matrix $\frac{1}{2}(M - M')$; terminate the last $\delta[Z]$ ports in unit inductors to yield N

state-space synthesis procedure is computationally simpler, but it has the disadvantage that, in the form we have stated it, gyrators will always appear in the synthesis, even for symmetric $Z(s)$.

We now summarize in Table 9.2.1 the steps in the synthesis of an arbitrary lossless impedance and an arbitrary positive real impedance using the reactance-extraction approach. Following below are illustrative examples.

Example 9.2.1 Consider the positive real impedance

$$Z(s) = \frac{s^2 + 2s + 4}{s^2 + s + 1} = 1 + \frac{s + 3}{s^2 + s + 1}$$

A minimal realization of $Z(s)$ is

$$F_0 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad G_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad J = [1]$$

We then calculate a triple P, L, and W_0 satisfying the positive real lemma equations:

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad L = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \quad W_0 = [\sqrt{2}]$$

Note that the matrices L' and W_0 have only one row, and this is obviously also the normal rank of $Z(s) + Z'(-s)$. The positive definite symmetric P has a factorization

$$P = T'T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, we obtain

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

and a minimal realization $\{F, G, H, J\}$ of $Z(s)$ satisfying (9.2.3) is

$$F = TF_0T^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \quad G = TG_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$H = (T^{-1})'H_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad J = [1]$$

The network N_c then has a positive real impedance matrix

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Next, form the impedance matrices

$$\tfrac{1}{2}(M + M') = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} [1] [1 \quad 0 \quad -1]$$

and

$$\tfrac{1}{2}(M - M') = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

Figure 9.2.2 shows a synthesis for the nondynamic network N_c by

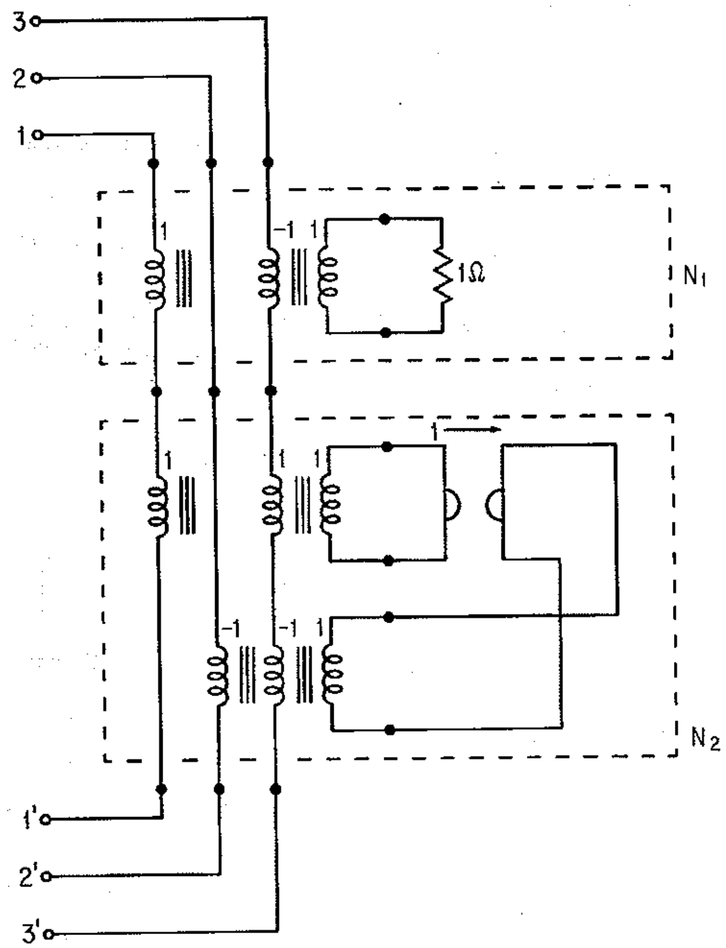


FIGURE 9.2.2. Synthesis of Nondynamic Coupling Network N_c for Example 9.2.1.

series connection of networks of impedance $\frac{1}{2}(M + M')$ and $\frac{1}{2}(M - M')$. Termination of the last two ports in unit inductors yields a complete synthesis of the prescribed positive real $Z(s)$, shown in Figure 9.2.3.

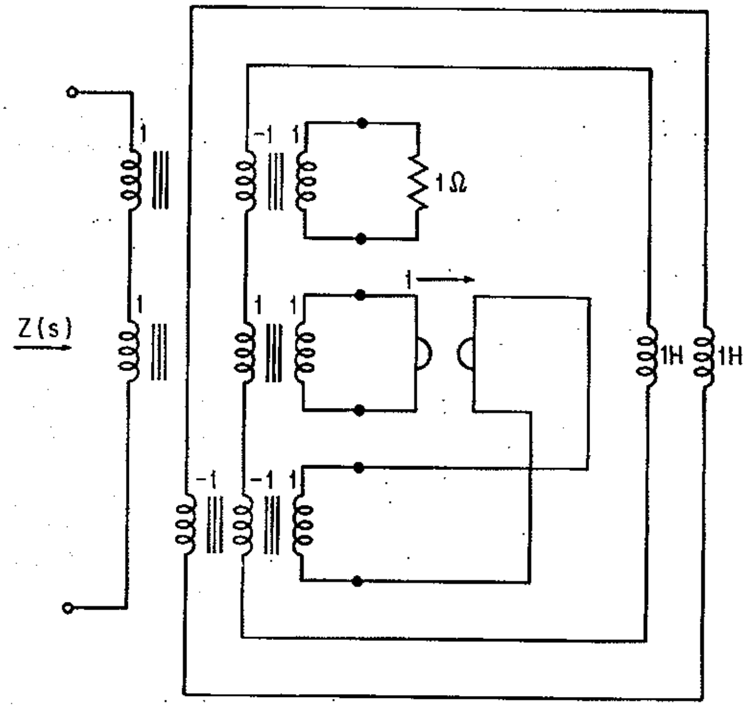


FIGURE 9.2.3. Complete Synthesis of $Z(s)$ of Example 9.2.1.

Example 9.2.2 Consider the positive real matrix (actually symmetric)

$$Z(s) = \begin{bmatrix} 1 & 1 \\ 1 & \frac{3s+3}{s+2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{-3}{s+2} \end{bmatrix}$$

A minimal realization of $Z(s)$ is given by

$$F = [-2] \quad G = [0 \quad 1] \quad H = [0 \quad -3] \quad J = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

A solution triple P, L, and W_0 of the positive real lemma equations is

$$P = [1] \qquad L = [0 \quad -2] \qquad W_0 = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2 \end{bmatrix}$$

(This triple may be easily found by inspection.) The minimal realization above satisfies (9.2.3) since P is the identity matrix, and no coordinate-basis change is therefore necessary. An impedance M for the nondynamic coupling network N_c is then given by

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Then we have the symmetric part and the skew symmetric part of M as follows:

$$\tfrac{1}{2}(M + M') = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & \sqrt{2} & \sqrt{2} \end{bmatrix}$$

and

$$\tfrac{1}{2}(M - M') = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The network N_c of impedance M is thus obtained by a series connection of networks of impedances $\frac{1}{2}(M + M')$ and $\frac{1}{2}(M - M')$, as shown in Fig. 9.2.4. A complete network N for the prescribed $Z(s)$ is found by terminating the last port of N_c in a 1-H inductance, as illustrated in Fig. 9.2.5.

From the examples above we observe that both syntheses use simultaneously the minimum number of reactances and resistances, as expected, at the cost however of the inclusion of a gyrator in the syntheses. (We know that both impedances being symmetric may be synthesized without the use of gyrators.)

We shall give yet another synthesis for the second symmetric positive real $Z(s)$ that arises from a situation in which the number of rows of the matrices L' and W_0 in (9.2.3) exceed the minimal number, i.e., normal rank $[Z(s) + Z'(-s)]$. As a consequence, the synthesis uses more than the minimum number of resistances. The example also illustrates the fact that by using more than the minimum number of resistances (while retaining the minimum number of reactances), it may be possible to achieve a reciprocal synthesis, i.e., one using no gyrators.

Example 9.2.3 Consider once again the symmetric positive real matrix in Example 9.2.2:

$$Z(s) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \dfrac{-3}{s+2} \end{bmatrix}$$

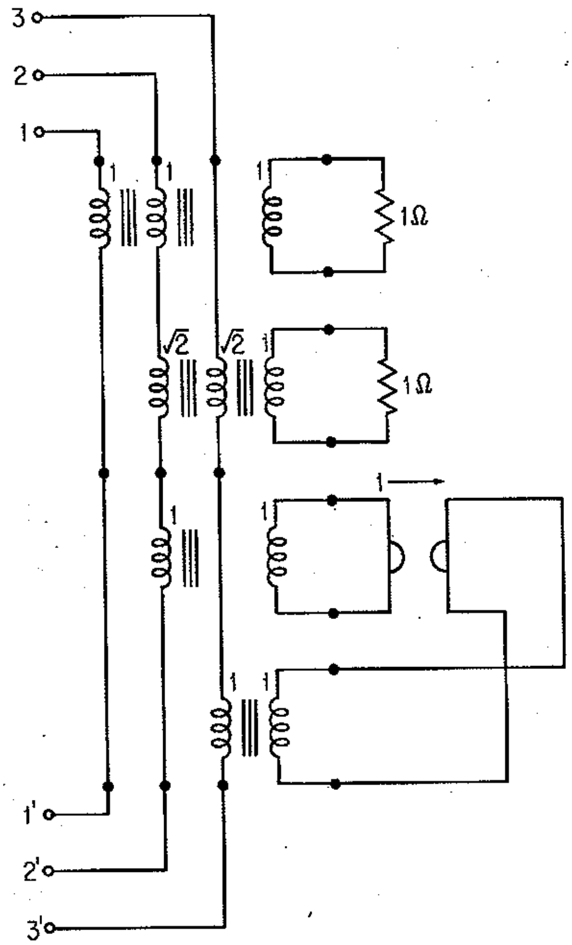


FIGURE 9.2.4. Synthesis of M of Example 9.2.2.

It can be easily verified that the quadruple $\{F, G, H, J\}$ with

$$F = [-2] \qquad G = [0 \quad \sqrt{3}] \qquad H = [0 \quad -\sqrt{3}] \qquad J = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$$

is a minimal realization of $Z(s)$. Further, it is easy to check by direct calculation that $F, G, H,$ and J above satisfy (9.2.3) with

$$L = [0 \quad -\sqrt{3} \quad 1] \qquad W_0 = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

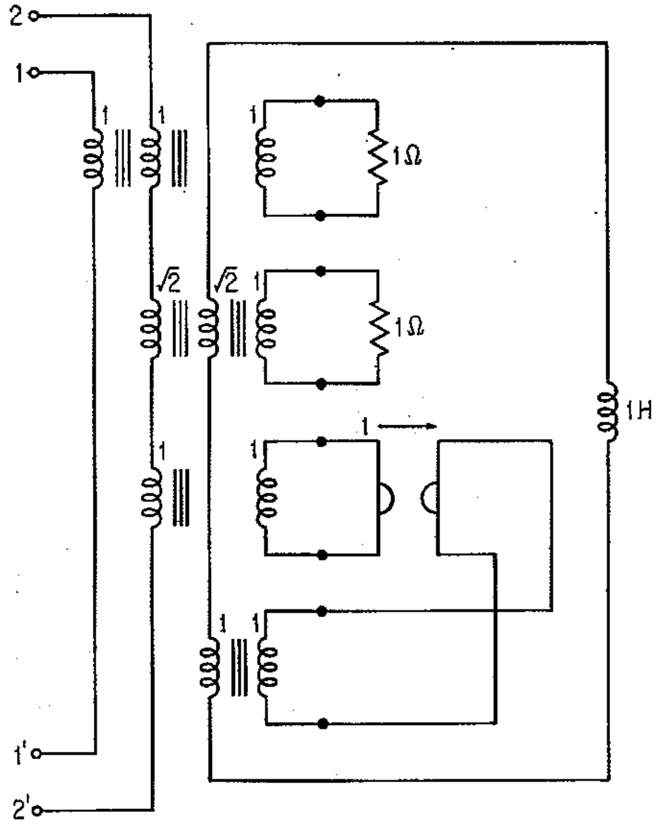


FIGURE 9.2.5. A Final Synthesis of $Z(s)$ for Example 9.2.2.

Notice that L' and W_0 now have three rows while normal rank $Z(s) + Z'(-s)$ is two. The nondynamic coupling network has a positive real impedance M of

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & \sqrt{3} \\ 0 & \sqrt{3} & 2 \end{bmatrix}$$

Obviously M, as well as being positive real, is also symmetric, which implies that M can be synthesized by a transformer-coupled resistor network. We have then

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \sqrt{2} & 0 \\ 0 & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & \sqrt{2} & \sqrt{\frac{3}{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and a synthesis N_c of M therefore requires three resistances (equal to the number of rows of L' and W_0). Terminating N_c in a 1-H inductor at the last port then yields a (reciprocal) synthesis of $Z(s)$ using three resistances, but no gyrators, as shown in Fig. 9.2.6.

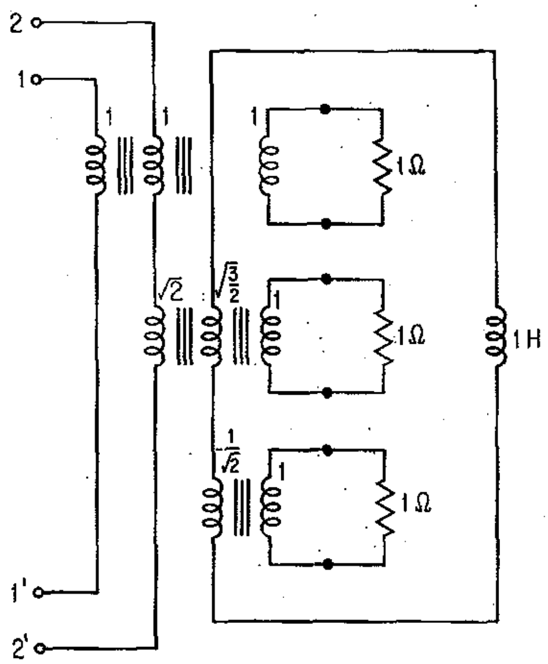


FIGURE 9.2.6. A Final Synthesis of $Z(s)$ for Example 9.2.3.

Though the example illustrates a reciprocal synthesis for a prescribed positive real $Z(s)$ that is symmetric, the theory considered in this section does not allow us to *guarantee* that when $Z(s)$ is symmetric, M is too, permitting a reciprocal synthesis. Neither have we shown in the example how we managed to arrive at a symmetric M, and one might ask if the gyratorless synthesis obtained above resulted by pure chance. The answer is neither yes nor no. A reciprocal synthesis was obtained in Example 9.2.3 without modification of the present theory simply because the impedance is characteristic of an RL network, in the sense that only one type of reactive element (in this case, inductances) is sufficient to give a reciprocal synthesis; on the other hand, one must know how to seek a reciprocal synthesis—simple increase of the number of resistances in the synthesis will not necessarily result in a reciprocal network. Later we shall consider procedures that always lead to a reciprocal synthesis for a positive real impedance matrix that is symmetric.

Minimal Gyrator Synthesis

Until recently, it was an open question as to what the minimum number of gyrators is in any synthesis of a prescribed nonsymmetric $m \times m$ $Z(s)$. A lower bound was known to be one half the rank of $Z(s) - Z'(s)$ [6], while an upper bound for lossless $Z(s)$ was known to be $m - 1$ (see [8] for a classical treatment and [2] for a straightforward state-space treatment). The conjectured lower bound was established for lossless synthesis in [9] and for lossy synthesis in [10]. A state-space treatment of the lossless case appears in [11]; see also Problem 9.2.7. A state-space treatment of the lossy case has not yet been achieved, but see Problem 9.2.6 for a restricted result.

Problem 9.2.1 Synthesize the lossless positive real impedance matrix

$$Z(s) = \frac{1}{s^3 + 4s}\begin{bmatrix} 5s^2 + 4 & 2s^3 - 3s^2 + 4s - 12 \\ -2s^3 - 3s^2 - 4s - 12 & 16s^2 + 40 \end{bmatrix}$$

Problem 9.2.2 Synthesize the lossless bounded real matrix

$$S(s) = \frac{1}{10s + 7}\begin{bmatrix} -1 & -10s + 4\sqrt{3} \\ 10s + 4\sqrt{3} & 1 \end{bmatrix}$$

[*Hint:* Convert $S(s)$ into an impedance matrix $Z(s)$ by using $Z = (I + S)(I - S)^{-1}$.]

Problem 9.2.3 Synthesize the positive real impedance matrix

$$Z(s) = \begin{bmatrix} \dfrac{2s^2 + 2s + 2}{s^2 + 1} & 4\dfrac{s - 1}{s + 1} \\ 0 & 2 \end{bmatrix}$$

by (a) extracting the $j\omega$-axis pole initially, and (b) without initial extraction of the $j\omega$-axis pole. Compare the two syntheses obtained and give a discussion of the computational aspects.

Problem 9.2.4 Using reactance-extraction arguments, show that every (nonreciprocal) minimal reactive synthesis of a positive real impedance $Z(s)$ defines a minimal realization $\{F, G, H, J\}$ such that (9.2.3) holds for certain matrices. (Assume that the nondynamic network resulting from a reactance extraction possesses an impedance matrix.)

Problem 9.2.5 (Constant hybrid matrix synthesis) Let M be a constant $p \times p$ hybrid matrix, with the top left $p_1 \times p_1$ corner of M an impedance matrix and the bottom right $p_2 \times p_2$ corner an admittance matrix. Here, $p = p_1 + p_2$. Show how to synthesize M.

Problem 9.2.6 (Minimal gyrator synthesis for constant matrices) Let M be as in Problem 9.2.5. Show that M can be synthesized with $\frac{1}{2}$ rank $(\Sigma M - M'\Sigma)$ gyrators, but no fewer, where $\Sigma = I_{p_1} + (-1)I_{p_2}$.

Problem 9.2.7 (Minimal gyrator lossless synthesis) Show that in the problem of synthesising a prescribed lossless positive real $Z(s)$ using simultaneously a minimum number of gyrators and reactive elements, it may be assumed without loss of generality that $Z(\infty)$ is nonsingular and that no element of $Z(s)$ possesses a pole at $s = 0$. Form a minimal realization $\{F, G, H, J\}$ of $Z(s)$ with $F + F' = 0$ and $G = H$. Let V be an orthogonal matrix such that the top left $r \times r$ submatrix of $V'(GJ^{-1}G' - F)V$ and bottom right $r \times r$ submatrix of $V'FV$ are zero, where $2r = \delta[Z]$. (Existence of V is a consequence simply of the skewness of $GJ^{-1}G' - F$ and F; see [11].) Use V to change the coordinate basis and apply the ideas of Problem 9.2.6 and of the discussion in Chapter 8 on reciprocal reactance extraction to derive a synthesis using both the minimum number of gyrators and the minimum number of reactive elements.

9.3 RESISTANCE-EXTRACTION SYNTHESIS

We recall that the problem of giving a nonreciprocal synthesis for an $m \times m$ positive real impedance $Z(s)$ via resistance extraction is as follows. Find real constant matrices F_L, G_L, H_L, and J_L that define the state-space equations

$$\dot{x} = F_L x + G_L \begin{bmatrix} u \\ u_1 \end{bmatrix} \tag{9.3.1a}$$

$$\begin{bmatrix} y \\ y_1 \end{bmatrix} = H_L' x + J_L \begin{bmatrix} u \\ u_1 \end{bmatrix} \tag{9.3.1b}$$

and are such that the following two conditions hold:

1. The matrices F_L, G_L, H_L, and J_L satisfy the lossless positive real lemma equations for some positive definite symmetric matrix P.
2. On setting

$$y_1 = -u_1 \tag{9.3.2}$$

(9.3.1) simplifies to

$$\dot{x} = Fx + Gu \qquad y = H'x + Ju \tag{9.3.3}$$

where the transfer-function matrix relating $\mathcal{L}[u(\cdot)]$ to $\mathcal{L}[y(\cdot)]$, which is $J + H'(sI - F)^{-1}G$, is the prescribed matrix $Z(s)$.

The vector u is an m vector and u_1 an r vector, so that Eqs. (9.3.1) are state-space equations for a lossless $m + r$ port. Physically, u and y represent current and voltage, as is evident from the fact that (9.3.3) constitute state-space equations for the prescribed impedance, while u_1 and y_1 are such that if the ith entry of u_1 is a current, the ith entry of y_1 is a voltage, and vice versa. In this section, however, we shall take every entry of u_1 to be a current and every entry of y_1 to be a voltage. Thus $J_L + H_L'(sI - F_L)^{-1}G_L$ becomes an impedance matrix, as distinct from a hybrid matrix. If a lossless network N_L is found synthesizing this impedance, termination of the last r ports of N_L in unit resistances yields a synthesis of $Z(s)$.

We shall impose two more restrictions, each with a physical meaning that will be explained. *The first requires the quadruple* $\{F, G, H, J\}$ *in* (9.3.3) *to be a minimal* realization of $Z(s)$. This means therefore that the size of the state vector x in (9.3.3), and hence (9.3.1), is to be exactly $\delta[Z]$. Consequently, the lossless coupling network N_L may be synthesized with $\delta[Z]$ reactive elements, and, in turn, the network N synthesizing $Z(s)$ may be obtained using $\delta[Z]$ reactive elements—the minimum number possible. The second restriction is that the size r of the vectors u_1 and y_1 is to be $\rho =$ normal rank $Z(s) + Z'(-s)$. Since ρ is the minimum possible number of resistances in any synthesis of $Z(s)$, while r is the actual number of resistances used in synthesizing $Z(s)$, this restriction is therefore equivalent to demanding a synthesis of $Z(s)$ that uses a minimum number of resistances. Actually, the latter restriction, in contrast with the first, is not essential for the method to work; as we shall see later, the dimension of u_1 and y_1 may be greater than ρ, corresponding to syntheses using a nonminimal number of resistances.

Proceeding now with the synthesis procedure, suppose that we have on hand a *minimal* realization $\{F, G, H, J\}$ of $Z(s)$, such that

$$\begin{aligned} F + F' &= -LL' \\ G &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{9.3.4}$$

with L' and W_0 having ρ rows, where $\rho =$ normal rank $[Z(s) + Z'(-s)]$. We claim that matrices F_L, G_L, H_L, and J_L with the properties stated are given by

$$F_L = \frac{1}{2}(F - F') \qquad G_L = \left[\frac{1}{2}(G + H) \,\middle|\, \frac{-1}{\sqrt{2}}L\right] = H_L$$
$$J_L = \begin{bmatrix} \frac{1}{2}(J - J') & \frac{1}{\sqrt{2}}W_0' \\ -\frac{1}{\sqrt{2}}W_0 & 0 \end{bmatrix} \tag{9.3.5}$$

First we shall observe that $\{F_L, G_L, H_L, J_L\}$ defined in (9.3.5) satisfies condition 1 listed earlier. By direct calculation, we have

$$\begin{aligned} F_L + F_L' &= 0 \\ G_L &= H_L \\ J_L + J_L' &= 0 \end{aligned} \tag{9.3.6}$$

Clearly, (9.3.6) are exactly the lossless positive real lemma equations with $P = I$, the identity matrix.

Now we shall prove that the quadruple of (9.3.5) satisfies condition 2. Substituting (9.3.5) into (9.3.1) yields, on expansion, the following equations:

$$\dot{x} = \frac{1}{2}(F - F')x + \frac{1}{2}(G + H)u - \frac{1}{\sqrt{2}}Lu_1 \tag{9.3.7a}$$

$$y = \frac{1}{2}(G + H)'x + \frac{1}{2}(J - J')u + \frac{1}{\sqrt{2}}W_0'u_1 \tag{9.3.7b}$$

$$y_1 = -\frac{1}{\sqrt{2}}L'x - \frac{1}{\sqrt{2}}W_0u \tag{9.3.7c}$$

Setting (9.3.2) *in* (9.3.7a) *and* (9.3.7b) to eliminate u_l, and then substituting for y_1 from (9.3.7c) leads to

$$\dot{x} = \tfrac{1}{2}(F - F' - LL')x + \tfrac{1}{2}[G + H - LW_0]u$$
$$y = \tfrac{1}{2}[G + H + LW_0]'x + \tfrac{1}{2}[J - J' + W_0'W_0]u$$

On using (9.3.4), these equations become

$$\dot{x} = Fx + Gu \qquad y = H'x + Ju \tag{9.3.3}$$

and the impedance matrix relating $\mathcal{L}[u(\cdot)]$ to $\mathcal{L}[y(\cdot)]$ is clearly $Z(s)$.

We conclude therefore that $\{F_L, G_L, H_L, J_L\}$ of (9.3.5) defines a lossless impedance matrix for the coupling network N_L. A synthesis of N_L, and hence the network N synthesiszing $Z(s)$, may be obtained using available classical methods [6], the technique of Chapter 8, or the state-space approach as discussed in the previous section. The latter method is preferable in this case since one avoids computation of the lossless impedance matrix $Z_L(s) = J_L + H_L'(sI - F_L)^{-1}G_L$, and, more significantly, the matrices F_L, G_L, H_L, and J_L are such that a synthesis of N_L is immediately derivable (without the need to change the state-space coordinate basis). To see this, observe that the real constant matrix M defined by

$$M = \begin{bmatrix} J_L & -H_L' \\ G_L & -F_L \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(J - J') & \frac{1}{\sqrt{2}}W_0' & -\frac{1}{2}(G + H)' \\ -\frac{1}{\sqrt{2}}W_0 & 0 & \frac{1}{\sqrt{2}}L' \\ \frac{1}{2}(G + H) & -\frac{1}{\sqrt{2}}L & \frac{1}{2}(F' - F) \end{bmatrix} \tag{9.3.8}$$

is skew, and therefore lossless positive real. In other words, M is the impedance of an easily synthesized $(m + p + n)$-port lossless nondynamic network of transformer-coupled gyrators, and termination of this network at its last n ports in 1-H inductances yields a synthesis for N_L. Resistive termination of all but the first m ports of N_L finally gives the desired synthesis for

$Z(s)$ in which a minimal number of resistances as well as reactive elements is used.

A study of the preceding synthesis procedure will quickly convince the reader that the minimality of the number of rows of the matrices L' and W_0 played no role in the manipulations, save that, of course, of guaranteeing a minimal number of resistors in the synthesis. Indeed, it is easy to see that any minimal realization $\{F, G, H, J\}$ and real matrices L' and W_0 *not* necessarily minimal in their numbers of rows, such that (9.3.4) is satisfied, will yield a synthesis of $Z(s)$. The number of resistances required is the dimension of the vectors u_1 and y_1, which, in turn, is precisely the number of rows in L' and W_0.

Summary of Synthesis

A summary of the synthesis procedure outlined in this section is as follows:

1. Ensure that no element of $Z(s)$ has a pole at infinity by carrying out, if necessary, a series extraction of transformer-coupled inductors.
2. Compute a minimal realization $\{F, G, H, J\}$ of $Z(s)$ such that (9.3.4) holds for some real constant matrices L and W_0.
3. Form the skew constant impedance matrix

$$M = \begin{bmatrix} \frac{1}{2}(J - J') & \frac{1}{\sqrt{2}}W_0' & -\frac{1}{2}(G + H)' \\ -\frac{1}{\sqrt{2}}W_0 & 0 & \frac{1}{\sqrt{2}}L' \\ \frac{1}{2}(G + H) & -\frac{1}{\sqrt{2}}L & \frac{1}{2}(F' - F) \end{bmatrix} \tag{9.3.8}$$

   and give a synthesis of M as a transformer-coupled gyrator network by using a factorization $M = T_g'[E \dotplus \cdots \dotplus E]T_g$, with E the impedance matrix of a unit gyrator.
4. Terminate the last n ($= \delta[Z] =$ dimension of the matrix F) ports of the above network in 1-H inductances, and then all but the first m ports of the resulting network in 1-Ω resistances to give a synthesis of $Z(s)$. The number of inductances used, being $n = \delta[Z]$, is minimal, while the number of resistances used is equal to the number of rows in L' and W_0, and may be minimal if desired.

As an alternative to steps 3 and 4, the following may be used.

3′. Compute matrices F_L, G_L, H_L, and J_L from (9.3.5) and form the lossless $(m + r) \times (m + r)$ impedance matrix $J_L + H_L'(sI - F_L)^{-1}G_L$ (here, r is the number of rows in L' and W_0).

4′. Synthesize the lossless impedance matrix $J_L + H_L'(sI - F_L)^{-1}G_L$ by any procedure, classical or state-space, and terminate the last r ports of the network synthesizing this matrix in unit resistors to yield the impedance matrix $Z(s)$ at the first m ports.

Example 9.3.1 Consider the positive real impedance

$$Z(s) = \frac{s^2 + s + 4}{s^2 + s + 1} = 1 + \frac{3}{s^2 + s + 1}$$

A minimal realization of $Z(s)$ is

$$F_0 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad G_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad J = [1]$$

A solution triple of the positive real lemma equations may be found to be

$$P = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$L_0' = [\sqrt{2} \quad -\sqrt{2}] \qquad W_0 = [\sqrt{2}]$$

With $T = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$, a quadruple $\{F, G, H, J\}$ and the associated L and W_0 satisfying (9.3.4) is therefore given by

$$F = TF_0T^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \qquad G = TG_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$H = (T^{-1})'H_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad L = (T^{-1})'L_0 = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

$$J = [1] \qquad W_0 = [\sqrt{2}]$$

Using (9.3.8) as a guide, we form the skew matrix

$$M = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

whence

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & \sqrt{2} & 0 \\ -1 & 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

A synthesis of M is shown in Fig. 9.3.1; a synthesis of $Z(s)$ is obtained by terminating ports 3 and 4 of the network synthesizing M in 1-H inductances and port 2 in a 1-Ω resistance. The desired network is shown

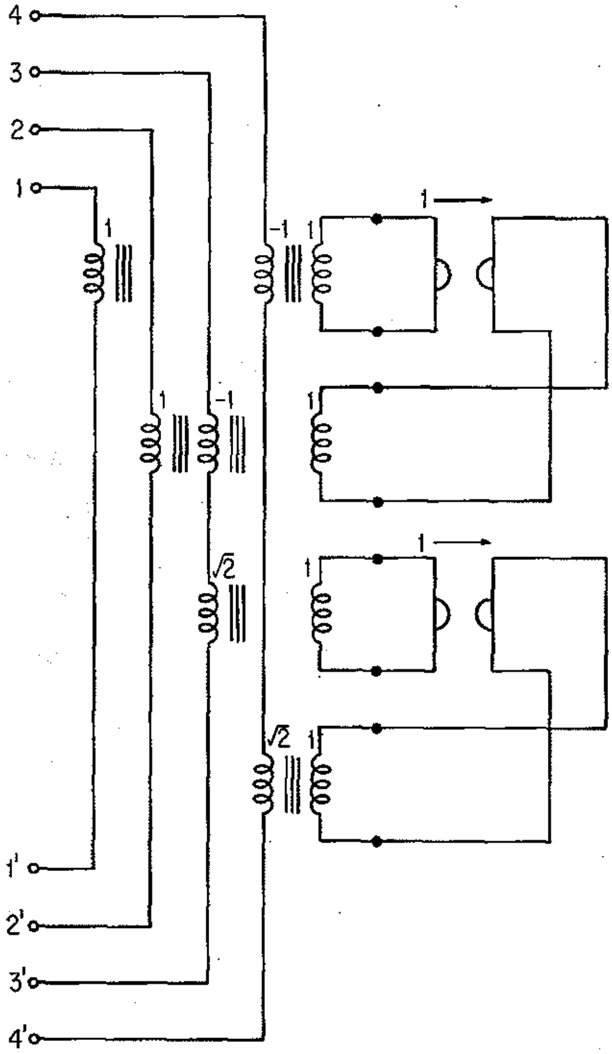


FIGURE 9.3.1. Synthesis of M in Example 9.3.1.

in Fig. 9.3.2. The synthesis obviously uses a minimal number of inductances as well as a minimal number of resistances.

To close this section, we offer some remarks comparing the reactance-extraction and resistance-extraction methods. These remarks carry over mutatis mutandis to syntheses to be presented later, viz., reciprocal immittance synthesis and nonreciprocal and reciprocal scattering synthesis. First, the only awkward computations are those requiring the construction of a minimal realization $\{F, G, H, J\}$ of the prescribed $Z(s)$ satisfying (9.3.4).

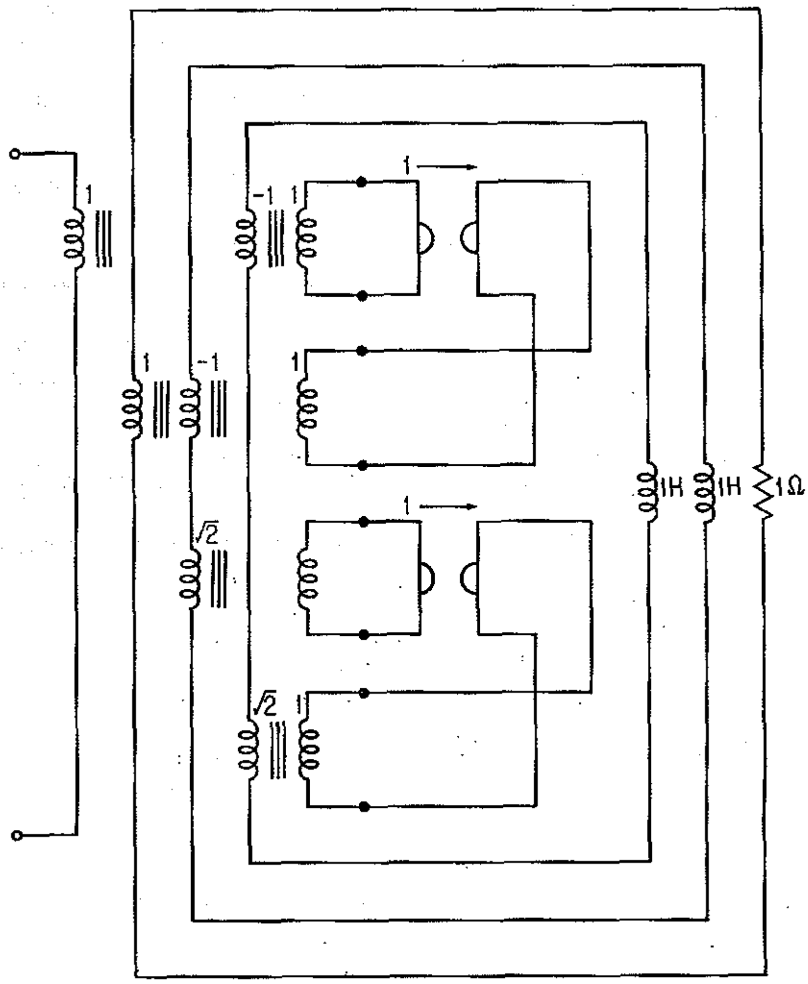


FIGURE 9.3.2. A Synthesis of $Z(s)$ of Example 9.3.1.

This construction is required in both the reactance- and resistance-extraction approaches, while the remaining calculations in both methods are very straightforward. Second, if one adopts the reactance-extraction approach to synthesizing the lossless impedance derived in the course of the resistance-extraction synthesis, one is led to the skew constant impedance matrix M of (9.3.8). Now, as may be checked easily, if a network with impedance M in (9.3.8) is terminated at ports $m + 1$ through $m + p$ in unit resistors, the resulting network has impedance matrix

$$\begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$$

This is the matrix that appears in the normal reactance extraction, and its occurrence here means that the structures resulting from the two syntheses are the same.

One might well ask why two approaches to the same end result should be discussed. There are several answers.

1. The two approaches give different conceptual insights into the synthesis problem.
2. The resistance-extraction approach, while marginally more complex, offers more variety in synthesis structures because there is variety in lossless synthesis procedures.
3. While the reactance-extraction approach is a more immediate consequence of state-space ideas than the resistance-extraction approach, the latter is better known from classical synthesis.

Problem 9.3.1 Show that the procedure considered in this section always yields a synthesis that uses nonreciprocal elements, i.e., gyrators, even if $Z(s)$ is symmetric.

Can you state what modifications might be made so that a reciprocal synthesis could be obtained?

Problem 9.3.2 Give two syntheses for the positive real impedance

$$Z(s) = \frac{s^2 + 4s + 9}{s^2 + 2s + 1}$$

one with a minimal number of resistors (one in this case) and the other with more than one resistor. Compare the two realizations in the number of gyrators used.

9.4 A RECIPROCAL (NONMINIMAL RESISTIVE AND REACTIVE) IMPEDANCE SYNTHESIS

In this section we shall look at a reciprocal synthesis procedure for a prescribed $m \times m$ positive real impedance $Z(s)$, which is also symmetric. The method is based on the reactance-extraction concept. It is always successful in eliminating gyrators in a synthesis, given that $Z(s)$ is symmetric, and there is no real increase in the amount of major computation involved over the nonreciprocal synthesis given. The major computational effort lies in the determination of a minimal realization $\{F, G, H, J\}$ of $Z(s)$, with $Z(\infty) < \infty$, such that

$$\begin{aligned} F + F' &= -LL' \\ G &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{9.4.1}$$

for some real matrices L and W_0, which is a calculation that has been required in the various syntheses considered so far. As may be expected, we do not gain something for nothing; in eliminating gyrators while retaining the same level of computation in achieving a synthesis, the cost paid is in terms of extra numbers of reactive elements and resistances appearing in the synthesis. That additional elements are needed should not be surprising at all in the light of Example 9.2.3, in which the only gyrator was eliminated at the cost of introducing an extra resistance.

We shall make use in discussing the method of a rather trivial identity. (Considered by Koga in [7], this identity provided a key to a related synthesis problem.) The identity is

$$Z(s) = \tilde{J} + \frac{\tilde{H}'}{\sqrt{2}}(sI - \tilde{F})^{-1}\frac{\tilde{G}}{\sqrt{2}} + \frac{\tilde{G}'}{\sqrt{2}}(sI - \tilde{F}')^{-1}\frac{\tilde{H}}{\sqrt{2}} \tag{9.4.2}$$

where $\{\tilde{F}, \tilde{G}, \tilde{H}, \tilde{J}\}$ is *any* state-space realization of $Z(s)$ with $Z(s) = Z'(s)$. Equation (9.4.2) may be readily verified on noting that since $Z(s) = Z'(s)$, one has $Z(s) = \frac{1}{2}[Z(s) + Z'(s)]$.

A sketch of the development leading to the reciprocal synthesis procedure is as follows. We shall first set up a constant square matrix, in turn made up from matrices that form a special realization (nonminimal in fact) of $Z(s)$. This constant matrix will be such that its positive real character can be easily recognised. We shall then define from this matrix another matrix via a simple orthogonal transformation, so that while the positive real character is retained, other additional properties arise. These properties then make it apparent that if the constant matrix is regarded as a hybrid matrix, it has a reciprocal passive synthesis; moreover, terminating certain ports of a network synthesizing the matrix appropriately in inductances and capacitances then yields a synthesis of $Z(s)$.

To begin, it is easy to see from (9.4.2) that matrices F_0, G_0, H_0, and J, defined by

$$F_0 = \begin{bmatrix} F & 0 \\ 0 & F' \end{bmatrix} \qquad G_0 = \begin{bmatrix} \dfrac{G}{\sqrt{2}} \\ \dfrac{H}{\sqrt{2}} \end{bmatrix} \qquad H_0 = \begin{bmatrix} \dfrac{H}{\sqrt{2}} \\ \dfrac{G}{\sqrt{2}} \end{bmatrix} \tag{9.4.3}$$

[with F, G, H, and J a minimal realization of $Z(s)$ satisfying (9.4.1)] constitute a state-space realization (certainly nonminimal) of $Z(s)$. In other words, as may be checked readily,

$$Z(s) = J + H_0'(sI - F_0)^{-1}G_0 \tag{9.4.4}$$

The dimension of F_0, or the size of the associated state vector, is obviously $2n$, i.e., twice the degree of $Z(s)$, which indicates that if we are to synthesize $Z(s)$ via, say, reactance extraction through the realization $\{F_0, G_0, H_0, J\}$ of $Z(s)$, then $2n$ reactive elements will presumably be needed. In fact, we shall show that a reciprocal synthesis for $Z(s)$ is readily obtained from $\{F_0, G_0, H_0, J\}$ through a simple orthogonal matrix basis change; exactly n inductances and n capacitances—a total of $2n$ reactive elements—are always used in the synthesis.

Let us consider a constant matrix M defined by

$$M = \begin{bmatrix} J & -H_0' \\ G & -F_0 \end{bmatrix} = \begin{bmatrix} J & \dfrac{-H'}{\sqrt{2}} & \dfrac{-G'}{\sqrt{2}} \\ \dfrac{G}{\sqrt{2}} & -F & 0 \\ \dfrac{H}{\sqrt{2}} & 0 & -F' \end{bmatrix} \tag{9.4.5}$$

Note that we have not so far given physical significance to M by, for example, demanding that it be the impedance matrix of a nondynamic network. But we can conclude that M is positive real from the following sequence of simply derived equalities:

$$\begin{aligned} M + M' &= \begin{bmatrix} J + J' & \dfrac{1}{\sqrt{2}}(G - H)' & \dfrac{1}{\sqrt{2}}(H - G)' \\ \dfrac{1}{\sqrt{2}}(G - H) & -(F + F') & 0 \\ \dfrac{1}{\sqrt{2}}(H - G) & 0 & -(F + F') \end{bmatrix} \\ &= \begin{bmatrix} W_0'W_0 & \dfrac{-1}{\sqrt{2}}(LW_0)' & \dfrac{1}{\sqrt{2}}(LW_0)' \\ \dfrac{-1}{\sqrt{2}}LW_0 & LL' & 0 \\ \dfrac{1}{\sqrt{2}}LW_0 & 0 & LL' \end{bmatrix} \\ &= \begin{bmatrix} W_0' & 0 \\ \dfrac{-L}{\sqrt{2}} & \dfrac{L}{\sqrt{2}} \\ \dfrac{L}{\sqrt{2}} & \dfrac{L}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} W_0 & \dfrac{-L'}{\sqrt{2}} & \dfrac{L'}{\sqrt{2}} \\ 0 & \dfrac{L'}{\sqrt{2}} & \dfrac{L'}{\sqrt{2}} \end{bmatrix} \\ &\geq 0 \end{aligned} \tag{9.4.6}$$

The first equality follows immediately from (9.4.5), the second on using (9.4.1), and the third from an obvious factorization.

We observe also another important fact from this sequence of equalities, which is that

$$\operatorname{rank}[M + M'] = 2r \tag{9.4.7}$$

where r is the number of rows of L' and W_0.

We shall now introduce yet another constant matrix M_1, like M in that it is positive real and consists of matrices of a realization of $Z(s)$, but possessing one additional property to be outlined. Consider the following matrix:

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & -I_n \\ I_n & I_n \end{bmatrix} \tag{9.4.8}$$

with I_n an $n \times n$ identity matrix and $n = \delta[Z]$. It is easy to verify that T is an *orthogonal matrix*. Now define another realization $\{F_1, G_1, H_1, J\}$ of $Z(s)$, using the matrix T of (9.4.8), via

$$F_1 = TF_0T' \qquad G_1 = TG_0 \qquad H_1 = TH_0 \tag{9.4.9}$$

The realization $\{F_1, G_1, H_1, J\}$ defines a constant matrix M_1 by

$$M_1 = \begin{bmatrix} J & -H_1' \\ G_1 & -F_1 \end{bmatrix} = \begin{bmatrix} J & -H_0'T' \\ TG_0 & -TF_0T' \end{bmatrix} \tag{9.4.10}$$

such that

1. $M_1 + M_1'$ is nonnegative definite, or M_1 is positive real, and $M_1 + M_1'$ has rank $2r$, where r is as defined previously following (9.4.7).
2. $[I_m \dotplus I_n \dotplus (-I_n)]M_1$ is symmetric.

To prove claim 1, we note that M_1 of (9.4.10) is

$$\begin{aligned} M_1 &= (I_n \dotplus T)\begin{bmatrix} J & -H_0' \\ G_0 & -F_0 \end{bmatrix}(I_n \dotplus T') \\ &= (I_n \dotplus T)M(I_n \dotplus T') \end{aligned}$$

so that

$$M_1 + M_1' = (I_n \dotplus T)(M + M')(I_n \dotplus T')$$

Since $M + M'$ is nonnegative definite, as noted in (9.4.6), it follows immediately that $M_1 + M_1'$ is also. Further, the above equality implies from the nonsingularity of $I_n \dotplus T$ and (9.4.7) that

$$\text{rank}\,(M_1 + M_1') = \text{rank}\,(M + M') = 2r \tag{9.4.11}$$

We now prove claim 2. By direct calculation, using (9.4.3), (9.4.8), and (9.4.9), we obtain

$$F_1 = \begin{bmatrix} \frac{1}{2}(F + F') & \frac{1}{2}(F - F') \\ \frac{1}{2}(F - F') & \frac{1}{2}(F + F') \end{bmatrix} \qquad G_1 = \begin{bmatrix} \frac{1}{2}(G - H) \\ \frac{1}{2}(G + H) \end{bmatrix} \qquad H_1 = \begin{bmatrix} \frac{1}{2}(H - G) \\ \frac{1}{2}(G + H) \end{bmatrix} \tag{9.4.12}$$

and therefore

$$M_1 = \begin{bmatrix} J & -H_1' \\ G_1 & -F_1 \end{bmatrix} = \begin{bmatrix} J & \frac{1}{2}(G-H)' & -\frac{1}{2}(G+H)' \\ \frac{1}{2}(G-H) & -\frac{1}{2}(F+F') & \frac{1}{2}(-F+F') \\ \frac{1}{2}(G+H) & \frac{1}{2}(-F+F') & -\frac{1}{2}(F+F') \end{bmatrix} \tag{9.4.13}$$

That $[I_m \dotplus I_n \dotplus (-I_n)]M_1$ is symmetric is evident by inspection.

Let us summarize the major facts derived above:

1. $\{F_1, G_1, H_1, J\}$ as given in (9.4.12) is a realization of $Z(s)$ with F_1 being $2n \times 2n$.
2. The constant matrix $M_1 = \begin{bmatrix} J & -H_1' \\ G_1 & -F_1 \end{bmatrix}$ is positive real, and ΣM_1 is symmetric, where $\Sigma = [I_m \dotplus I_n \dotplus (-I_n)]$.

Point 2 implies that, *if we consider the matrix M_1 as a hybrid matrix* of an $(m + 2n)$-port network, where the first $m + n$ ports are current-excited ports corresponding to the first $m + n$ rows of Σ that have a $+1$ on the diagonal, and the remaining n ports are voltage-excited ports corresponding to the rows of Σ that have a -1 on the diagonal, then a network realizing M_1 can be found that uses only passive reciprocal elements.

Further, from our discussion of the reactance-extraction procedure we can see that 1 and 2 together imply that termination of the last $2n$ ports of a synthesis of M_1 in n unit inductances for the current-excited ports, and n unit capacitances for the voltage-excited ports yields a network synthesizing $Z(s)$. Since a synthesis of M_1 that is passive and reciprocal is guaranteed by point 2 to exist, and since we know how to derive a passive reciprocal synthesis of $Z(s)$ from a passive reciprocal synthesis of M_1, it follows that a reciprocal passive synthesis of $Z(s)$ can be achieved.

So far, we have not yet shown how to go about synthesizing M_1. This we shall now consider, thereby completing the task of synthesizing $Z(s)$.

Synthesis of a Resistive Network Described by a Constant Hybrid Matrix M

Consider the following equation describing a reciprocal $(p + q)$ port N_m having a constant hybrid matrix M:

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} M_{11} & -M_{21}' \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \tag{9.4.14}$$

with the $(p + q) \times (p + q)$ constant matrix M positive real, or $M + M' \geq 0$. The matrix M_{11} is $p \times p$ and symmetric, and M_{22} is $q \times q$ and symmetric. The first p ports of N_m are the current-excited or open-circuit ports, while the remaining q ports are the voltage-excited or short-circuit ports. It is easy to see that the positive real character of M, or the nonnegative definiteness of $M + M'$ together with the form of M, implies that the *symmetric principal submatrices* M_{11} *and* M_{22} *are individually nonnegative definite, or positive real.*

The positive real $p \times p$ symmetric matrix M_{11}, corresponding to the current-excited ports of M, is therefore an impedance matrix and is obviously realizable as a p-port transformer-coupled resistance network. The positive real $q \times q$ symmetric matrix M_{22}, corresponding to the voltage-excited ports of M, is an admittance matrix and is realizable as a q-port transformer-coupled conductance network. The remaining portion of M, that is, $\begin{bmatrix} 0 & -M'_{21} \\ M_{21} & 0 \end{bmatrix}$, represents, as we may recall from an earlier section, the hybrid-matrix description of a multiport transformer with a turns-ratio matrix $-M_{21}$. We may therefore rightly expect that the $q \times p$ matrix M_{21} represents some sort of transformer coupling between the two subnetworks described by the impedance M_{11} and the admittance M_{22}. A complete synthesis of M is therefore achieved by an appropriate series-parallel connection. In fact, it can be readily verified by straightforward analysis that the network N_m *in Fig. 9.4.1 synthesizes* M; i.e., the hybrid matrix of N_m is M. (Problem 9.4.1 explores the hybrid matrix synthesis problem further.)

Synthesis of $Z(s)$

A synthesis of the hybrid matrix M_1 of (9.4.13) is evidently provided by the connection of Fig. 9.4.1 with

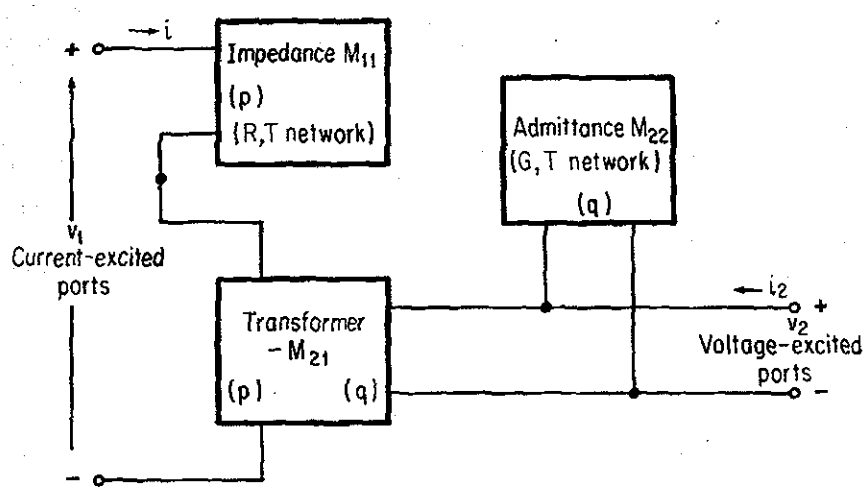


FIGURE 9.4.1. Synthesis of the Hybrid Matrix M of Equation (9.4.14).

$$M_{11} = \begin{bmatrix} J & \frac{1}{2}(G-H)' \\ \frac{1}{2}(G-H) & \frac{1}{2}(-F-F') \end{bmatrix} \qquad M_{22} = \tfrac{1}{2}[-F-F'] \qquad (9.4.15)$$
$$M_{21} = [\tfrac{1}{2}(G+H) \quad \tfrac{1}{2}(-F+F')]$$

A synthesis of $Z(s)$ then follows from that for M_1, by terminating all current-excited ports other than the first m ports of the network N_m synthesizing M_1 in 1-H inductances, and all the voltage-excited ports in 1-F capacitances.

Observe that exactly n inductors and n capacitors are always used, so a total of $2n = 2\delta[Z]$ reactive elements appear. Also, the total number of resistances used is the sum of rank M_{11} and rank M_{22}, which is the same as

$$\text{rank}\,[M_{11} + M'_{11}] + \text{rank}\,[M_{22} + M'_{22}] = \text{rank}\,[M_1 + M'_1] = 2r$$

Therefore, the synthesis uses $2r$ resistances, where r is the number of rows of the matrices L' and W_0 appearing in (9.4.1).

Arising from these observations, the following points should be noted. First, before carrying out a synthesis for a positive real $Z(s)$ with $Z(\infty) < \infty$, it is worthwhile to apply those parts of the preliminary simplification procedure described earlier that permit extraction of reciprocal lossless sections. The motivation in doing so should be clear, for the positive real impedance that remains to be synthesized after such extractions will have a degree, n' say, that is less than the original degree $\delta[Z]$ by the number of reactive elements extracted, d say. Thus the number of reactive elements used in the overall synthesis becomes $d + 2n'$; in contrast, if no initial lossless extractions were made, this number would be $2\delta[Z]$ or $2(n' + d)$—hence a saving of d reactive elements results. Moreover, in computing a solution triple P, L, and W_0 satisfying the positive real lemma equations [which is necessary in deriving (9.4.1)] some of the many available methods described earlier in any case require the simplification process to be carried out (so that $J + J' = 2J$ is nonsingular).

Second, the number of rows r of the matrices L' and W_0 associated with (9.4.1) has a minimum value of $\rho = \text{normal rank } [Z(s) + Z'(-s)]$. Thus the minimum number of resistors that it is possible to achieve using the synthesis procedure just presented is 2ρ.

The synthesis procedure will now be summarized, and then a simple synthesis example will follow.

Summary of Synthesis

The synthesis procedure falls into the following steps:

1. By lossless extractions, reduce the problem of synthesizing a prescribed symmetric positive real impedance matrix to one of synthesizing a symmetric positive real $m \times m$ $Z(s)$ with $Z(\infty) < \infty$.

2. Find a minimal realization for $Z(s)$ and solve the positive real lemma equations.
3. Obtain a minimal realization $\{F, G, H, J\}$ of $Z(s)$ and associated real matrices L' and W_0 possessing the minimum possible number of rows, such that (9.4.1) holds.
4. Form the symmetric positive real impedance M_{11}, admittance M_{22}, and transformer turns-ratio matrix $-M_{21}$ using (9.4.15).
5. Synthesize M_{11}, M_{22}, and $-M_{21}$ and connect these together as shown in Fig. 9.4.1. Terminate all of the voltage-excited ports in 1-F capacitances, and all but the first m current-excited ports in 1-H inductances to yield a synthesis of the $m \times m$ $Z(s)$.

Example 9.4.1 Consider the positive real impedance of Example 9.3.1:

$$Z(s) = \frac{s^2 + s + 4}{s^2 + s + 1} = 1 + \frac{3}{s^2 + s + 1}$$

A minimal realization $\{F, G, H, J\}$ satisfying (9.4.1) is

$$F = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \quad G = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad J = [1]$$

and the associated matrices L' and W_0, with L' and W_0 possessing the minimal number of rows, are

$$L' = [0 \quad \sqrt{2}] \qquad W_0 = [\sqrt{2}]$$

First, we form the impedance M_{11}, the admittance M_{22}, and the turns-ratio matrix M_{21} as follows:

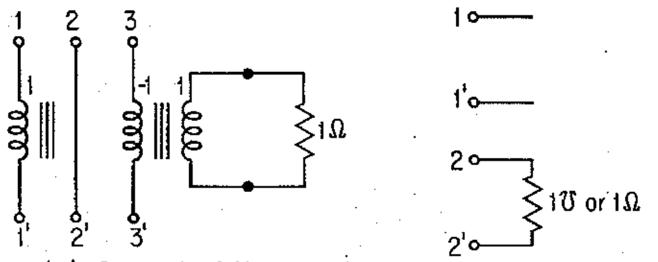
$$M_{11} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} [1][1 \quad 0 \quad -1]$$

$$M_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

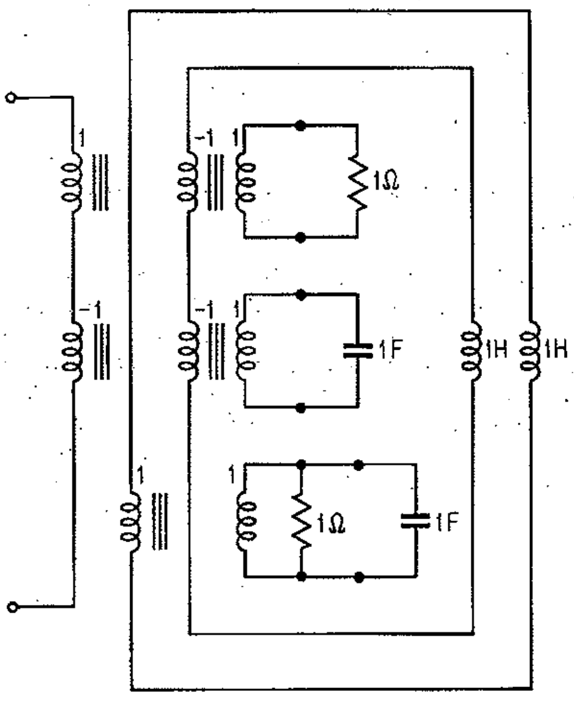
$$M_{21} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

A synthesis of M_{11} is shown in Fig. 9.4.2a, and M_{22} in Fig. 9.4.2b. Note that M_{22} is nothing more than an open circuit for port 1 and a 1-Ω resistor for port 2. A final synthesis of $Z(s)$ is obtained by using the connection of Fig. 9.4.1 and terminating this network appropriately in 1-F capacitances and 1-H inductances, as shown in Fig. 9.4.2c.

By inspecting Fig. 9.4.2, we observe that, as expected, two resistances, two inductances, and two capacitances are used in the synthesis. But if we examine them more closely, we can observe an interesting fact: the circuit



(a) Synthesis of M_{11} (b) Synthesis of M_{22}



(c) Final Synthesis of Z(s) of Example 9.4.1

FIGURE 9.4.2 Synthesis of Example 9.4.1.

of Fig. 9.4.2 contains unnecessary elements that may be eliminated without affecting the terminal behavior, and it actually reduces to the simplified circuit arrangement shown in Fig. 9.4.3. Problem 9.4.2 asks the reader to verify this. Observe now that the synthesis in Fig. 9.4.3 uses one resistance, inductance, and capacitance, and no gyrators. Thus,

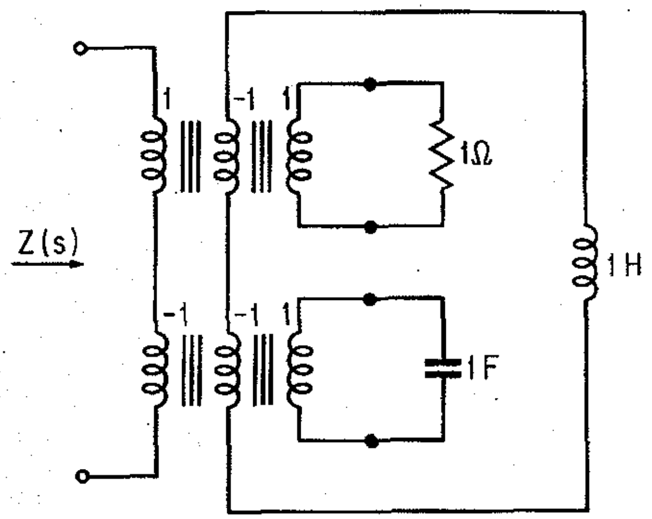


FIGURE 9.4.3. A Simplified Circuit Arrangement for Fig. 9.4.2.

in this rather special case, we have been able to obtain a reciprocal synthesis that uses simultaneously the minimal number of resistances and the minimal number of reactances! One should not be tempted to think that this might also hold true in general; in fact (see [6]), it is not possible in general to achieve a reciprocal synthesis with simultaneously a minimum number of resistances as well as reactive elements, although it is true that a reciprocal synthesis can always be obtained that uses a minimal number of resistances and a nonminimal number of reactive elements, or vice versa. This will be shown in Chapter 10.

The example above also illustrates the following point: although the reciprocal synthesis method in this section theoretically requires twice the minimal number of resistances ($2p$) and twice the minimal number of reactive-elements ($2\delta[Z]$) for a reciprocal synthesis of a positive real symmetric $Z(s)$, [at least, when $Z(s)$ is the residual impedance obtained after preliminary synthesis simplifications have been applied to the original impedance matrix], cases may arise such that these numbers may be reduced.

As already noted, the method in this section yields a reciprocal synthesis in a fashion as simple as the nonreciprocal synthesis obtained from the methods of the earlier sections of this chapter. In fact, the major computation task is the same, requiring the derivation of (9.4.1). However, the cost paid for the elimination of gyrators is that a nonminimal number of resistances as well as a *nonminimal* number of reactive elements are needed. In Chapter 10, we shall consider synthesis procedures that are able to achieve reciprocal syntheses with a minimum number of resistances *or* a minimal number of reactive elements. The computation necessary for these will be more complex than that required in this chapter.

One feature of the synthesis of this section, as well as the syntheses of Chapter 10, is that transformers are in general required. State-space techniques have so far made no significant impact on transformerless synthesis problems; in fact, there is as yet not even a state-space parallel of the classical transformerless synthesis of a positive real impedance function. This synthesis, due to R. Bott and R.J. Duffin, is described in, e.g., [8]; it uses a number of resistive and reactive elements which increases exponentially with the degree of the impedance function being synthesized, and is certainly almost never minimal in the number of either reactive or resistive elements. It is possible, however, to make a very small step in the direction of state-space transformerless synthesis. The step amounts to converting the problem of synthesizing (without transformers) an arbitrary positive real $Z(s)$ to the problem of synthesizing a constant positive real Z_0. If the reciprocal synthesis approach of this section is used, one can argue that a transformerless synthesis will result if the constant matrix M_1 of (9.4.13), which is a hybrid matrix, is synthesizable with no transformers. Therefore, one simply demands that not only should F, G, H, and J satisfy (9.4.1), but also that they be such that the nondynamic hybrid matrix of (9.4.13) possess a transformerless synthesis. Similar problem statements can be made in connection with the reciprocal synthesis procedures of the next section. (Note that if gyrators are allowed, transformerless synthesis is trivial. So it only makes sense to consider reciprocal transformerless synthesis.)

Finally we comment that there exists a resistance extraction approach, paralleling closely the reactance extraction approach of this section, which will synthesize a symmetric positive real $Z(s)$ with twice the minimum numbers of reactive and resistive elements [4, 5].

Problem 9.4.1 Show that the hybrid matrix M of the circuit arrangement in Fig. 9.4.1 is

$$M = \begin{bmatrix} M_{11} & -M_{21}' \\ M_{21} & M_{22} \end{bmatrix} \begin{matrix} \}p \\ \}q \end{matrix}$$
$$\qquad \underbrace{\phantom{M_{11}}}_{p} \quad \underbrace{\phantom{M_{22}}}_{q}$$

Consider specifically a hybrid matrix M_1 of the form of (9.4.13) with the first $m + n$ ports current-excited and the remaining n ports voltage-excited ports. Now it is known that the last n of the $m + n$ current-excited ports are to be terminated in inductances, while all the n voltage-excited ports are to be terminated in capacitances. Can you find an alternative circuit arrangement to Fig. 9.4.1 for M_1 that exhibits clearly the last n current-excited ports separated from the first m current-excited ports (Fig. 9.4.1 shows these combined together) in terms of the submatrices appearing in (9.4.13)?

Problem 9.4.2 Show that the circuit arrangement of Fig. 9.4.3 is equivalent to that of Fig. 9.4.2.

[*Hint:* Show that the impedance $Z(s)$ is $(s^2 + s + 4)/(s^2 + s + 1)$.]

Problem 9.4.3 Synthesize the symmetric positive real impedance

$$Z(s) = \frac{s^2 + 2s + 4}{s^2 + s + 1}$$

using the method described in this section. Then note that two resistances, two capacitances, and two inductances are required.

Problem 9.4.4 Combine the ideas of this section with those of Section 9.3 to derive a resistance-extraction reciprocal synthesis of a symmetric positive real $Z(s)$. The synthesis will use twice the minimum numbers of reactive and resistive elements [4, 5].

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10

Reciprocal Impedance Synthesis*

10.1 INTRODUCTION

In this chapter we shall continue our discussion of impedance synthesis techniques. More specifically, we shall be concentrating entirely on the problem of reciprocal or gyratorless passive synthesis.

In Section 9.4 a method of deriving a reciprocal synthesis was given. This method, however, requires excess numbers of both resistances and energy storage elements. In the sequel we shall be studying reciprocal synthesis methods that require only the minimum possible number of energy storage elements, and also a reciprocal synthesis method that requires only the minimum possible number of resistances.

A brief summary of the material in this chapter is as follows. In Section 10.2 we consider a simple technique for lossless reciprocal synthesis based on [1]. In Section 10.3 general reciprocal synthesis via reactance extraction is considered, while a synthesis via the resistance-extraction technique is given in Section 10.4. All methods use the minimum possible number of energy storage elements. In Section 10.5 we examine from the state-space viewpoint the classical Bayard synthesis. The synthesis procedure uses the minimum possible number of resistances and is based on the material in [2]. The results of Section 10.3 may be found in [3], with extensions in [4].

*This chapter may be omitted at a first reading.

10.2 RECIPROCAL LOSSLESS SYNTHESIS

We shall consider here the problem of generating a state-space synthesis of a prescribed $m \times m$ symmetric lossless positive real impedance $Z(s)$. Of course, $Z(s)$ must satisfy all the positive real constraints and also the constraint that $Z(s) = -Z'(-s)$.

We do not assume that any preliminary lossless section extractions, as considered in an earlier chapter, are carried out—for this would completely solve the problem! We shall however assume (with no loss of generality) that $Z(\infty) < \infty$. In fact, $Z(\infty)$ must then be zero. The reasoning is as follows: since $Z(s)$ is lossless,

$$Z(\infty) + Z'(-\infty) = 0$$

But the left-hand side is simply $Z(\infty) + Z'(\infty) = 2Z(\infty)$, by the symmetry of $Z(s)$. Hence

$$Z(\infty) = 0$$

We recall that the lossless positive real properties of $Z(s)$ guarantee the existence of a positive definite symmetric matrix P such that

$$\begin{aligned} PF + F'P &= 0 \\ PG &= H \end{aligned} \tag{10.2.1}$$

where $\{F, G, H\}$ is any minimal realization of $Z(s)$. With any T such that $T'T = P$, another minimal realization $\{F_1, G_1, H_1\}$ of $Z(s)$ defined by $\{TFT^{-1}, TG, (T^{-1})'H\}$ satisfies the equations

$$\begin{aligned} F_1 + F_1' &= 0 \\ G_1 &= H_1 \end{aligned} \tag{10.2.2}$$

Now the symmetry property of $Z(s)$ implies (see Chapter 7) the existence of a symmetric nonsingular matrix A uniquely defined by the minimal realization $\{F_1, G_1, H_1\}$ of $Z(s)$, such that

$$\begin{aligned} AF_1 &= F_1'A \\ AG_1 &= -H_1 \end{aligned} \tag{10.2.3}$$

We shall show that A possesses a decomposition

$$A = U'\Sigma U \tag{10.2.4}$$

where U is an orthogonal matrix and

$$\Sigma = [I_{n_1} \dotplus (-I_{n_2})] \tag{10.2.5}$$

with, of course, $n_1 + n_2 = n = \delta[Z(s)]$. To see this, observe that (10.2.2) and (10.2.3) yield $AF_1' = F_1A$ and $AH_1 = -G_1$, or

$$\begin{aligned} F_1'A^{-1} &= A^{-1}F_1 \\ A^{-1}G_1 &= -H_1 \end{aligned} \tag{10.2.6}$$

From (10.2.3) and (10.2.6) we see that both A and A^{-1} satisfy the equations $XF_1 = F_1'X$ and $XG_1 = -H_1$. Since the solution of these equations is unique, it follows that $A = A^{-1}$. This implies that $A^2 = I$, and consequently all eigenvalues of A must be $+1$ or -1. As A is symmetric also, there exists an orthogonal matrix U such that $A = U'\Sigma U$, where, by absorbing a permutation matrix in U, the diagonal matrix Σ with $+1$ and -1 entries only may be assumed to be $\Sigma = [I_{n_1} \dotplus (-1)I_{n_2}]$.

With the orthogonal matrix U defined above, we obtain another minimal realization of $Z(s)$ given by

$$\{F_2, G_2, H_2\} = \{UF_1U', UG_1, UH_1\} \tag{10.2.7}$$

It now follows from (10.2.2) and (10.2.3) that

$$\begin{aligned} F_2 + F_2' &= 0 \\ G_2 &= H_2 \\ \Sigma F_2 &= F_2'\Sigma \\ \Sigma G_2 &= -H_2 \end{aligned} \tag{10.2.8}$$

From the minimal realization $\{F_2, G_2, H_2\}$, a reciprocal synthesis for $Z(s)$ follows almost immediately; by virtue of (10.2.8), the constant matrix

$$M = \begin{bmatrix} 0 & -H_2' \\ G_2 & -F_2 \end{bmatrix} \tag{10.2.9}$$

satisfies

$$M + M' = 0 \qquad (I_m \dotplus \Sigma)M = M'(I_m \dotplus \Sigma) \tag{10.2.10}$$

The matrix M is therefore synthesizable as the hybrid matrix of a reciprocal lossless network, with the current-excited ports corresponding to the rows of $(I_m \dotplus \Sigma)$ that have a $+1$ on the diagonal, and the voltage-excited ports corresponding to those rows that have a -1 on the diagonal. Now because M satisfies (10.2.10), it must have the form

$$M = \begin{bmatrix} 0_{m+n_1} & M_{12} \\ -M'_{12} & 0_{n_2} \end{bmatrix} \tag{10.2.11}$$

We conclude that *the hybrid matrix M is realizable as a multiport transformer.*

A synthesis of $Z(s)$ of course follows from that for M by terminating the primary ports (other than the first m ports) of M in unit inductances, and all the secondary ports in unit capacitors.

Those familiar with classical synthesis procedures will recall that there are two standard lossless syntheses for lossless impedance functions—the Foster and Cauer syntheses [8], with multiport generalizations in [6]. The Foster synthesis can be obtained by taking in (10.2.8)

$$F_2 = \dotplus_i \begin{bmatrix} 0 & -\gamma_i \\ \gamma_i & 0 \end{bmatrix} \qquad G'_2 = H'_2 = [a_1 \quad 0 \quad a_2 \quad 0 \quad \cdots]$$

(It is straightforward to show the existence of a coordinate-basis change producing this structure of F_2, G_2, and H_2.) The Cauer synthesis can be obtained by using the preliminary extraction procedures of Chapter 8 to carry out the entire synthesis.

Problem 10.2.1. Synthesize the impedance matrix

$$Z(s) = \frac{1}{s}\begin{bmatrix} s^2+1 & s^2-3 & s^2+2 \\ s^2-3 & s^2+10 & s^2-7 \\ s^2+2 & s^2-7 & s^2+5 \end{bmatrix}$$

10.3 RECIPROCAL SYNTHESIS VIA REACTANCE EXTRACTION

The problem of synthesizing a prescribed $m \times m$ positive real symmetric impedance $Z(s)$ without gyrators is now considered. We shall assume, with no loss of generality, that $Z(\infty) < \infty$. The synthesis has the feature of always using the minimum possible number of reactive elements (inductors and capacitors); this number is precisely $n = \delta[Z(s)]$. Further, the synthesis is based on the technique of reactance extraction. As should be expected, the number of resistors used will generally exceed rank $[Z(s) + Z'(-s)]$.

We recall that the underlying problem is to obtain a state-space realization $\{F, G, H, J\}$ of $Z(s)$ such that the constant matrix

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} \tag{10.3.1}$$

has the properties

$$M + M' \geq 0 \tag{10.3.2}$$

$$[I_m + \Sigma]M = M'[I_m + \Sigma] \tag{10.3.3}$$

where Σ is a diagonal matrix with diagonal entries $+1$ or -1. Furthermore, if (10.3.1) through (10.3.3) hold with F possessing the *minimum* possible size $n \times n$, then a synthesis of $Z(s)$ is readily obtained that uses the minimum number n of inductors and capacitors.

The constant matrix M in (10.3.1) that satisfies (10.3.2) and (10.3.3) is the *hybrid* matrix of a nondynamic network; the current-excited ports are the first m ports together with those ports corresponding to diagonal entries of Σ with a value of $+1$, while the remaining ports are of course voltage-excited ports. On terminating the nondynamic network (which is always synthesizable) at the voltage-excited ports in 1-F capacitances and at all but the first m current-excited ports in 1-H inductances, the prescribed impedance $Z(s)$ is observed.

More precisely, the first condition, (10.3.2), says that the hybrid matrix M possesses a *passive* synthesis, while the second condition, (10.3.3), says that it possesses a *reciprocal* synthesis. Obviously, any matrix M that satisfies both conditions simultaneously possesses a synthesis that is passive and reciprocal. In Chapter 9 we learned that any minimal realization $\{F, G, H, J\}$ of $Z(s)$ that satisfies the (special) positive real lemma equations

$$\begin{aligned} F + F' &= -LL' \\ G &= H - LW_0 \\ J + J' &= W_0'W_0 \end{aligned} \tag{10.3.4}$$

results in M of (10.3.1) fulfilling (10.3.2). [In fact if M satisfies (10.3.2) for any $\{F, G, H, J\}$, then there exist L and W_0 satisfying (10.3.4).] The procedure required to derive $\{F, G, H, J\}$ satisfying (10.3.4) was also covered. We shall now derive, from a minimal realization of $Z(s)$ satisfying (10.3.4), another minimal realization $\{F_1, G_1, H_1, J\}$ for which both conditions (10.3.2) and (10.3.3) hold simultaneously. In essence, the reciprocal synthesis problem is then solved.

Suppose that $\{F, G, H, J\}$ is a state-space *minimal* realization of $Z(s)$ such that (10.3.4) holds. The symmetry (or reciprocity) property of $Z(s)$ guarantees the existence of a unique symmetric nonsingular matrix P with

$$\begin{aligned} PF &= F'P \\ PG &= -H \end{aligned} \tag{10.3.5}$$

As explained in Chapter 7 (see Theorem 7.4.3), *any* matrix T appearing in a decomposition of P of the form

$$P = T'\Sigma T \qquad \Sigma = [I_{n_1} \dotplus (-1)I_{n_2}] \tag{10.3.6}$$

generates another minimal realization $\{TFT^{-1}, TG, (T^{-1})'H, J\}$ of $Z(s)$ that satisfies the reciprocity condition (10.3.3). However, there is no guarantee that the passivity condition is still satisfied. We shall now exhibit a specific state-space basis change matrix T from among all possible matrices in the above decompositon of P such that in the new state-space basis, passivity is preserved.

The matrix T is described in Theorem 10.3.1; first, we require two preliminary lemmas.

Lemma 10.3.1. Let R be a real matrix similar to a real symmetric matrix S. Then S is positive definite (nonnegative definite) if $R + R'$ is positive definite (nonnegative definite), but not necessarily conversely.

The proof of the above lemma is straightforward, depending on the fact that matrices that are similar to one another possess the same eigenvalues, and the fact that a real symmetric matrix has real eigenvalues only. A formal proof to the lemma is requested in a problem.

Second, we have the following result:

Lemma 10.3.2. Let P be the symmetric nonsingular matrix satisfying (10.3.5). Then P can be represented as

$$P = BV\Sigma V' = V\Sigma V'B \tag{10.3.7}$$

where B is symmetric positive definite, V is orthogonal, and $\Sigma = [I_{n_1} \dotplus (-1)I_{n_2}]$. Further, $V\Sigma V'$ commutes with $B^{1/2}$, the unique positive definite square root of B; i.e.,

$$V\Sigma V'B^{1/2} = B^{1/2}V\Sigma V' \tag{10.3.8}$$

Proof. It is a standard result (see, e.g., [5]), that for a nonsingular matrix P, there exist unique matrices B and U, such that B is symmetric positive definite, U is orthogonal, and $P = BU$. From the symmetry of P,

$$BU = U'B = (U'BU)U'$$

This shows that B and $(U'BU)$ are both symmetric positive-definite square roots of the symmetric positive-definite matrix $K = PP' = P^2$. It follows from the uniqueness of positive-definite square roots that

$$B = U'BU$$

Hence, multiplying on the left by U,

$$UB = BU = P$$

Next, we shall show that the orthogonal matrix U has a decomposition $V\Sigma V'$. To do this, we need to note that U is symmetric. Because P is symmetric, $P = BU = U'B$. Also, $P = UB$. The nonsingularity of B therefore implies that $U = U'$. Because U is symmetric, it has real eigenvalues, and because it is orthogonal, it has eigenvalues with unity modulus. Hence the eigenvalues of U must be $+1$ or -1. Therefore, U may be written as

$$U = V\Sigma V'$$

where V is orthogonal and $\Sigma = [I_{n_1} \dotplus (-1)I_{n_2}]$. Now to prove (10.3.8), we proceed as follows. By premultiplying (10.3.7) by $\Sigma V'$ and postmultiplying by $V\Sigma$, we have

$$\Sigma V'BV = V'BV\Sigma$$

on using the fact that V is orthogonal and $\Sigma^2 = I$. The above relation shows that $V'BV$ commutes with Σ. Because $V'BV$ is symmetric, this means that $V'BV$ has the form

$$V'BV = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where B_1 and B_2 are symmetric positive-definite matrices of dimensions $n_1 \times n_1$ and $n_2 \times n_2$, respectively. Now $(V'B^{1/2}V)^2 = V'BV$ and $[B_1^{1/2} \dotplus B_2^{1/2}]^2 = [B_1 \dotplus B_2]$. By the uniqueness of positive-definite square roots, we have $V'B^{1/2}V = [B_1^{1/2} \dotplus B_2^{1/2}]$. Therefore

$$\Sigma V'B^{1/2}V = V'B^{1/2}V\Sigma$$

from which (10.3.8) is immediate. $\nabla\nabla\nabla$

With the above lemmas in hand and with all relevant quantities defined therein, we can now state the following theorem.

Theorem 10.3.1. Let $\{F, G, H, J\}$ be a minimal realization of a positive real symmetric $Z(s)$ satisfying the special positive real lemma equations (10.3.4). Let V, B, and Σ be defined as in Lemma 10.3.2, and define the nonsingular matrix T by

$$T = V'B^{1/2} \tag{10.3.9}$$

Then the minimal realization $\{F_1, G_1, H_1, J\} = \{TFT^{-1}, TG, (T^{-1})'H, J\}$ of $Z(s)$ is such that the hybrid matrix

$$M_1 = \begin{bmatrix} J & -H_1' \\ G_1 & -F_1 \end{bmatrix}$$

satisfies *the following* properties:

$$M_1 + M_1' \geq 0 \tag{10.3.10}$$

$$[I_m + \Sigma]M_1 = M_1'[I_m + \Sigma] \tag{10.3.11}$$

Proof. From the theorem statement, we have

$$\begin{aligned} T'\Sigma T &= B^{1/2}V\Sigma V'B^{1/2} \\ &= BV\Sigma V' \\ &= P \end{aligned} \tag{10.3.6}$$

The second and third equalities follow from Lemma 10.3.2.

It follows from the above equation by a simple direct calculation or by Theorem 7.4.3 that T defines a new minimal realization $\{F_1, G_1, H_1, J\}$ of $Z(s)$ such that

$$[I_m + \Sigma]M_1 = M_1'[I_m + \Sigma] \tag{10.3.11}$$

holds. It remains to check that (10.3.10) is also satisfied.

Recalling the definition of M as $\begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$, it is clear that

$$M_1 = [I \dotplus T]M[I \dotplus T^{-1}]$$

or, on rewriting using (10.3.9),

$$\begin{aligned} M_1 &= [I \dotplus V'B^{1/2}]M[I \dotplus B^{-1/2}V] \\ &= [I \dotplus V'B^{1/2}V][I \dotplus V']M[I \dotplus V][I \dotplus V'B^{-1/2}V] \\ &= [I \dotplus V'B^{1/2}V]M_0[I \dotplus V'B^{-1/2}V] \end{aligned} \tag{10.3.12}$$

where M_0 denotes $[I \dotplus V']M[I \dotplus V]$. From the fact that $M + M' \geq 0$, it follows readily that

$$M_0 + M_0' \geq 0 \tag{10.3.13}$$

Now in the proof of Lemma 10.3.2 it was established that $V'B^{1/2}V$ has the special form $[B_1^{1/2} \dotplus B_2^{1/2}]$, where $B_1^{1/2}$ and $B_2^{1/2}$ are symmetric positive-definite matrices of dimensions $n_1 \times n_1$ and $n_2 \times n_2$, respectively. For simplicity, let us define symmetric positive-definite matrices Q_1 and Q_2 of dimensions, respectively, $(m + n_1) \times (m + n_1)$ and $n_2 \times n_2$, by

$$Q_1 = [I_m \dotplus B_1^{1/2}]$$
$$Q_2 = B_2^{1/2}$$

and let M_0 be partitioned conformably as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

i.e., M_{11} is $(m + n_1) \times (m + n_1)$ and M_{22} is $n_2 \times n_2$. Then we can rewrite (10.3.12), after simple calculations, as

$$M_1 = \begin{bmatrix} Q_1^{-1}M_{11}Q_1 & Q_1^{-1}M_{12}Q_2 \\ Q_2^{-1}M_{21}Q_1 & Q_2^{-1}M_{22}Q_2 \end{bmatrix}$$

The proven symmetry property of $[I_m \dotplus \Sigma]M_1$ implies that the submatrices $Q_1^{-1}M_{11}Q_1$ and $Q_2^{-1}M_{22}Q_2$ are symmetric, while $Q_2^{-1}M_{21}Q_1 = -(Q_1^{-1}M_{12}Q_2)'$. Consequently

$$M_1 + M_1' = \begin{bmatrix} 2Q_1^{-1}M_{11}Q_1 & 0 \\ 0 & 2Q_2^{-1}M_{22}Q_2 \end{bmatrix}$$

Now M_{11} and M_{22} are principal submatrices of M_0, and so $M_{11} + M_{11}'$ and $M_{22} + M_{22}'$ are principal submatrices of $M_0 + M_0'$. By (10.3.13), $M_0 + M_0'$ is nonnegative definite, and so $M_{11} + M_{11}'$ and $M_{22} + M_{22}'$ are also nonnegative definite. On identifying the symmetric matrix $Q_1^{-1}M_{11}Q_1$ with S and the matrix M_{11} with R in Lemma 10.3.1, it follows by Lemma 10.3.1 that $Q_1^{-1}M_{11}Q_1$ is nonnegative definite. The same is true for $Q_2^{-1}M_{22}Q_2$. Hence

$$M_1 + M_1' \geq 0 \tag{10.3.10}$$

and the proof is now complete. $\nabla\nabla\nabla$

With Theorem 10.3.1 on hand, a passive reciprocal synthesis is readily obtained. The constant hybrid matrix M_1 is synthesizable with a network of multiport ideal transformers and positive resistors only, as explained in Chapter 9. A synthesis of the prescribed $Z(s)$ then follows by appropriately terminating this network in inductors and capacitors.

A brief summary of the synthesis procedure follows:

1. Find a minimal realization $\{F_0, G_0, H_0, J\}$ for the prescribed $Z(s)$, which is assumed to be positive real, symmetric, and such that $Z(\infty) < \infty$.
2. Solve the positive real lemma equations and compute another minimal realization $\{F, G, H, J\}$ satisfying (10.3.4).
3. Compute the matrix P for which $PF = F'P$ and $PG = -H$, and from P find matrices B, V, and Σ with properties as defined in Lemma 10.3.2.
4. With $T = V'B^{1/2}$, form another minimal realization of $Z(s)$ by $\{TFT^{-1}, TG, (T^{-1})'H, J\}$ and obtain the hybrid matrix M_1, given by

$$M_1 = \begin{bmatrix} J & -H'T^{-1} \\ TG & -TFT^{-1} \end{bmatrix}$$

5. Give a reciprocal passive synthesis for M_1; terminate appropriate ports of this network in unit inductors and unit capacitors to yield a synthesis of $Z(s)$.

The calculations involved in obtaining B, V, and Σ from P are straightforward. We note that B is simply the symmetric positive-definite square root of the matrix $PP' = P^2$; the matrix $B^{-1}P$ (or PB^{-1}), which is symmetric and orthogonal, can be easily decomposed as $V\Sigma V'$ [5].

Example 10.3.1 Let the symmetric positive real impedance matrix $Z(s)$ to be synthesized be

$$Z(s) = \frac{s^2 + s + 4}{s^2 + s + 1} = 1 + \frac{3}{s^2 + s + 1}$$

A minimal realization for $Z(s)$ satisfying (10.3.4) is given by

$$F = \begin{bmatrix} -0.5 & 1.5 \\ -0.5 & -0.5 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \quad H = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \quad J = [1]$$

The solution P of the equations $PF = F'P$ and $PG = -H$ is easily computed to be

$$P = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

for which B, V, and Σ are

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad V = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The required state-space basis change $T = V'B^{1/2}$ is then

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

yielding another minimal realization $\{F_1, G_1, H_1, J\}$ of $Z(s)$:

$$F_1 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \quad G_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad H_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad J = [1]$$

We then form the hybrid matrix

$$M_1 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

It is easy to see that $M_1 + M_1'$ is nonnegative definite, and $(I \dotplus \Sigma)M_1$ is symmetric. A synthesis of M_1 is shown in Fig. 10.3.1; a synthesis of $Z(s)$ is obtained by terminating ports 2 and 3 of this network in a unit inductance and unit capacitance, respectively, as shown in Fig. 10.3.2.

The reader will have noted that the general theme of the synthesis given in this section is to start with a realization of $Z(s)$ from which a passive synthesis results, and construct from it a realization from which a passive *and* reciprocal synthesis results. One can also proceed (see [3] and Problem

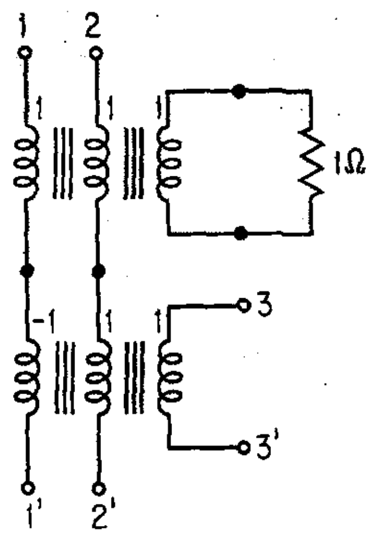

FIGURE 10.3.1. A Synthesis of the Hybrid Matrix M_1.

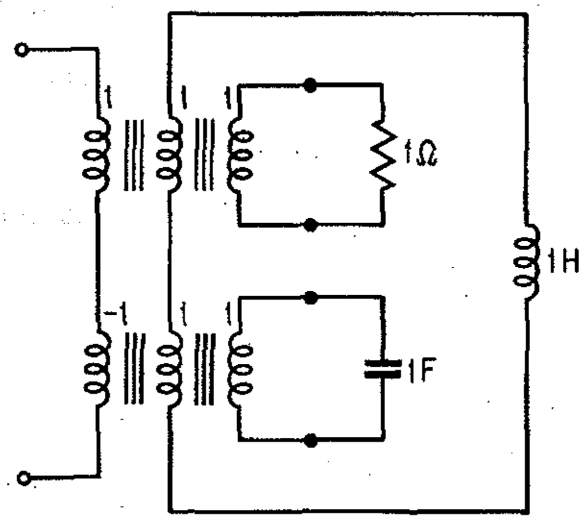

FIGURE 10.3.2. A Synthesis of $Z(s)$.

10.3.5) by starting with a realization from which a reciprocal, but not necessarily passive, synthesis results, and constructing from it a realization from which a reciprocal and passive synthesis results. The result of Problem 10.3.1 actually unifies both procedures, which at first glance look quite distinct; [4] should be consulted for further details.

Problem 10.3.1 Let $\{F, G, H, J\}$ be an arbitrary minimal realization of a symmetric positive real matrix. Let Q be a positive-definite symmetric matrix satisfying the positive real lemma equations and let P be the unique symmetric nonsingular matrix satisfying the reciprocity equation, i.e., Eqs. (10.3.5). Prove that if a nonsingular matrix T can be found such that

$$T'\Sigma T = P \qquad \text{with } \Sigma = [I_{n_1} \dotplus (-1)I_{n_2}]$$

and

$$T'\Psi T = Q$$

for some arbitrary symmetric positive definite matrix Ψ having the form $[\Psi_1 \dotplus \Psi_2]$, where Ψ_1 is $n_1 \times n_1$ and Ψ_2 is $n_2 \times n_2$, then the hybrid matrix M formed from the minimal realization $\{TFT^{-1}, TG, (T^{-1})'H, J\}$ satisfies the equations

$$M + M' \geq 0$$

and

$$[I \dotplus \Sigma]M = M'[I \dotplus \Sigma]$$

Note that this problem is concerned with proving a theorem offering a more general approach to reciprocal synthesis than that given in the text; an effective computational procedure to accompany the theorem is suggested in Problem 10.3.8.

Problem 10.3.2 Show that Theorem 10.3.1 is a special case of the result given in Problem 10.3.1.

Problem 10.3.3 Prove Lemma 10.3.1.

Problem 10.3.4 Synthesize the following impedance matrices using the method of this section:

(a) $Z(s) = \dfrac{s^2 + 4s + 9}{s^2 + 2s + 1}$

(b) $Z(s) = \begin{bmatrix} \dfrac{3(s^2 + 4)}{s^2 + 6s + 4} & \dfrac{2s^2 + 4}{s^2 + 6s + 4} \\ \dfrac{2s^2 + 4}{s^2 + 6s + 4} & \dfrac{2s^2 + 6s + 4}{s^2 + 6s + 4} \end{bmatrix}$

Problem 10.3.5 Let $\{F, G, H, J\}$ be a minimal realization of an $m \times m$ symmetric positive real $Z(s)$ such that

$$\Sigma F = F'\Sigma \qquad \Sigma G = -H$$

with $\Sigma = [I_{n_1} \dotplus (-1)I_{n_2}]$. Suppose that P is a positive-definite symmetric matrix satisfying the positive real lemma equations for the above minimal realization, and let P be partitioned conformably as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P'_{12} & P_{22} \end{bmatrix}$$

where P_{11} is $n_1 \times n_1$ and P_{22} is $n_2 \times n_2$. Define now an $n_1 \times n_2$ matrix Q as any solution of the quadratic equation

$$QP'_{12}Q - P_{11}Q - QP_{22} + P_{12} = 0$$

with the extra property that $I - Q'Q$ is nonsingular. (It is possible to demonstrate the existence of such Q.)

Show that, with

$$T = \begin{bmatrix} (I_{n_1} - QQ')^{-1/2} & (I_{n_1} - QQ')^{-1/2}Q \\ (I_{n_2} - Q'Q)^{-1/2}Q' & (I_{n_2} - Q'Q)^{-1/2} \end{bmatrix}$$

then the minimal realization $\{F_1, G_1, H_1, J\} = \{TFT^{-1}, TG, (T^{-1})'H, J\}$ of $Z(s)$ is such that the hybrid matrix

$$M_1 = \begin{bmatrix} J & -H'_1 \\ G_1 & -F_1 \end{bmatrix}$$

satisfies the following properties:

$$M_1 + M_1' \geq 0$$
$$[I_m + \Sigma]M_1 = M_1'[I_m + \Sigma]$$

(*Hint:* Use the result of Problem 10.3.1.)

Can you give a computational procedure to solve the quadratic equation for a particular Q with the stated property? (See [3, 4] for further details.)

Problem 10.3.6 Suppose that it is known that there exists a synthesis of a prescribed symmetric positive real $Z(s)$ using resistors and inductors only. Explain what simplifications result in the synthesis procedure of this section.

Problem 10.3.7 Let $Z(s)$ be a symmetric-positive-real matrix with minimal realization $\{F, G, H, J\}$. [Assume that $Z(\infty) < \infty$.] Show that the number of inductors and capacitors in a minimal reactive reciprocal synthesis of $Z(s)$ can be determined by examining the rank and signature of the symmetric Hankel matrix

$$\begin{bmatrix} H'G & H'FG & \cdots & H'F^{n-1}G \\ H'FG & H'F^2G & \cdots & H'F^nG \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ H'F^{n-1}G & H'F^nG & \cdots & H'F^{2n-2}G \end{bmatrix}$$

Problem 10.3.8 Show that the following procedure will determine a matrix T satisfying the equations of Problem 10.3.1. Let S_1 be such that $Q = S_1'IS_1$, and let S_2 be an orthogonal matrix such that $S_2(S_1')^{-1}PS_1'S_2'$ is diagonal. Set $S = S_2S_1$. Then S will simultaneously diagonalize P and Q. From S and the diagonalized P, find T. Discuss how all T might be found.

Problem 10.3.9 Let $Z(s)$ be a symmetric positive real matrix with minimal realization $\{F, G, H, J\}$. Suppose that $\Sigma F = F'\Sigma$ and $\Sigma G = -H$ where Σ has the usual meaning. Suppose also that $R = 2J$ is nonsingular. Let P be the minimum solution of

$$PF + F'P = -(PG - H)R^{-1}(PG - H)'$$

Show that the maximum solution is $(\Sigma P \Sigma)^{-1}$.

10.4 RECIPROCAL SYNTHESIS VIA RESISTANCE EXTRACTION

In the last section a technique yielding a reciprocal synthesis for a prescribed symmetric positive real impedance matrix was considered, the synthesizing network being viewed as a nondynamic reciprocal network terminated in inductors and capacitors. Another technique—the resistance-extraction technique—for achieving a reciprocal synthesis will be dealt with in this section. The synthesizing network is now thought of as a reciprocal lossless network terminated in positive resistors. The synthesis also uses a minimal possible number of reactive elements.

With the assistance of the method given in the last section, we take as our starting point the knowledge of a minimal realization $\{F, G, H, J\}$ of $Z(s)$ with all the properties stated in Theorem 10.3.1; i.e.,

$$\begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} + \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}' \geq 0 \tag{10.4.1}$$

and

$$\begin{bmatrix} I & 0 \\ 0 & \Sigma \end{bmatrix}\begin{bmatrix} J & -H' \\ G & -F \end{bmatrix} = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}'\begin{bmatrix} I & 0 \\ 0 & \Sigma \end{bmatrix} \tag{10.4.2}$$

Suppose that Σ is $[I_{n_1} \dotplus (-1)I_{n_2}]$. Let us partition the matrices F, G, and H conformably as shown below:

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \qquad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \qquad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$

Hence, F_{11} is $n_1 \times n_1$, F_{22} is $n_2 \times n_2$, G_1 and H_1 are $n_1 \times m$, and G_2 and H_2 are $n_2 \times m$. Then (10.4.2) may be easily seen to imply that

$$J = J' \quad G_1 = -H_1 \quad G_2 = H_2 \quad F_{11} = F'_{11} \quad F_{22} = F'_{22} \quad F_{21} = -F'_{12} \tag{10.4.3}$$

On substituting the above equalities into (10.4.1), we have

$$\begin{bmatrix} 2J & 2G'_1 & 0 \\ 2G_1 & -2F_{11} & 0 \\ 0 & 0 & -2F_{22} \end{bmatrix} \geq 0$$

Hence the principal submatrices $\begin{bmatrix} 2J & 2G'_1 \\ 2G_1 & -2F_{11} \end{bmatrix}$ and $[-2F_{22}]$ are individually nonnegative definite. Therefore, there exists an $(m + n_1) \times r_1$ matrix R and an $n_2 \times r_2$ matrix S, where r_1 is the rank of $\begin{bmatrix} 2J & 2G'_1 \\ 2G_1 & -2F_{11} \end{bmatrix}$ and r_2 is the rank of $[-2F_{22}]$, such that

$$\begin{bmatrix} 2J & 2G'_1 \\ 2G_1 & -2F_{11} \end{bmatrix} = RR' \tag{10.4.4}$$

$$[-2F_{22}] = SS' \tag{10.4.5}$$

If we partition the matrix R as

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$$

with R_1 $m \times r_1$ and R_2 $n_1 \times r_1$, then (10.4.4) reduces to

$$2J = R_1 R_1' \qquad 2G_1 = R_2 R_1' \qquad -2F_{11} = R_2 R_2' \tag{10.4.6}$$

With these preliminaries and definitions, we now state the following theorem, which, although it is not obvious from the theorem statement, is the crux of the synthesis procedure.

Theorem 10.4.1. Let $Z(s)$ be an $m \times m$ symmetric positive real impedance matrix with $Z(\infty) < \infty$. Let $\{F, G, H, J\}$ be a minimal realization of $Z(s)$ such that (10.4.1) and (10.4.2) hold. Then the following state-space equations define an $(m + r)$-port lossless network:

$$\dot{x} = \frac{1}{2}(F - F')x + \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}R_2 & 0 \\ G_2 & 0 & \frac{1}{\sqrt{2}}S \end{bmatrix} \begin{bmatrix} u \\ u_1 \end{bmatrix}$$

$$\begin{bmatrix} y \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & G_2' \\ \frac{1}{\sqrt{2}}R_2' & 0 \\ 0 & \frac{1}{\sqrt{2}}S' \end{bmatrix} x + \begin{bmatrix} 0 & \frac{R_1}{\sqrt{2}} & 0 \\ \frac{-R_1'}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ u_1 \end{bmatrix} \tag{10.4.7}$$

[Here x is an n vector, u and y are m vectors, and u_1 and y_1 are vectors of dimension $r = r_1 + r_2$.] Moreover, if the last r ports of the lossless network are terminated in unit resistors, corresponding to setting

$$u_1 = -y_1 \tag{10.4.8}$$

then the prescribed impedance $Z(s)$ will be observed at the remaining m ports.

Proof. We shall establish the second claim first. From (10.4.8) and the second equation in (10.4.7), we have

$$y_1 = \begin{bmatrix} \frac{1}{\sqrt{2}}R_2' & 0 \\ 0 & \frac{1}{\sqrt{2}}S' \end{bmatrix} x + \begin{bmatrix} \frac{-R_1'}{\sqrt{2}} \\ 0 \end{bmatrix} u = -u_1$$

Substituting the above into (10.4.7) yields

$$\dot{x} = \frac{1}{2}(F - F')x + \begin{bmatrix} 0 \\ G_2 \end{bmatrix} u - \begin{bmatrix} \frac{1}{2}R_2R_2' & 0 \\ 0 & \frac{1}{2}SS' \end{bmatrix} x + \begin{bmatrix} \frac{1}{2}R_2R_1' \\ 0 \end{bmatrix} u$$

$$y = [0 \quad G_2']x - \begin{bmatrix} \frac{1}{2}R_1R_2' & 0 \end{bmatrix} x + \begin{bmatrix} \frac{1}{2}R_1R_1' \end{bmatrix} u$$

or, on using Eqs. (10.4.6) and collecting like terms

$$\dot{x} = \left\{ \frac{1}{2}(F - F') + \begin{bmatrix} F_{11} & 0 \\ 0 & F_{22} \end{bmatrix} \right\} x + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} u$$

$$y = [-G_1' \quad G_2']x + Ju$$

Using now (10.4.3), it is easily seen that the above equations are the same as

$$\dot{x} = Fx + Gu$$

$$y = H'x + Ju$$

and evidently the transfer-function matrix relating $U(s)$ to $Y(s)$ is $J + H'(sI - F)^{-1}G$, which is the prescribed $Z(s)$.

Next we prove the first claim, viz., that the state-space equations (10.4.7) define a lossless network. We observe that the transfer-function matrix relating the Laplace transforms of input variables to those of output variables has a state-space realization, in fact a minimal realization, $\{F_L, G_L, H_L, J_L\}$ given by

$$F_L = \frac{1}{2}(F - F') = \begin{bmatrix} 0 & F_{12} \\ -F_{12}' & 0 \end{bmatrix}$$

$$G_L = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}R_2 & 0 \\ G_2 & 0 & \frac{1}{\sqrt{2}}S \end{bmatrix} = H_L$$

$$J_L = \begin{bmatrix} 0 & \frac{R_1}{\sqrt{2}} & 0 \\ -\frac{R_1'}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly then,

$$F_L + F'_L = 0$$
$$G_L = H_L$$
$$J_L + J'_L = 0$$

Since these are the equations of the lossless positive real lemma with P the identity matrix I, (10.4.7) defines a lossless network.

∇ ∇ ∇

With the above theorem, we have on hand therefore a passive synthesis of $Z(s)$, provided we can give a lossless synthesis of (10.4.7). Observe that though u is constrained to be a vector of currents by the fact that $Z(s)$ is an impedance matrix (with y being the corresponding vector of voltages), u_1 however can have any element either a current or voltage. [Irrespective of the definition of u_1, (10.4.8) always holds and (10.4.7) still satisfies the lossless positive real lemma.] Thus there are many passive syntheses of $Z(s)$ corresponding to different assignments of the variables u_1 (and y_1). [*Note:* A lossless network synthesizing (10.4.7) may be obtained using any known methods, for example, by the various classical methods described in [6] or by the reactance-extraction method considered earlier.]

By making an appropriate assignment of variables, we shall now exhibit a particular lossless network that is reciprocal. In effect, we are thus presenting a reciprocal synthesis of $Z(s)$.

Consider the constant matrix

$$M = \begin{bmatrix} J_L & -H'_L \\ G_L & -F_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{R_1}{\sqrt{2}} & 0 & 0 & -G'_2 \\ -\frac{R'_1}{\sqrt{2}} & 0 & 0 & -\frac{R'_2}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{S'}{\sqrt{2}} \\ 0 & \frac{R_2}{\sqrt{2}} & 0 & 0 & -F_{12} \\ G_2 & 0 & \frac{S}{\sqrt{2}} & F'_{12} & 0 \end{bmatrix} \tag{10.4.9}$$

It is evident on inspection that M is skew, or

$$M + M' = 0$$

and ΣM is symmetric with

$$\Sigma = [I_m \dotplus (-1)I_{r_1} \dotplus I_{r_2} \dotplus I_{n_1} \dotplus (-1)I_{n_2}]$$

These equations imply that if M is thought of as a hybrid matrix, where ports

$m + 1$ to $m + r_1$ and the last n_2 ports are voltage-excited ports with all others being current-excited ports, then M has a nondynamic lossless synthesis using reciprocal elements only. In addition, termination of this synthesis of M at the first n_1 of the last n ports in unit inductances and the remaining n_2 of the last n ports in unit capacitances yields a reciprocal lossless synthesis of the state-space equations (10.4.7) of Theorem 10.4.1, in which the first r_1 entries of u_1 are voltages, while the last r_2 entries are currents. Taking the argument one step further, it follows from Theorem 10.4.1 that termination of this reciprocal lossless network at its last r ports in unit resistances then yields a reciprocal synthesis of $Z(s)$.

We may combine the above result with those contained in Theorem 10.4.1 to give a *reciprocal synthesis via resistance extraction.*

> **Theorem 10.4.2.** With the quantities as defined in Theorem 10.4.1, and with the first r_1 entries of the excitation r vector u_1 defined to be voltages and the last r_2 entries defined to be currents, the state-space equations of (10.4.7) define an $(m + r)$-port lossless network that is reciprocal.

As already noted, the reciprocal lossless network defined in the above theorem may be synthesized by many methods. Before closing this section, we shall note one procedure. Suppose that the reciprocal lossless network is to be obtained via the reactance-extraction technique; then the synthesis rests ultimately on a synthesis of the constant hybrid matrix M of (10.4.9). Now the hybrid matrix M has a synthesis that is nondynamic and lossless, as well as reciprocal. Consequently, the network synthesizing M must be a multiport ideal transformer. This becomes immediately apparent on permuting the rows and columns of M so that the current-excited ports are grouped together and the voltage-excited ports together:

$$M_p = \left[\begin{array}{ccc|cc} & & & \frac{R_1}{\sqrt{2}} & -G_2' \\ & 0 & & 0 & -\frac{S'}{\sqrt{2}} \\ & & & \frac{R_2}{\sqrt{2}} & -F_{12} \\ \hline -\frac{R_1'}{\sqrt{2}} & 0 & -\frac{R_2'}{\sqrt{2}} & & \\ G_2 & \frac{S}{\sqrt{2}} & F_{12}' & & 0 \end{array}\right]$$

Evidently M_p is realizable as an $(m + r_2 + n_1) \times (r_1 + n_2)$-port transformer with an $(r_1 + n_2) \times (m + r_2 + n_1)$ turns-ratio matrix of

$$T_c = \begin{bmatrix} \dfrac{R'_1}{\sqrt{2}} & 0 & \dfrac{R'_2}{\sqrt{2}} \\ -G_2 & -\dfrac{S}{\sqrt{2}} & -F'_{12} \end{bmatrix} \tag{10.4.10}$$

The above argument shows that a synthesis of a symmetric positive real $Z(s)$ using passive and reciprocal elements may be achieved simply by terminating an $(m + r_2 + n_1) \times (r_1 + n_2)$-port transformer of turns-ratio matrix T_c in (10.4.10) at the last n_2 secondary ports in units capacitances, at the remaining r_1 secondary ports in unit resistances, at the last n_1 primary ports in unit inductances, and at all but the first m of the remaining primary ports in r_2 unit resistances.

Let us illustrate the above approach with the following simple example.

Example 10.4.1 Consider the (symmetric) positive real impedance

$$Z(s) = \frac{s^2 + 2s + 9}{8s^2 + 16s + 8} = \frac{1}{8} + \frac{1}{s^2 + 2s + 1}$$

A minimal state-space realization of $Z(s)$ that satisfies the conditions in (10.4.1) and (10.4.2) with $\Sigma = [1 \dotplus -1]$ is

$$F = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \quad G = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad H = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad J = [\tfrac{1}{8}]$$

as may be easily checked. Now

$$\begin{bmatrix} 2J & 2G'_1 \\ 2G_1 & -2F_{11} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} [\tfrac{1}{2} \quad 2] = RR'$$

$$[-2F_{22}] = 0 = SS'$$

Hence

$$R = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix} \qquad S = 0$$

and the 2×3 turns-ratio matrix T_c in (10.4.10) is therefore

$$T_c = \begin{bmatrix} \dfrac{1}{2\sqrt{2}} & 0 & \sqrt{2} \\ -\dfrac{1}{2} & 0 & 1 \end{bmatrix}$$

Note that the second column of T_c can in fact be deleted; for the moment we retain it to illustrate more effectively the synthesis procedure. A multiport transformer realizing T_c is shown in Fig. 10.4.1, and with appropriate terminations at certain primary ports and all the secondary ports the network of Fig. 10.4.2 yields a synthesis of $Z(s)$.

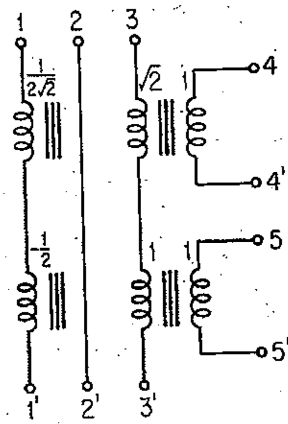

FIGURE 10.4.1. A Multiport Transformer Realizing T_c of Example 10.4.1.

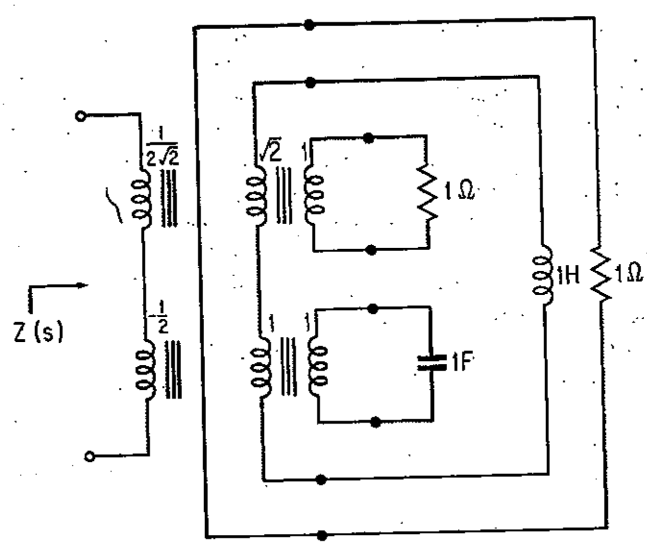

FIGURE 10.4.2. A Synthesis of $Z(s)$ for Example 10.4.1.

It is evident that one resistor is unnecessary and the circuit in Fig. 10.4.2 may be reduced to the one shown in Fig. 10.4.3. By inspection, we see that the synthesis uses the minimal number of reactive elements as predicted, and, in this particular example, the minimal number of resistances as well.

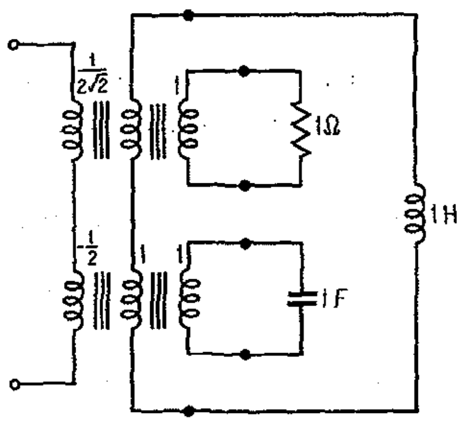

FIGURE 10.4.3. A Simplified Version of the Network of Fig. 10.4.2.

This concludes our discussion of reciprocal synthesis using a minimal number of reactive elements. The reader with a background in classical synthesis will have perceived that one famous classical synthesis, the Brune synthesis [8], has not been covered. It is at present an open problem as to how this can be obtained via state-space methods. It is possible to analyze a network resulting from a Brune synthesis and derive state-space equations via the analysis procedures presented much earlier; the associated matrices F, G, H, and J satisfy (10.4.1) and (10.4.2) and have additional structure. The synthesis problem is of course to find a quadruple $\{F, G, H, J\}$ satisfying (10.4.1) and (10.4.2) and possessing this structure.

Another famous classical synthesis procedure is the reciprocal Darlington synthesis. Its multiport generalization, the Bayard synthesis, is covered in the next section.

Problem 10.4.1 Consider the positive real impedance function

$$Z(s) = \frac{s^2 + 4s + 9}{s^2 + 2s + 1}$$

Using the method of the last section, compute a realization of $Z(s)$ satisfying (10.4.1) and (10.4.2). Then form the state-space equations defined in Theorem 10.4.1, and verify that on setting $u_1 = -y_1$, the resulting state-space equations have a transfer-function matrix that is $Z(s)$.

Problem 10.4.2 With the same impedance function as given in Problem 10.4.1, and with the state-space equations defined in Theorem 10.4.1, give all possible syntheses of $Z(s)$ (corresponding to differing assignments for the variables u_1 and y_1), including the particular reciprocal synthesis given in the text. Compare the various syntheses obtained.

10.5 A STATE-SPACE BAYARD SYNTHESIS*

In earlier sections of this chapter, reciprocal synthesis methods were considered resulting in syntheses for a prescribed symmetric positive real impedance matrix using the minimum possible number of energy storage elements. The number of resistors required varies however from case to case and often exceeds the minimum possible. Of course the lower limit on the number of resistors necessary in a synthesis of $Z(s)$ is well defined and is given by the normal rank of $Z(s) + Z'(-s)$.

A well-known classical reciprocal synthesis technique that requires the minimum number of resistors is the Bayard synthesis (see, e.g., [6, 7]), the basis of which is the *Gauss factorization procedure*. We shall present in this section a state-space version of the Bayard synthesis method, culled from [2].

Minimality of the number of resistors and absence of gyrators in general means that a nonminimal number of reactive elements are used in the synthesis. This means that the state-space equations encountered in the synthesis may be nonminimal, and, accordingly, many of the powerful theorems of linear system theory are not directly applicable. This is undoubtedly a complicating factor, making the description of the synthesis procedure more complex than any considered hitherto.

As remarked, an essential part of the Bayard synthesis is the application of the Gauss factorization procedure, of which a detailed discussion may be found in [6]. Here, we shall content ourselves merely with quoting the result.

Statement of the Gauss Factorization Procedure

Suppose that $Z(s)$ is positive real. As we know, there exist an infinity of matrix spectral factors $W(s)$ of real rational functions such that

$$Z(s) + Z'(-s) = W'(-s)W(s) \tag{10.5.1}$$

Of the spectral factors $W(s)$ derivable by the procedures so far given, almost all have the property that $W(s)$ has minimal degree; i.e., $\delta[W(\cdot)] = \delta[Z(\cdot)]$. The Gauss factorization procedure in general yields a *nonminimal degree* $W(s)$. But one attractive feature of the Gauss factorization procedure is the ease with which $W(s)$ may be found.

*This section may be omitted at a first reading of this chapter.

The Gauss Factorization. Let $Z(s)$ be an $m \times m$ positive real matrix; suppose that no element of $Z(s)$ possesses a purely imaginary pole, and that $Z(s) + Z'(-s)$ has rank r almost everywhere. Then there exists an $r \times r$ diagonal matrix N_1 of real Hurwitz* polynomials, and an $m \times r$ matrix N_2 of real polynomials such that

$$Z(s) + Z'(-s) = N_2(-s)[N_1(-s)]^{-1}[N_1(s)]^{-1}N_2'(s) \qquad (10.5.2)$$

Moreover, if $Z(s) = Z'(s)$, there exist N_1 and N_2 as defined above with, also, $N_2(s) = N_2(-s)$. The matrices $N_1(s)$ and $N_2(s)$ are computable via a technique explained in [6].†

Observe that there is no claim that N_1 and N_2 are unique; indeed, they are not, and neither is the spectral factor

$$W(s) = [N_1(s)]^{-1}N_2'(s) \qquad (10.5.3)$$

In the remainder of the section we discuss how the spectral factor $W(s)$ in (10.5.3) can be used in developing a state-space synthesis of $Z(s)$. The discussion falls into two parts:

1. The construction of a state-space realization (which is generally nonminimal) of $Z(s)$ from a state-space realization of $W(s)$. This realization of $Z(s)$ yields readily a nonreciprocal synthesis of $Z(s)$ that uses a minimum number of resistors.
2. An interpretation is given in state-space terms of the constraint in the Gauss factorization procedure owing to the symmetry of $Z(s)$. An orthogonal matrix is derived that takes the state-space realization of $Z(s)$ found in part 1 into a new state-space basis from which a reciprocal synthesis of $Z(s)$ results. The synthesis of course uses the minimum number of resistors.

Before proceeding to consider step 1 above, we first note that the Gauss factorization requires that no element of $Z(s)$ possesses a purely imaginary pole; later when we consider step 2, it will be further required that $Z(\infty)$ be nonsingular. Of course, these requirements, as we have explained earlier, are inessential in that $Z(s)$ may always be assumed to possess these properties without any loss of generality.

*For our purposes, a Hurwitz polynomial will be defined as one for which all zeros have negative real parts.

†The examples and problems of this section will involve matrices $N_1(\cdot)$ and $N_2(\cdot)$ that can be computed without need for the formal technique of [6].

Construction of a State-Space Realization of $Z(s)$ via the Gauss Spectral Factorization

Suppose that $Z(s)$ is an $m \times m$ positive real impedance matrix to be synthesized, such that no element possesses a purely imaginary pole. Constraints requiring $Z(s)$ to be symmetric or $Z(\infty)$ to be nonsingular will not be imposed for the present. Assuming that $Z(s) + Z'(-s)$ has normal rank r, then by the use of the Gauss factorization procedure, we may assume that we have on hand a diagonal $r \times r$ matrix of Hurwitz polynomials $N_1(s)$ and an $m \times r$ matrix of polynomials $N_2(s)$ such that a spectral factor $W(s)$ of

$$Z(s) + Z'(-s) = W'(-s)W(s) \tag{10.5.1}$$

is

$$W(s) = [N_1(s)]^{-1}N_2'(s) \tag{10.5.3}$$

The construction of a state-space realization of $Z(s)$ will be preceded by construction of a realization of $W(s)$. Thus matrices $F, G, L,$ and W_0 are determined, in a way we shall describe, such that

$$W'(s) = N_2(s)[N_1(s)]^{-1} = W_0' + G'(sI - F')^{-1}L \tag{10.5.4}$$

[Subsequently, F and G will be used in a realization of $Z(s)$.]

The calculation of W_0 is straightforward by setting $s = \infty$ in (10.5.3). Of course, $W_0 = W(\infty) < \infty$ since $Z(\infty) < \infty$. The matrices F and L will be determined next. We note that

$$[N_1(s)]^{-1} = \operatorname{diag}\left\{\frac{1}{p_1(s)}, \frac{1}{p_2(s)}, \dots, \frac{1}{p_r(s)}\right\}$$

where each $p_i(s)$ is a Hurwitz polynomial. *Define $\hat{F}_i'$ to be the companion matrix with $p_i(s)$ as characteristic polynomial;* i.e.,

$$\hat{F}_i' = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & & 1 \\ -p_{ir_i} & & \cdots & & -p_{i1} \end{bmatrix} \qquad p_i(s) = s^{r_i} + \sum_{j=1}^{r_i} p_{ij}s^{r_i - j}$$

and define $\hat{l}_i = [0\ 0 \cdots 0\ 1]'$. As we know, $[\hat{F}_i', \hat{l}_i]$ is completely controllable for each i; further,

$$(sI - \hat{F}'_i)^{-1}\hat{l}_i = \frac{1}{p_i(s)}[1 \quad s \quad \cdots \quad s^{r_i-1}]'$$

as may be readily checked. Define now $\hat{F}$ and $\hat{L}$ as

$$\hat{F}' = \begin{bmatrix} \hat{F}'_1 & 0 & \cdots & 0 \\ 0 & \hat{F}'_2 & \cdots & \cdot \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{F}'_r \end{bmatrix} \qquad \hat{L} = \begin{bmatrix} \hat{l}_1 & 0 & \cdots & 0 \\ 0 & \hat{l}_2 & \cdots & \cdot \\ & & \ddots & \vdots \\ 0 & & \cdots & \hat{l}_r \end{bmatrix}$$

Obviously, the pair $[\hat{F}', \hat{L}]$ is completely controllable because each $[\hat{F}'_i, \hat{l}_i]$ is.

Having fixed the matrices $\hat{F}$ and $\hat{L}$, we now seek a matrix $\hat{G}$ such that

$$\hat{G}'(sI - \hat{F}')^{-1}\hat{L} = W'(s) - W'_0$$

Subsequently, we shall use the triple $\{\hat{F}, \hat{G}, \hat{L}\}$ to define the desired triple $\{F, G, L\}$ satisfying (10.5.4). From (10.5.3) we observe that the (i, j)th entry of $V(s) = W'(s)$ is

$$v_{ij}(s) = \frac{n_{2ij}(s)}{p_j(s)} = v_{0ij} + \frac{\hat{n}_{ij}(s)}{p_j(s)} \tag{10.5.5}$$

where $n_{2ij}(s)$ is the (i, j)th entry of $N_2(s)$ of (10.5.3), v_{0ij} is $v_{ij}(\infty)$, and the degree of $\hat{n}_{ij}(s)$ is less than the degree of $p_j(s)$ with $\hat{n}_{ij}(s)$ the residue of the polynomial $n_{2ij}(s)$ modulo $p_j(s)$. Define $\hat{\mathbf{n}}'_{ij}$ to be *the row vector obtained from the coefficients of $\hat{n}_{ij}(s)$, arranged in ascending power of s.* Then because $(sI - \hat{F}')^{-1}\hat{L}$ can be checked to have the obvious form

$$\begin{bmatrix} \dfrac{1}{p_1(s)}\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{r_1-1} \end{bmatrix} & 0 & 0 & \cdots \\ 0 & \dfrac{1}{p_2(s)}\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{r_2-1} \end{bmatrix} & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

it follows that

$$\hat{G}' = \begin{bmatrix} \hat{\mathbf{n}}'_{11} & \hat{\mathbf{n}}'_{12} & \cdots & \hat{\mathbf{n}}'_{1r} \\ \hat{\mathbf{n}}'_{21} & \hat{\mathbf{n}}'_{22} & \cdots & \hat{\mathbf{n}}'_{2r} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \hat{\mathbf{n}}'_{m1} & \hat{\mathbf{n}}'_{m2} & \cdots & \hat{\mathbf{n}}'_{mr} \end{bmatrix}$$

A few words of caution are appropriate here. One should bear in mind that in constructing $\hat{F}'$ and $\hat{L}$, *if there exists a $p_i(s)$ that is simply a constant, then the corresponding $\hat{F}'_i$ block must evanesce,* so that the rows and columns corresponding to the block $\hat{F}'_i$ in the $\hat{F}'$ matrix must be missing. Correspondingly, the associated rows in the $\hat{L}$ matrix occupied by the vector $\hat{l}_i$ will evanesce, but the associated single column of zeros must remain in order to give the correct dimension, because if $\hat{F}'$ is $n \times n$, then $\hat{L}$ is $n \times r$, where r is the normal rank of $Z(s) + Z'(-s)$. Consequently, the number of columns in $\hat{L}$ is always r. The matrix $\hat{G}'$ will have $\hat{\mathbf{n}}'_{qi}$ zero for all q, and the corresponding columns will also evanesce because $\hat{G}'$ has n columns.

The matrices $\hat{F}$, $\hat{G}$, and $\hat{L}$ defined above constitute a realization of $W(s) - W_0$. We shall however derive another realization via a simple state-space basis change. For each i, define P_i as the symmetric-positive-definite solution of $P_i\hat{F}_i + \hat{F}'_iP_i = -\hat{l}_i\hat{l}'_i$. Of course, such a P_i always exists because the eigenvalues of $\hat{F}_i$ lie in $\text{Re}\,[s] < 0$ [*i.e.*, the characteristic polynomial $p_i(s)$ of $\hat{F}_i$ is Hurwitz], and the pair $[\hat{F}'_i, \hat{l}_i]$ is completely controllable. Take T_i to be any square matrix such that $T'_iT_i = P_i$, and define T as the direct sum of the T_i. With the definitions $F_i = T_i\hat{F}_iT_i^{-1}$, $l_i = (T'_i)^{-1}\hat{l}_i$, $F = T\hat{F}T^{-1}$, and $L = (T')^{-1}\hat{L}$, it follows that $F_i + F'_i = -l_il'_i$ and

$$F + F' = -LL' \tag{10.5.6}$$

Finally, define $G = T\hat{G}$.

Then $\{F, G, L, W_0\}$ *is a realization of $W(s)$, with the additional properties that* (10.5.6) *holds and that* $[F, L]$ *is completely observable* (the latter property following from the complete observability of $[\hat{F}, \hat{L}]$ or, equivalently, the complete controllability of $[\hat{F}', \hat{L}]$).

Define now

$$H = G + LW_0 \tag{10.5.7}$$

We claim that *with F, G, and H defined as above, and with $J = Z(\infty)$, $\{F, G, H, J\}$ constitutes a realization for $Z(s)$.* To verify this, we observe that

$$\begin{aligned}W'(-s)W(s) &= W_0'W_0 + W_0'L'(sI-F)^{-1}G + G'(-sI-F')^{-1}LW_0 \\ &\quad + G'(-sI-F')^{-1}LL'(sI-F)^{-1}G \\ &= W_0'W_0 + W_0'L'(sI-F)^{-1}G + G'(-sI-F')^{-1}LW_0 \\ &\quad + G'(-sI-F')^{-1}[-sI-F'+sI-F](sI-F)^{-1}G \\ &\qquad\qquad\text{using (10.5.6)} \\ &= W_0'W_0 + (G'+W_0'L')(sI-F)^{-1}G \\ &\quad + G'(-sI-F')^{-1}(LW_0+G) \\ &= W_0'W_0 + H'(sI-F)^{-1}G + G'(-sI-F')^{-1}H \\ &\qquad\qquad\text{using (10.5.7)}\end{aligned}$$

Now the right-hand quantity is also $Z(s) + Z'(-s)$. The fact that F is a direct sum of matrices all of which have a Hurwitz characteristic polynomial means that $H'(sI - F)^{-1}G$ is such that every element can only have left-half-plane poles. This guarantees that $Z(s) = Z(\infty) + H'(sI - F)^{-1}G$, because every element of $Z(s)$ can only have left-half-plane poles.

The above constructive procedure for obtaining a realization of $Z(s)$ up to this point is valid for symmetric and nonsymmetric $Z(s)$. The procedure yields immediately a passive synthesis of an arbitrary positive real $Z(s)$, in general nonreciprocal, via the reactance-extraction technique. This is quickly illustrated.

Consider the constant impedance matrix

$$M = \begin{bmatrix} J & -H' \\ G & -F \end{bmatrix}$$

It follows on using (10.5.6), (10.5.7), and the constraint $W_0'W_0 = J + J'$ that

$$M + M' = \begin{bmatrix} W_0' \\ -L \end{bmatrix} [W_0 \quad -L']$$

Therefore, $M + M'$ is nonnegative definite, implying that M is synthesizable with passive elements; a passive synthesis of $Z(s)$ results when all but the first m ports are terminated in unit inductors. The synthesis is in general nonreciprocal, since, in general, $M \neq M'$.

In the following subsection we shall consider the remaining problem—that of giving a reciprocal synthesis for symmetric $Z(s)$.

Reciprocal Synthesis of $Z(s)$

As indicated before, there is no loss of generality in assuming that $Z(\infty) + Z'(\infty)$ is nonsingular and that no element of $Z(s)$ has a purely imaginary pole. Using the symmetry of $Z(s)$ and letting $\omega = \infty$, it follows that $Z(\infty)$ is positive definite.

Now *normal* rank $[Z(s) + Z'(-s)] \geq$ rank $[Z(\infty) + Z'(-\infty)] =$ rank $[2Z(\infty)]$. Since the latter has full rank, normal rank $[Z(s) + Z'(-s)]$ is m. Then $W(s)$, $N_1(s)$, and $N_2(s)$, as defined in the Gauss factorization procedure, are $m \times m$ matrices. Further, $W_0'W_0 = 2J$, and therefore W_0 is nonsingular. As indicated in the statement of the factorization procedure, $N_2(s)$ may be assumed to be even. We shall make this assumption.

Our goal here is to replace the realization $\{F, G, H, J\}$ of $Z(s)$ by a realization $\{TFT^{-1}, TG, (T^{-1})'H, J\}$ such that the new realization possesses additional properties helpful in incorporating the reciprocity constraint. To generate T, which in fact will be an orthogonal matrix, we shall define a symmetric matrix P. Then P will be shown to satisfy a number of constraints, including a constraint that all its eigenvalues be $+1$ or -1. Then T will be taken to be any orthogonal matrix such that $P = T'\Sigma T$, where Σ is a diagonal matrix, with diagonal entries being $+1$ or -1.

Lemma 10.5.1. With F, G, H, L, and W_0 as defined in this section, the equations

$$\begin{aligned} FP &= PF' \\ PL &= L \end{aligned} \tag{10.5.8}$$

define a unique symmetric nonsingular matrix P. Moreover, the following equations also hold:

$$\begin{aligned} PH &= -G \\ PG &= -H \\ PF &= F'P \\ P &= P^{-1} \end{aligned} \tag{10.5.9}$$

To prove the lemma, we shall have to call upon the identities

$$L'(sI - F)^{-1}L = L'(sI - F')^{-1}L \tag{10.5.10}$$

and

$$-H'(sI - F)^{-1}L = G'(sI - F')^{-1}L \tag{10.5.11}$$

The proof of these identities is requested in Problem 10.5.2. The proof of the second identity relies heavily on the fact that the polynomial matrix $N_2(s)$ appearing in the Gauss factorization procedure is even.

Proof. If $\{F, L, L\}$ is a minimal realization of $L'(sI - F)^{-1}L$, the existence, uniqueness, symmetry, and nonsingularity of P follow from the important result giving a state-space characterization of the symmetry property of a transfer-function matrix. Certainly, $[F, L]$ is completely observable. That $[F, L]$ is completely controllable can be argued as follows: because $[F', L]$ is completely controllable, $[F' - LK', L]$ is completely controllable for all K. Taking $K = -L$ and using the relation $F + F' = -LL'$, it follows that $[F, L]$ is completely controllable.

The equation yielding P explicitly is, we recall,

$$P[L \;\; F'L \;\; \cdots \;\; (F')^{n-1}L] = [L \;\; FL \;\; \cdots \;\; F^{n-1}L] \tag{10.5.12}$$

Next, we establish (10.5.9). From the identity (10.5.11), we have

$$\begin{aligned} G'(sI - F')^{-1}L &= -H'(sI - F)^{-1}L \\ &= -H'P(sI - P^{-1}FP)^{-1}P^{-1}L \\ &= -H'P(sI - F')^{-1}L \end{aligned}$$

Consequently, using the complete controllability of $[F', L]$,

$$PH = -G$$

This is the first of (10.5.9), and together with the definition of H as $G + LW_0$, it yields

$$-G = PH = P(G + LW_0) = PG + LW_0$$

or, again, using (10.5.7),

$$PG = -H$$

This is the second equation of (10.5.9). Next, from $F + F' = -LL'$, we have the third equation of (10.5.9):

$$\begin{aligned} PF &= -PF' - PLL' \\ &= -FP - LL'P \qquad \text{by (10.5.8)} \\ &= -(F + LL')P \\ &= F'P \qquad \text{using (10.5.6)} \end{aligned}$$

This equation and (10.5.8) yield that $P^{-1}L = L$ and $FP^{-1} = P^{-1}F'$, or, in other words, P^{-1} satisfies $FX = XF'$ and $XL = L$. Hence the last equation of (10.5.9) holds, proving the lemma. $\nabla\nabla\nabla$

Equation (10.5.9) implies that $P^2 = I$, and thus the eigenvalues of P must be $+1$ or -1. Since P is symmetric, there exists an orthogonal matrix T such that $P = T\Sigma T'$, with Σ consisting of $+1$ and -1 entries in the diagonal positions and zeros elsewhere.

We now define a realization $\{F_1, G_1, H_1, J\}$ of $Z(s)$ by $F_1 = TFT'$, $G_1 = TG$, and $H_1 = TH$; the associated hybrid matrix is

$$M = \begin{bmatrix} J & -H_1' \\ G_1 & -F_1 \end{bmatrix} \tag{10.5.13}$$

This hybrid matrix is the key to a reciprocal synthesis.

Theorem 10.5.1. Let $Z(s)$ be an $m \times m$ symmetric positive real impedance matrix with $Z(\infty)$ nonsingular and no element of $Z(s)$ possessing a purely imaginary pole. With a realization $\{F_1, G_1, H_1, J\}$ of $Z(s)$ as defined above, the hybrid matrix M of of (10.5.13) satisfies the following constraints:

1. $M + M' \geq 0$, and has rank m.
2. $(I_m \dotplus \Sigma)M$ is symmetric.

Proof. By direct calculation

$$\begin{aligned} M + M' &= (I_m \dotplus T)\begin{bmatrix} 2J & -H' + G' \\ G - H & -F - F' \end{bmatrix}(I_m \dotplus T') \\ &= (I_m \dotplus T)\begin{bmatrix} W_0'W_0 & -W_0'L' \\ -LW_0 & LL' \end{bmatrix}(I_m \dotplus T') \\ &= (I_m \dotplus T)\begin{bmatrix} W_0' \\ -L \end{bmatrix}[W_0 \quad -L'](I_m \dotplus T') \end{aligned}$$

Hence $M + M'$ is nonnegative definite. The rank condition is obviously fulfilled, since $[W_0 \quad -L']$ has rank no less than W_0, which has rank m.

It remains to show that $J = J'$, $\Sigma G_1 = -H_1$, and $\Sigma F_1 = F_1'\Sigma$. The first equation is obvious from the symmetry of $Z(s)$. From (10.5.9), $T'\Sigma TG = -H$ or $\Sigma(TG) = -TH$. That is, $\Sigma G_1 = -H_1$. Also, from (10.5.9), $T'\Sigma TF = F'T'\Sigma T$ or $\Sigma TFT' = TF'T'\Sigma$. That is, $\Sigma F_1 = F_1'\Sigma$. $\nabla\nabla\nabla$

With the above theorem in hand, the synthesis of $Z(s)$ is immediate.

The matrix M is realizable as a hybrid matrix, the current-excited ports corresponding to the rows of $I_m \dotplus \Sigma$ that have a $+1$ on the diagonal, and the voltage-excited ports corresponding to those rows of $I_m \dotplus \Sigma$ that have a -1 on the diagonal. The theorem guarantees that M is synthesizable using only passive and reciprocal components. A synthesis of $Z(s)$ then follows on terminating current-excited ports other than the first m in unit inductances, and the voltage-excited ports in unit capacitances. Since $M + M'$ has rank m, the synthesis of M and hence of $Z(s)$ can be achieved with m resistances—the minimum number possible, as aimed for.

As a summary of the synthesis procedure, the following steps may be noted:

1. By transformer, inductance, and capacitance extractions, reduce the general problem to one of synthesizing an $m \times m$ $Z(s)$, such that $Z(\infty)$ is nonsingular, rank $[Z(s) + Z'(-s)] = m$, no element of $Z(s)$ possesses a pole on the $j\omega$ axis, and of course $Z(s) = Z'(s)$.
2. Perform a Gauss factorization, as applicable for symmetric $Z(s)$. Thus $W(s)$ is found such that $Z'(-s) + Z(s) = W'(-s)W(s)$ and $W(s) = [N_1(s)]^{-1}N_2'(s)$, where $N_1(\cdot)$ is a diagonal $m \times m$ matrix of Hurwitz polynomials, and $N_2(s)$ is an $m \times m$ matrix of even polynomials.
3. Construct $\hat{F}$ using companion matrices, and $\hat{L}$ and $\hat{G}$ from $W(s)$. From $\hat{F}, \hat{L}$, and $\hat{G}$ determine F, L, and G by solving a number of equations of the kind $P_i\hat{F}_i + \hat{F}_i'P_i = -\hat{l}_i\hat{l}_i'$ [see (10.5.6) and associated remarks].
4. Find $H = G + LW_0$, and P as the unique solution of $FP = PF'$ and $PL = L$.
5. Compute an orthogonal matrix T reducing P to a diagonal matrix Σ of $+1$ and -1 elements: $P = T'\Sigma T$.
6. With $F_1 = TFT'$, $G_1 = TG$, and $H_1 = TH$, define the hybrid matrix M of Eq. (10.5.13) and give a reciprocal synthesis. Terminate appropriate ports in unit inductors and capacitors to generate a synthesis of $Z(s)$.

Example 10.5.1 Let us synthesize the scalar positive real impedance

$$z(s) = \frac{s^2 + 2s + 4}{s^2 + s + 1}$$

As $z(s)$ has no poles on the $j\omega$ axis and $z(\infty)$ is not zero, we compute

$$\begin{aligned} z(s) + z(-s) &= \frac{2(s^4 + 3s^2 + 4)}{(s^2 + s + 1)(s^2 - s + 1)} \\ &= \frac{2(s^2 + s + 2)(s^2 - s + 2)}{(s^2 + s + 1)(s^2 - s + 1)} \end{aligned}$$

It is readily seen that candidates for $W(s)$ are $\sqrt{2}(s^2 + s + 2)/(s^2 + s + 1)$ or $\sqrt{2}(s^2 - s + 2)/(s^2 + s + 1)$. But the numerators are not even functions. To get the numerator of a $W(s)$ to be an even function, multiply $z(s) + z(-s)$ by $(s^4 + 3s^2 + 4)/(s^4 + 3s^2 + 4)$; thus

$$W(s) = \frac{\sqrt{2}(s^2 - s + 2)(s^2 + s + 2)}{(s^2 + s + 1)(s^2 + s + 2)} = \frac{\sqrt{2}(s^4 + 3s^2 + 4)}{s^4 + 2s^3 + 4s^2 + 3s + 2}$$

The process is equivalent to modifying the impedance $z(s)$ through multiplication by $(s^2+s+2)/(s^2+s+2)$, a standard technique used in the classical Darlington synthesis [8]. (The Gauss factorization that gives $N_2(s) = N_2(-s)$ for $Z(s) = Z'(s)$ in multiports, often only by inserting common factors into $N_1(s)$ and $N_2(s)$, may be regarded as a generalization of the classical one-port Darlington procedure.)

We write

$$W'(s) = \sqrt{2} + \frac{-2\sqrt{2}s^3 - \sqrt{2}s^2 - 3\sqrt{2}s + 2\sqrt{2}}{s^4 + 2s^3 + 4s^2 + 3s + 2}$$

a realization of which is

$$\hat{F}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -4 & -2 \end{bmatrix} \qquad \hat{L} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\hat{G}' = [2\sqrt{2} \quad -3\sqrt{2} \quad -\sqrt{2} \quad -2\sqrt{2}] \qquad W_0 = \sqrt{2}$$

Obtain P_1 through $P_1\hat{F} + \hat{F}'P_1 = -\hat{L}\hat{L}'$ to give

$$P_1 = \begin{bmatrix} \frac{5}{28} & 0 & \frac{-1}{7} & 0 \\ 0 & \frac{1}{7} & 0 & \frac{-3}{14} \\ \frac{-1}{7} & 0 & \frac{3}{14} & 0 \\ 0 & \frac{-3}{14} & 0 & \frac{4}{7} \end{bmatrix}$$

One matrix T_1 such that $T_1'T_1 = P$ is

$$T_1 = \begin{bmatrix} \frac{1}{2\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ -\sqrt{\frac{2}{21}} & 0 & \sqrt{\frac{3}{14}} & 0 \\ 0 & \frac{-3}{4\sqrt{7}} & 0 & \frac{2}{\sqrt{7}} \end{bmatrix}$$

$$T_1^{-1} = \begin{bmatrix} 2\sqrt{3} & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ \frac{4}{\sqrt{3}} & 0 & \sqrt{\frac{14}{3}} & 0 \\ 0 & \frac{3}{2} & 0 & \frac{\sqrt{7}}{2} \end{bmatrix}$$

Therefore, matrices of a new state-space realization of $W(s)$ are

$$F = T_1 \hat{F} T_1^{-1} = \begin{bmatrix} 0 & \frac{-\sqrt{3}}{2} & 0 & \frac{-\sqrt{7}}{2\sqrt{3}} \\ \frac{\sqrt{3}}{2} & \frac{-9}{8} & 0 & \frac{-3\sqrt{7}}{8} \\ 0 & 0 & 0 & \frac{-2\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{7}}{2\sqrt{3}} & \frac{-3\sqrt{7}}{8} & 2\sqrt{\frac{2}{3}} & \frac{-7}{8} \end{bmatrix}$$

$$L = (T_1^{-1})'\hat{L} = \begin{bmatrix} 0 \\ \frac{3}{2} \\ 0 \\ \frac{\sqrt{7}}{2} \end{bmatrix} \qquad G = T_1 \hat{G} = \begin{bmatrix} \sqrt{\frac{2}{3}} \\ \frac{-3}{2\sqrt{2}} \\ -\sqrt{\frac{7}{3}} \\ \frac{-\sqrt{7}}{2\sqrt{2}} \end{bmatrix}$$

Define

$$H = G + LW_0 = \begin{bmatrix} \sqrt{\frac{2}{3}} \\ \frac{3}{2\sqrt{2}} \\ -\sqrt{\frac{7}{3}} \\ \frac{\sqrt{7}}{2\sqrt{2}} \end{bmatrix}$$

At this stage, nonreciprocal synthesis is possible. For reciprocal synthesis, we proceed as follows. The solution of $FP = PF'$ and $PL = L$ is

$$P = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Hence

$$T = T' = T^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad \Sigma = [1 \dotplus 1 \dotplus -1 \dotplus -1]$$

Next, we form the hybrid matrix

$$M = \begin{bmatrix} J & -H'T' \\ TG & -TFT' \end{bmatrix} = \begin{bmatrix} 1 & \frac{-\sqrt{7}}{2\sqrt{2}} & \frac{-3}{2\sqrt{2}} & \sqrt{\frac{7}{3}} & -\sqrt{\frac{2}{3}} \\ \frac{-\sqrt{7}}{2\sqrt{2}} & \frac{7}{8} & \frac{3\sqrt{7}}{8} & -2\sqrt{\frac{2}{3}} & \frac{-\sqrt{7}}{2\sqrt{3}} \\ \frac{-3}{2\sqrt{2}} & \frac{3\sqrt{7}}{8} & \frac{9}{8} & 0 & \frac{-\sqrt{3}}{2} \\ -\sqrt{\frac{7}{3}} & 2\sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ \sqrt{\frac{2}{3}} & \frac{\sqrt{7}}{2\sqrt{3}} & \frac{\sqrt{3}}{2} & 0 & 0 \end{bmatrix}$$

for which the first three ports correspond to current-excited ports and the last two ports correspond to voltage-excited ports. It can be verified that $(1 \dotplus \Sigma)M$ is symmetric; $M + M'$ is nonnegative definite and has rank 1. A synthesis of $Z(s)$ is shown in Fig. 10.5.1. Evidently, the synthesis uses one resistance.

Problem 10.5.1 Synthesize the symmetric-positive-real impedance

$$z(s) = \begin{bmatrix} \frac{3s+3}{s+2} & 1 \\ 1 & 1 \end{bmatrix}$$

using the procedure outlined in this section.

Problem 10.5.2 With matrices as defined in this section, prove the identity

$$L'(sI - F)^{-1}L = L'(sI - F')^{-1}L$$

by showing that $L'(sI - F)^{-1}L$ is diagonal.

Prove the identity

$$-H'(sI - F)^{-1}L = G'(sI - F')^{-1}L$$

[*Hint:* The (i, j)th element of $W'(s) = W_0 + G'(sI - F')^{-1}L$ is

$$\frac{n_{2ij}(s)}{p_j(s)} = v_{0ij} + \frac{\hat{n}_{ij}(s)}{p_j(s)} = v_{0ij} + \mathbf{n}'_{ij}(sI - F'_j)l_j$$

with $\mathbf{n}'_{ij} = \hat{\mathbf{n}}'_{ij}T_j$. The polynomial $n_{2ij}(s)$ is even if $Z(s)$ is symmetric. Show that the dimension of F_j is even and that

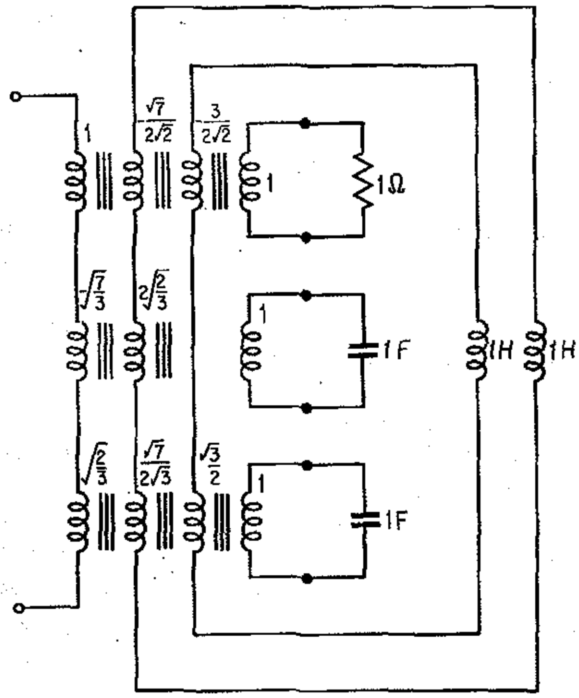

FIGURE 10.5.1. A Final Synthesis for $Z(s)$ in Example 10.5.1.

$$\begin{aligned} n_{2ij}(s) &= v_{0ij}\det(-sI - F'_j) + \mathbf{n}'_{ij}[\operatorname{adj}(-sI - F'_j)]l_j \\ &= v_{0ij}\det(sI + F'_j) + n'_{ij}[\operatorname{adj}(sI + F'_j)]l_j \\ &= \det(sI - F_j)[v_{0ij} - \mathbf{n}'_{ij}(sI + F'_j)^{-1}l_j][1 - l'_j(sI - F_j)^{-1}l_j] \\ &= \det(sI - F_j)[v_{0ij} - (v_{0ij}l'_j + \mathbf{n}'_{ij})(sI - F_j)^{-1}l_j] \end{aligned}$$

from which the result follows.]

Problem 10.5.3 (For those familiar with the classical Darlington synthesis.) Given the impedance function

$$z(s) = \frac{s^2 + s + 2}{2s^2 + s + 1}$$

give a synthesis for this $z(s)$ by (a) the procedure of this section, and (b) the Darlington method, if you are familiar with this technique. Compare the number of elements of the networks.

Problem 10.5.4 It is known that the Gauss factorization does not yield a unique $W(s)$. Consider the symmetric positive real impedance matrix

$$Z(s) = \frac{1}{s^2 + 6s + 4}\begin{bmatrix} 3(s^2 + 4) & 2s^2 + 4 \\ 2s^2 + 4 & 2s^2 + 6s + 4 \end{bmatrix}$$

Using the Gauss factorization, obtain two different factors $W(s)$ and the corresponding state-space syntheses for $Z(s)$.

Compare the two syntheses.

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11

Scattering Matrix Synthesis

11.1 INTRODUCTION

In Chapter 8 we formulated two different approaches to the problem of synthesizing a bounded real $m \times m$ scattering matrix $S(s)$; the first was based on the reactance-extraction concept, while the second was based on the resistance-extraction concept. In the present chapter we shall discuss solutions to the synthesis problem using both of these approaches.

The reader should note in advance the following points. First, it will be assumed throughout that the prescribed $S(s)$ is a *normalized* scattering matrix; i.e., the normalization number at each port of a network of scattering matrix $S(s)$ is unity [1, 2].

Second, the bounded real property always implies that $S(s)$ is such that $S(\infty) < \infty$. Consequently, an arbitrary bounded real $S(s)$ *always* possesses a state-space realization $\{F, G, H, J\}$ such that

$$S(s) = J + H'(sI - F)^{-1}G \tag{11.1.1}$$

Third, we learned earlier that the algebraic characterization of the bounded real property in state-space terms, as contained in the bounded real lemma, applies only to *minimal* realizations of $S(s)$. Since the synthesis of $S(s)$ in state-space terms depends primarily on this algebraic characterization of its bounded real property, it follows that the synthesis procedures to be considered will be stated in terms of a particular minimal realization of $S(s)$.

Fourth, with no loss of generality, we shall also assume the availability of minimal realization $\{F, G, H, J\}$ of $S(s)$ with the property that for some real matrices L and W_0, the following equations hold:

$$\begin{aligned} F + F' &= -HH' - LL' \\ -G &= HJ + LW_0 \\ I - J'J &= W_0'W_0 \end{aligned} \tag{11.1.2}$$

That this is a valid assertion is guaranteed by the following theorem, which follows easily from the bounded real lemma. Proof of the theorem will be omitted.

Theorem 11.1.1. Let $\{F_0, G_0, H_0, J\}$ be an arbitrary minimal realization of a rational bounded real matrix $S(s)$. Suppose that $\{P, L_0, W_0\}$ is a triple that satisfies the bounded real lemma for the above minimal realization. Let T be any matrix ranging over the set of matrices for which

$$T'T = P \tag{11.1.3}$$

Then the minimal realization $\{F, G, H, J\}$ given by $\{TF_0T^{-1}, TG_0, (T^{-1})'H_0, J\}$ satisfies Eq. (11.1.2), where $L = (T^{-1})'L_0$.

Note that F, G, H, L, and W_0 are in no way unique. This is because there are an infinity of matrices P satisfying the bounded real lemma equations for any one minimal realization $\{F_0, G_0, H_0, J\}$ of $S(s)$. Further, with any one matrix P, there is associated an infinity of L and W_0 in the bounded real lemma equations and an infinity of T satisfying (11.1.3). Note also that the dimensions of L and W_0 are not unique, with the number of rows of L' and W_0 being bounded below by ρ = normal rank $[I - S'(-s)S(s)]$.

Except for lossless synthesis, (11.1.2) will be taken as the starting point in all the following synthesis procedures. In the case of a *lossless* scattering matrix, we know from earlier considerations that the matrices L and W_0 are actually zero and (11.1.2) reduces therefore to

$$\begin{aligned} F + F' &= -HH' \\ -G &= HJ \\ I - J'J &= 0 \end{aligned} \tag{11.1.4}$$

A minimal realization of a lossless $S(s)$ satisfying (11.1.4) is much simpler to derive than is a minimal realization of a lossy $S(s)$ satisfying (11.1.2). This is because the lossless bounded real lemma equations are very easily solved.

The following is a brief outline of the chapter. In Section 11.2 we consider the synthesis of nonreciprocal networks via the reactance-extraction approach. Gyrators are allowed as circuit components. Lossless nonreciprocal synthesis is also treated here as a special case of the procedure. In Section 11.3 a synthesis procedure for reciprocal networks is considered, again based on reactance extraction. Also, a simple technique for lossless reciprocal synthesis is given. The material in these sections is drawn from [3] and [4]. Finally, in Section 11.4 syntheses of nonreciprocal and reciprocal networks are obtained that use the resistance-extraction technique; these syntheses are based on results of [5]. All methods in this chapter yield minimal reactive syntheses.

11.2 REACTANCE-EXTRACTION SYNTHESIS OF NONRECIPROCAL NETWORKS

As discussed earlier, the reactance extraction synthesis problem of finding a passive structure synthesizing a prescribed $m \times m$ bounded real $S(s)$ may be phrased as follows:

> From $S(s)$, derive a real-rational $W(p)$ via a change of complex variables
>
> $$s = \frac{p+1}{p-1} \tag{11.2.1}$$
>
> for which
>
> $$S(s) = S\left(\frac{p+1}{p-1}\right) = W(p) \tag{11.2.2}$$
>
> Then find a state-space realization $\{F_w, G_w, H_w, J_w\}$ for $W(p)$ such that the matrix
>
> $$M = \begin{bmatrix} J_w & H_w' \\ G_w & F_w \end{bmatrix} \tag{11.2.3}$$
>
> satisfies the inequality
>
> $$I - M'M \geq 0 \tag{11.2.4}$$

As seen from (11.2.2), the real rational matrix $W(p)$ is derived from the bounded real matrix $S(s)$ by the bilinear transformation $s = (p + 1)/(p - 1)$ of (11.2.1), which maps the left half complex s plane into the unit circle in the complex p plane. Of greater interest to us however is the relation between minimal state-space realizations of $S(s)$ and minimal state-space realizations of $W(p)$. Suppose that $\{F_0, G_0, H_0, J\}$ is a minimal realization of $S(s)$. Now

$$\begin{aligned} W(p) &= S\left(\frac{p+1}{p-1}\right) \\ &= J + H_0'\left(\frac{p+1}{p-1}I - F_0\right)^{-1}G_0 \\ &= J - H_0'(F_0 - I)^{-1}[pI - (F_0 + I)(F_0 - I)^{-1}]^{-1}(p-1)G_0 \end{aligned}$$

Consider only the term $[pI - (F_0 + I)(F_0 - I)^{-1}]^{-1}(p-1)$. We have

$$\begin{aligned} &[pI - (F_0 + I)(F_0 - I)^{-1}]^{-1}[pI - I] \\ &\quad = [pI - (F_0 + I)(F_0 - I)^{-1}]^{-1} \\ &\qquad \times [pI - (F_0 + I)(F_0 - I)^{-1} - I + (F_0 + I)(F_0 - I)^{-1}] \\ &\quad = [pI - (F_0 + I)(F_0 - I)^{-1}]^{-1} \\ &\qquad \times [pI - (F_0 + I)(F_0 - I)^{-1} + 2(F_0 - I)^{-1}] \\ &\quad = I + 2[pI - (F_0 + I)(F_0 - I)^{-1}]^{-1}(F_0 - I)^{-1} \end{aligned}$$

Therefore

$$\begin{aligned} W(p) = J &- H_0'(F_0 - I)^{-1}G_0 \\ &- 2H_0'(F_0 - I)^{-1}[pI - (F_0 + I)(F_0 - I)^{-1}]^{-1}(F_0 - I)^{-1}G_0 \end{aligned}$$

Evidently, a state-space realization $\{F_{w_0}, G_{w_0}, H_{w_0}, J_w\}$ may be readily identified to be

$$\begin{aligned} F_{w_0} &= (F_0 + I)(F_0 - I)^{-1} \\ G_{w_0} &= \sqrt{2}(F_0 - I)^{-1}G_0 \\ H_{w_0} &= -\sqrt{2}(F_0' - I)^{-1}H_0 \\ J_w &= J - H_0'(F_0 - I)^{-1}G_0 \end{aligned} \tag{11.2.5}$$

Several points should be noted.

1. The quantities in (11.2.5) are well defined; i.e., the matrix $(F_0 - I)^{-1}$ exists by virtue of the fact that $S(s)$ is analytic in $\text{Re}[s] > 0$ and, in particular, at $s = 1$. In other words, $+1$ can not be an eigenvalue of F_0.
2. By a series of simple manipulations using (11.2.5), it can be verified easily that the transformation defined by (11.2.5) is reversible. In fact, $\{F_0, G_0, H_0, J\}$ is given in terms of $\{F_{w_0}, G_{w_0}, H_{w_0}, J_w\}$ by exactly the same equations, (11.2.5), with $\{F_0, G_0, H_0, J\}$ and $\{F_{w_0}, G_{w_0}, H_{w_0}, J_w\}$ interchanged.
3. The transformations in (11.2.5) define a minimal realization of $W(p)$ if and only if $\{F_0, G_0, H_0, J\}$ is a minimal realization of $S(s)$. (This fact follows from 2 incidentally; can you show how?)

We now state the following lemma, which will be of key importance in solving the synthesis problem:

Lemma 11.2.1. *Suppose that* $\{F, G, H, J\}$ *is one minimal realization of* $S(s)$ *for which (11.1.2) holds for some real matrices* L *and* W_0. *Let* $\{F_w, G_w, H_w, J_w\}$ *be the associated matrices computed via the transformations defined in (11.2.5). Then* $\{F_w, G_w, H_w, J_w\}$ *are such that*

$$\begin{aligned} F_w'F_w - I &= -H_wH_w' - L_wL_w' \\ -F_w'G_w &= H_wJ_w + L_wW_w \\ I - J_w'J_w &= W_w'W_w + G_w'G_w \end{aligned} \tag{11.2.6}$$

for some real matrices L_w *and* W_w.

Proof. From (11.2.5) we have

$$\begin{aligned} F &= (F_w + I)(F_w - I)^{-1} \\ G &= \sqrt{2}(F_w - I)^{-1}G_w \\ H &= -\sqrt{2}(F_w' - I)^{-1}H_w \\ J &= J_w - H_w'(F_w - I)^{-1}G_w \end{aligned} \tag{11.2.7}$$

Define matrices L_w and W_w by

$$\begin{aligned} \sqrt{2}L_w &= -(F_w' - I)L \\ W_w &= W_0 + L_w'(F_w - I)^{-1}G_w \end{aligned} \tag{11.2.8}$$

We then substitute the above equations into (11.1.2). The first equation of (11.1.2) yields

$$\begin{aligned} (F_w + I)(F_w - I)^{-1} + (F_w' - I)^{-1}(F_w' + I) &= -2(F_w' - I)^{-1}H_wH_w'(F_w - I)^{-1} \\ &\quad - 2(F_w' - I)^{-1}L_wL_w'(F_w - I)^{-1} \end{aligned}$$

This reduces simply, on premultiplying by $F_w' - I$ and postmultiplying by $F_w - I$, to

$$F_w'F_w - I = -H_wH_w' - L_wL_w'$$

which is the first equation of (11.2.6). The second equation of (11.1.2) yields

$$\begin{aligned}-\sqrt{2}(F_w - I)^{-1}G_w \\ &= -\sqrt{2}(F'_w - I)^{-1}H_w[J_w - H'_w(F_w - I)^{-1}G_w] \\ &\quad - \sqrt{2}(F'_w - I)^{-1}L_w[W_w - L'_w(F_w - I)^{-1}G_w]\end{aligned}$$

Straightforward manipulation then gives

$$\begin{aligned}(F'_w - I)(F_w - I)^{-1}G_w \\ &= H_wJ_w + L_wW_w - (H_wH'_w + L_wL'_w)(F_w - I)^{-1}G_w\end{aligned}$$

Using the first equation of (11.2.6), the above reduces to

$$(F'_w - I - F'_wF_w + I)(F_w - I)^{-1}G_w = H_wJ_w + L_wW_w$$

or

$$-F'_wG_w = H_wJ_w + L_wW_w$$

This establishes the second equation of (11.2.6). Last, from the third relation of (11.1.2), we have

$$\begin{aligned}I - [J'_w - G'_w(F'_w - I)^{-1}H_w][J_w - H'_w(F_w - I)^{-1}G_w] \\ &= [W'_w - G'_w(F'_w - I)^{-1}L_w][W_w - L'_w(F_w - I)^{-1}G_w]\end{aligned}$$

On expansion we obtain

$$\begin{aligned}I - J'_wJ_w - W'_wW_w \\ &= -(H_wJ_w + L_wW_w)'(F_w - I)^{-1}G_w \\ &\quad - G'_w(F'_w - I)^{-1}(H_wJ_w + L_wW_w) \\ &\quad + G'_w(F'_w - I)^{-1}(H_wH'_w + L_wL'_w)(F_w - I)^{-1}G_w \\ &= G'_wF_w(F_w - I)^{-1}G_w + G'_w(F'_w - I)^{-1}F'_wG_w \\ &\quad + G'_w(F'_w - I)^{-1}(I - F'_wF_w)(F_w - I)^{-1}G_w \\ &= G'_w(F'_w - I)^{-1}[F'_w(F_w - I) + (F'_w - I)F_w \\ &\quad + I - F'_wF_w](F_w - I)^{-1}G_w \\ &= G'_wG_w\end{aligned}$$

The second equality above of course follows from the use of the first and the second relations of (11.2.6). Thus (11.2.6) is established, and the lemma proved. ∇ ∇ ∇

Suppose that we compute a minimal realization $\{F_w, G_w, H_w, J_w\}$ of $W(p)$ from a particular minimal realization $\{F, G, H, J\}$ satisfying (11.1.2). Then, using Lemma 11.2.1, we can see that $\{F_w, G_w, H_w, J_w\}$ possesses the desired property for synthesis, viz., the matrix M defined in (11.2.3) satisfies the inequality (11.2.4). The following steps prove this:

$$\begin{aligned} I - M'M &= \begin{bmatrix} I - J'_wJ_w - G'_wG_w & -(H_wJ_w + F'_wG_w)' \\ -(H_wJ_w + F'_wG_w) & I - H_wH'_w - F'_wF_w \end{bmatrix} \\ &= \begin{bmatrix} W'_wW_w & W'_wL'_w \\ L_wW_w & L_wL'_w \end{bmatrix} \\ &= \begin{bmatrix} W'_w \\ L_w \end{bmatrix} [W_w \quad L'_w] \end{aligned}$$

The second equality follows from application of (11.2.6). The last equality obviously implies that $I - M'M$ is nonnegative definite. Therefore, as discussed in detail in an earlier chapter, the problem of synthesizing $S(s)$ is reduced to the problem of synthesizing a real constant scattering matrix S_a, which in this nonreciprocal case is the same as M. Synthesis of a constant bounded real scattering matrix is straightforward; a simple approach is to apply the procedures discussed earlier,* which will convert the problem of synthesizing S_a to the problem of synthesizing a constant impedance matrix Z_a. We know the solution to this problem, and so we can achieve a synthesis of $S(s)$.

A brief summary of the synthesis procedure follows:

1. Find a minimal realization $\{F_0, G_0, H_0, J\}$ for the bounded real $m \times m$ $S(s)$.
2. Solve the bounded real lemma equations and compute another minimal realization $\{F, G, H, J\}$ satisfying (11.1.2).
3. Calculate matrices F_w, G_w, H_w, and J_w according to (11.2.5), and obtain the scattering matrix S_a given by

$$S_a = \begin{bmatrix} J_w & H'_w \\ G_w & F_w \end{bmatrix}$$

   The appropriate normalization numbers for S_a, other than the first m ports, are determined by the choice of the reactance values as explained in the next step.
4. Choose any set of desired inductance values. (As an alternative, choose all reactances to be capacitive; in this case the matrices G_w and F_w in S_a are replaced by $-G_w$ and $-F_w$, respectively.†) Then calculate the

*In the earlier discussion it was always assumed that the scattering matrix was normalized. If S_a is not initially normalized, it must first be converted to a normalized matrix (see Chapter 2).

†This point will become clear if the reader studies carefully the material in Section 8.3.

normalization numbers r_{m+l}, $l = 1, 2, \ldots, n$ with $r_{m+l} = L_l$ (or $r_{m+l} = 1/C_l$ if the reactances are capacitive). Of course, the normalization numbers at the first m ports are fixed by $S(s)$, being the same as those of $S(s)$, i.e., one.

5. Obtain a nondynamic network N_1 synthesizing the real constant scattering matrix S_a with the appropriate normalization numbers for S_a just defined. Finally, terminate the last n ports of N_1 appropriately in inductors (or capacitors) chosen in step 4 to yield a synthesis for the prescribed $S(s)$.

Note that at no stage does the transfer-function matrix $W(p)$ have to be explicitly computed; only the matrices of a state-space realization are required.

Lossless Scattering Matrix Synthesis

A special case of the above synthesis that is of significant interest is lossless synthesis. Let an $m \times m$ scattering matrix $S(s)$ be a prescribed lossless bounded real matrix; i.e.,

$$I - S'(-s)S(s) = 0 \tag{11.2.9}$$

Now one way of synthesizing $S(s)$ would be to apply the "preliminary" extraction procedures of Chapter 8. For lossless $S(s)$, they are not merely preliminary, but will provide a complete synthesis. Here, however, we shall assume that these preliminary procedures are *not* carried out. Our goal is instead one of deriving a real constant scattering matrix

$$M = \begin{bmatrix} J_w & H'_w \\ G_w & F_w \end{bmatrix} \tag{11.2.3}$$

where $\{F_w, G_w, H_w, J_w\}$ is a state-space realization of

$$W(p) = S\left(\frac{p+1}{p-1}\right) \tag{11.2.2}$$

with

$$s = \frac{p+1}{p-1} \tag{11.2.1}$$

such that M is *lossless* bounded real, or

$$I - M'M = 0 \tag{11.2.10}$$

Let $\{F, G, H, J\}$ be a minimal realization of $S(s)$. Assume that F is an $n \times n$ matrix, and that (11.1.4) holds; i.e.,

$$\begin{aligned} F + F' &= -HH' \\ -G &= HJ' \\ I - J'J &= 0 \end{aligned} \tag{11.2.11}$$

Let one minimal realization $\{F_w, G_w, H_w, J_w\}$ of $W(p)$ be computed via the transformations defined in (11.2.5). Then, following the proof of Lemma 11.2.1, it is easy to show that because of (11.2.11), the matrices F_w, G_w, H_w, and J_w are such that

$$\begin{aligned} F'_w F_w - I &= -H_w H'_w \\ -F'_w G_w &= H_w J_w \\ I - J'_w J_w &= G'_w G_w \end{aligned} \tag{11.2.12}$$

With the above $\{F_w, G_w, H_w, J_w\}$ on hand, we have reached our goal, since it is easily checked that (11.2.12) implies that the real constant matrix M of (11.2.3) is a lossless scattering matrix satisfying (11.2.10). Because M is lossless and constant, it can be synthesized by an $(m + n)$-port lossless nondynamic network N_c containing merely transformer-coupled gyrators. A synthesis of $S(s)$ then follows very simply.

Of course, the synthesis procedure for a general bounded real $S(s)$ stated earlier in this section also applies in this special case of lossless $S(s)$. The only modification necessary lies in step 2, which now reads

2′. Solve the lossless bounded real lemma equations and compute another minimal realization $\{F, G, H, J\}$ satisfying (11.2.11).

The following examples illustrate the reactance extraction synthesis procedure.

Example 11.2.1 Consider the bounded real scattering function (assumed normalized)

$$S(s) = \frac{1}{1 + 2s} = \frac{\frac{1}{2}}{s + \frac{1}{2}}$$

It is not difficult to verify that $\{-\frac{1}{2}, 1/\sqrt{2}, 1/\sqrt{2}, 0\}$ is a minimal realization of $S(s)$ that satisfies (11.1.2) with $L = 1/\sqrt{2}$ and $W_0 = -1$.

From (11.2.5) the realization $\{F_w, G_w, H_w, J_w\}$ is given by $\{-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\}$, so

$$S_a = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

It is readily checked that $I - S'_a S_a \geq 0$ holds.

Suppose that the only inductance required in the synthesis is chosen to be 1 H. Then S_a has unity normalizations at both ports. To synthesize S_a, we deal with the equivalent impedance:

$$\begin{aligned} Z_a &= (I + S_a)(I - S_a)^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= (Z_a)_{sy} + (Z_a)_{sk} \end{aligned}$$

The symmetric part $(Z_a)_{sy}$ has a synthesis in which ports 1 and 2 are decoupled, with port 1 simply a unit resistance and port 2 a short circuit; the skew-symmetric part $(Z_a)_{sk}$ has a synthesis that is simply a unit gyrator. The synthesis of Z_a or of S_a is shown in Fig. 11.2.1. The final synthesis for $S(s)$ shown in Fig. 11.2.2 is obtained by terminating port 2 of the synthesis for S_a in a 1-H inductance. On inspection and on noting

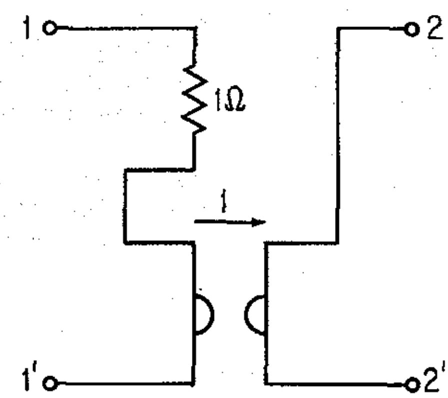

FIGURE 11.2.1. Synthesis of Z_a in Example 11.2.1.

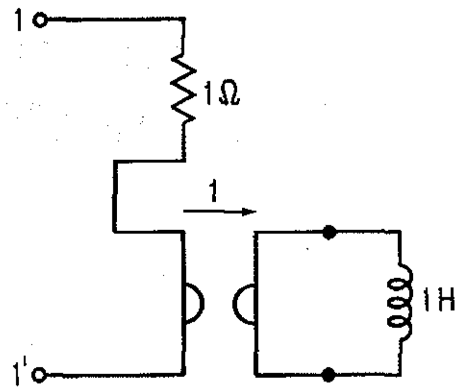

FIGURE 11.2.2. Synthesis of $S(s)$ of Example 11.2.1.

that a unit gyrator terminated in a unit inductance is equivalent to a unit capacitance, as Fig. 11.2.3 illustrates, Fig. 11.2.2 reduces simply to a series connection of a capacitance and a resistance, as shown in Fig. 11.2.4.

Example 11.2.2 As a further illustration, consider the same bounded real scattering function as in Example 11.2.1. But suppose that we now choose the only reactance to be a 1-F capacitance; then the real constant scattering matrix S_a is given by

$$S_a = \begin{bmatrix} J_w & H'_w \\ -G_w & -F_w \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

The matrix $I - S'_a S_a$ may be easily checked to be nonnegative definite. Since the capacitance chosen is 1 F, S_a has unity normalization numbers at all ports. Next, we convert S_a to an admittance function*

$$\begin{aligned} Y_a &= (I - S_a)(I + S_a)^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1 \quad -1] \end{aligned}$$

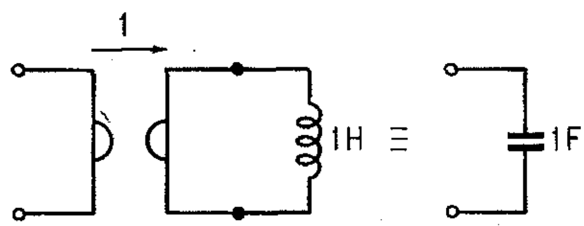

FIGURE 11.2.3. Equivalent Networks

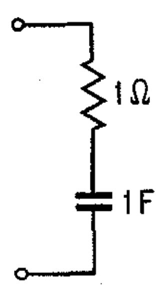

FIGURE 11.2.4. A Network Equivalent to that of Figure 11.2.2.

*Note that an impedance function for S_a does not exist because $I - S_a$ is singular. Of course, if we like, we can (with the aid of an orthogonal transformer) reduce S_a to a lower order $\hat{S}_a$ such that $I - \hat{S}_a$ is now nonsingular—see the earlier preliminary simplification procedures.

Figure 11.2.5 shows a synthesis for Y_a; termination of this network in a unit capacitance at the second port yields a final synthesis of $S(s)$, as shown in Fig. 11.2.6. Of course the network of Fig. 11.2.6 can be simplified to that of Fig. 11.2.4 obtained in the previous example.

Problem 11.2.1 Prove that the transformations defined in (11.2.5) are reversible in the sense that the matrices $\{F_0, G_0, H_0, J\}$ and $\{F_{w_0}, G_{w_0}, H_{w_0}, J_w\}$ may be interchanged.

Problem 11.2.2 Show that among the many possible nonreciprocal network syntheses for a prescribed $S(s)$ obtained by the method considered in this section, there exist syntheses that are minimal resistive; i.e. the number of resistors

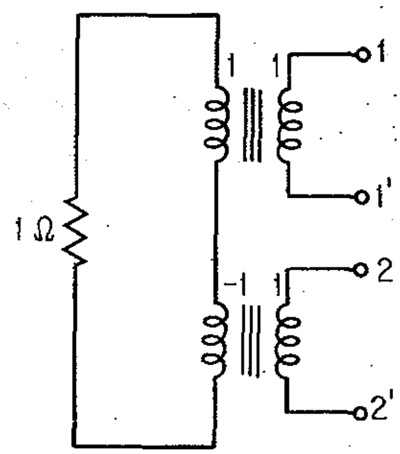

FIGURE 11.2.5. Synthesis of Y_a in Example 11.2.2.

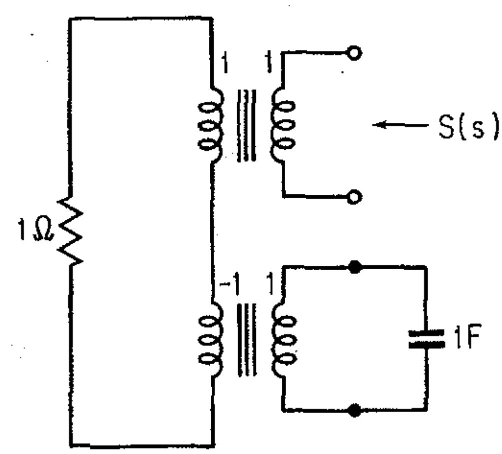

FIGURE 11.2.6. Synthesis of $S(s)$ in Example 11.2.2.

employed in the syntheses is normal rank $[I - S'(-s)S(s)]$. How are they to be obtained?

Problem 11.2.3 *Synthesize the lossless bounded real matrices*

(a) $$S(s) = \frac{1}{10s + 7}\begin{bmatrix} -1 & -10s + 4\sqrt{3} \\ 10s + 4\sqrt{3} & 1 \end{bmatrix}$$

(b) $$S(s) = \frac{1}{2s^2 + 3s + 5}\begin{bmatrix} s + 3 & 2s^2 - 2s + 4 \\ 2s^2 + 2s + 4 & s - 3 \end{bmatrix}$$

Problem 11.2.4 Give two syntheses for the bounded real scattering function $S(s) = (s^2 + s + 1)(2s^2 + 2s + 1)^{-1}$, one with all reactances inductive, and the other with all reactances capacitive.

11.3 REACTANCE-EXTRACTION SYNTHESIS OF RECIPROCAL NETWORKS*

In this section the problem of reciprocal or gyratorless passive synthesis of scattering matrices is discussed. The reactance-extraction technique is still used. One important feature of the synthesis is that it always requires only the minimum number of reactive or energy storage elements; this number n is precisely the degree $\delta[S(s)]$ of $S(s)$, the prescribed $m \times m$ bounded real scattering matrix. The number of dissipative elements generally exceeds rank $[I - S'(-s)S(s)]$ however.

Recall that the underlying idea in the synthesis is to find a state-space realization $\{F_w, G_w, H_w, J_w\}$ of $W(p) = S[(p + 1)/(p - 1)]$ such that the real constant matrix

$$M = \begin{bmatrix} J_w & H'_w \\ G_w & F_w \end{bmatrix} \tag{11.3.1}$$

possesses the properties

$$(1) \quad I - M'M \geq 0, \tag{11.3.2}$$

and

$$(2) \quad (I_m + \Sigma)M = M'(I_m + \Sigma). \tag{11.3.3}$$

Here Σ is a diagonal matrix with diagonal entries being $+1$ or -1 only. Recall also that if the above conditions hold with F_w possessing the minimum possible size $n \times n$, then a synthesis of $S(s)$ can be obtained that uses the minimum number n of inductors and capacitors.

The real constant matrix M defines a real constant matrix

*This section may be omitted at a first reading.

$$S_a = (I_m + \Sigma)M \tag{11.3.4}$$

which is the scattering matrix of a nondynamic network. The normalization numbers of S_a are determined by those of $S(s)$ and by the arbitrarily chosen values of inductances and capacitances that are used when terminating the nondynamic network to yield a synthesis of $S(s)$. In effect, the first condition (11.3.2) says that the scattering matrix S_a possesses a passive synthesis, while the second condition (11.3.3) implies that it has a reciprocal synthesis. When both conditions are fulfilled simultaneously, then a synthesis that is passive and reciprocal is obtainable.

In the previous section we showed that if a minimal realization $\{F_w, G_w, H_w, J_w\}$ of $W(p)$ satisfies

$$\begin{aligned} F'_wF_w - I &= -H_wH'_w - L_wL'_w \\ -F'_wG_w &= H_wJ_w + L_wW_w \\ I - J'_wJ_w &= W'_wW_w + G'_wG_w \end{aligned} \tag{11.3.5}$$

for some real constant matrices L_w and W_w, then the matrix M of (11.3.1) satisfies (11.3.2). Furthermore, the set of minimal realizations $\{F_w, G_w, H_w, J_w\}$ of $W(p)$ satisfying (11.3.5) derives actually from the set of minimal realizations $\{F, G, H, J\}$ of $S(s)$ satisfying the (special) bounded real lemma equations

$$\begin{aligned} F + F' &= -HH' - LL' \\ -G &= HJ + LW_0 \\ I - J'J &= W'_0W_0 \end{aligned} \tag{11.3.6}$$

where L and W_0 are some real matrices. We also noted the procedure required to derive a quadruple $\{F, G, H, J\}$ satisfying (11.3.6), and hence a quadruple $\{F_w, G_w, H_w, J_w\}$ fulfilling (11.3.5). Our objective now is to derive from a minimal realization $\{F_w, G_w, H_w, J_w\}$ satisfying (11.3.5) another minimal realization $\{F_{w1}, G_{w1}, H_{w1}, J_w\}$ for which both conditions (11.3.2) and (11.3.3) hold simultaneously; in doing so, the reciprocal synthesis problem will be essentially solved.

Assume that $\{F_w, G_w, H_w, J_w\}$ is a state-space minimal realization of $W(p)$ [derived from $S(s)$] such that (11.3.5) holds. Now $W(p)$ is obviously symmetric because $S(s)$ is. The symmetry property then guarantees the existence of a unique symmetric nonsingular matrix P with

$$\begin{aligned} PF_w &= F'_wP \\ PG_w &= H_w \end{aligned} \tag{11.3.7}$$

(The same matrix P can also be obtained from $PF = F'P$ and $PG = -H$

incidentally.) Recall that *any* matrix T appearing in a decomposition of P of the form

$$P = T'\Sigma T \qquad \Sigma = [I_{n_1} \dotplus (-1)I_{n_2}]$$

generates another minimal realization $\{TF_wT^{-1}, TG_w, (T^{-1})'H_w, J_w\}$ of $W(p)$ that satisfies the reciprocity condition of (11.3.3) (see Chapter 7). However, there is no guarantee that after the coordinate-basis change the passivity condition remains satisfied. Our aim is to exhibit a specific state-space basis change matrix T from among all possible matrices in the above decomposition of P such that in the new state-space basis, passivity is preserved. The required matrix T will be given in Theorem 11.3.1, but first we require the following two results.

Lemma 11.3.1. Let C be a real matrix similar to a real symmetric matrix D; then $I - D'D$ is positive definite (nonnegative definite) if $I - C'C$ is positive definite (nonnegative definite), but not necessarily conversely.

By using the facts that matrices that are similar to one another possess the same eigenvalues, and that a real symmetric matrix has real eigenvalues only [6], the above lemma can be easily proved. We leave the formal proof of the lemma as an exercise.

The second result we require has, in fact, been established in Lemma 10.3.2; for convenience, we shall repeat it here without proof.

Lemma 11.3.2. Let P be the symmetric nonsingular matrix satisfying (11.3.7), or equivalently $PF = F'P$ and $PG = -H$. Then P can be represented as

$$P = BV\Sigma V' = V\Sigma V'B \tag{11.3.8}$$

where B is symmetric positive definite, V is orthogonal, and $\Sigma = [I_{n_1} \dotplus (-1)I_{n_2}]$. Further, $V\Sigma V'$ commutes with $B^{1/2}$, the unique positive definite square root of B; i.e.,

$$V\Sigma V'B^{1/2} = B^{1/2}V\Sigma V' \tag{11.3.9}$$

These two lemmas provide the key results needed for the proof of the main theorem, which is as follows:

Theorem 11.3.1. Let $\{F, G, H, J\}$ be a minimal realization of a bounded real symmetric $S(s)$ satisfying the special bounded real lemma equations (11.3.6). Let $\{F_w, G_w, H_w, J_w\}$ denote the minimal realization of the associated real symmetric matrix $W(p) =$

$S[(p+1)/(p-1)]$ derived from $\{F, G, H, J\}$ through the transformations

$$\begin{aligned} F_w &= (F+I)(F-I)^{-1} \\ G_w &= \sqrt{2}(F-I)^{-1}G \\ H_w &= -\sqrt{2}(F'-I)^{-1}H \\ J_w &= J - H'(F-I)^{-1}G \end{aligned} \tag{11.3.10}$$

With V, B, and Σ defined as in Lemma 11.3.2, define the nonsingular matrix T by

$$T = V'B^{1/2} \tag{11.3.11}$$

Then the minimal realization $\{F_{w_1}, G_{w_1}, H_{w_1}, J_w\} = \{TF_wT^{-1}, TG_w, (T^{-1})'H_w, J_w\}$ of $W(p)$ is such that the constant matrix

$$M_1 = \begin{bmatrix} J_w & H'_{w_1} \\ G_{w_1} & F_{w_1} \end{bmatrix}$$

has the following properties:

$$I - M_1'M_1 \geq 0 \tag{11.3.12}$$

$$(I_m \dotplus \Sigma)M_1 = M_1'(I_m \dotplus \Sigma) \tag{11.3.13}$$

Proof. We have, from the theorem statement,

$$\begin{aligned} T'\Sigma T &= B^{1/2}V\Sigma V'B^{1/2} \\ &= BV\Sigma V' \\ &= P \end{aligned}$$

where the second and third equalities are implied by Lemma 11.3.2. As noted above, it follows that T defines a new minimal realization $\{F_{w_1}, G_{w_1}, H_{w_1}, J_w\}$ of $W(p)$ such that

$$(I_m \dotplus \Sigma)M_1 = M_1'(I_m \dotplus \Sigma) \tag{11.3.13}$$

holds.

Next, we shall prove that (11.3.12) is also satisfied. With M defined as $\begin{bmatrix} J_w & H'_w \\ G_w & F_w \end{bmatrix}$, it is clear that

$$M_1 = [I \dotplus T]M[I \dotplus T^{-1}]$$

or, on using (11.3.11),

$$\begin{aligned} M_1 &= [I + V'B^{1/2}]M[I + B^{-1/2}V] \\ &= [I + V'B^{1/2}V][I + V']M[I + V][I + V'B^{-1/2}V] \\ &= [I + V'B^{1/2}V]M_0[I + V'B^{-1/2}V] \end{aligned} \tag{11.3.14}$$

where M_0 denotes $[I + V']M[I + V]$. From the theorem statement we know that $I - M'M \geq 0$; it follows readily from the orthogonal property of $I + V$ that

$$I - M_0'M_0 \geq 0 \tag{11.3.15}$$

Equation (11.3.15) is to be used to establish (11.3.12).

On multiplying on the left of (11.3.14) by $I + \Sigma$, we have

$$[I + \Sigma]M_1 = [I + \Sigma][I + V'B^{1/2}V]M_0[I + V'B^{-1/2}V]$$

But from (11.3.9) of Lemma 11.3.2, it can be easily shown that $\Sigma V'B^{1/2}V = V'B^{1/2}V\Sigma$, or that Σ commutes with $V'B^{1/2}V$. Hence $I + \Sigma$ commutes with $I + V'B^{1/2}V$. Therefore, the above equation reduces to

$$[I + \Sigma]M_1 = [I + V'B^{1/2}V][I + \Sigma]M_0[I + V'B^{-1/2}V] \tag{11.3.16}$$

Observe that (11.3.15) can be rewritten as

$$I - M_0'M_0 = I - M_0'(I + \Sigma)(I + \Sigma)M_0 \geq 0$$

Observe also, using (11.3.16), that $(I + \Sigma)M_0$ is similar to $(I + \Sigma)M_1$, which has been shown to be symmetric. It follows from Lemma 11.3.1, on identifying $(I + \Sigma)M_0$ with C and $(I + \Sigma)M_1$ with D, that

$$I - M_1'(I + \Sigma)(I + \Sigma)M_1 \geq 0$$

whence

$$I - M_1'M_1 \geq 0 \qquad \nabla\nabla\nabla$$

Theorem 11.3.1 converts the problem of synthesizing a bounded real symmetric $S(s)$ to the problem of synthesizing a constant bounded real symmetric S_a. The latter problem may be tackled by, for example, converting it to an impedance synthesis problem using the procedures of Chapter 8. In effect, then, Theorem 11.3.1 solves the synthesis problem. A summary of the synthesis procedure is as follows:

1. Find a minimal realization $\{F_0, G_0, H_0, J\}$ for the prescribed $m \times m$ $S(s)$, which is assumed to be bounded real and symmetric.
2. Solve the bounded real lemma equations and compute another minimal realization $\{F, G, H, J\}$ satisfying (11.3.6).
3. Calculate matrices F_w, G_w, H_w, and J_w according to (11.3.10).
4. Compute the matrix P for which $PF_w = F'_wP$ and $PG_w = H_w$, and from P find matrices B, V, and Σ with properties as defined in Lemma 11.3.2.
5. With $T = V'B^{1/2}$, calculate $\{F_{w_1}, G_{w_1}, H_{w_1}, J_w\} = \{TF_wT^{-1}, TG_w, (T^{-1})'H_w, J_w\}$ and obtain the scattering matrix S_a given by

$$S_a = \begin{bmatrix} J_w & H'_{w_1} \\ \Sigma G_{w_1} & \Sigma F_{w_1} \end{bmatrix}$$

where the appropriate normalization number for S_a other than the first m ports is determined by the choice of desired reactances, as explained explicitly in the next step.
6. Choose any set of n_1 inductance values and n_2 capacitance values. Then define the normalization numbers r_{m+l}, $l = 1, 2, \ldots, n$, at the $(m + 1)$th through $(m + n)$th port by

$$r_{m+l} = L_l \qquad l = 1, 2, \ldots, n_1$$

and

$$r_{m+l} = C_l^{-1} \qquad l = n_1 + 1, \ldots, n$$

The normalization numbers at the first m ports are the same as those of $S(s)$.
7. Give a synthesis N_1 for the constant symmetric scattering matrix S_a taking into account the appropriate normalization numbers of S_a. Finally, terminate the last n ports of N_1 appropriately in n_1 inductors and n_2 capacitors to yield a synthesis for the prescribed $S(s)$.

Before illustrating this procedure with an example, we shall discuss the special case of lossless reciprocal synthesis.

Lossless Reciprocal Synthesis

Suppose now that the $m \times m$ symmetric bounded real scattering matrix $S(s)$ is also lossless; i.e., $S'(-s)S(s) = I$. Synthesis could be achieved by carrying out the "preliminary" lossless section extractions of Chapter 8, which for lossless matrices provide a complete synthesis. We assume here that these extractions are not made.

Let $\{F, G, H, J\}$ be a minimal realization of the prescribed lossless $S(s)$.

As pointed out in the previous section, we may take $\{F, G, H, J\}$ to be such that

$$\begin{aligned} F + F' &= -HH' \\ -G &= HJ \\ I - J'J &= I - J^2 = 0 \end{aligned} \tag{11.3.17}$$

[The last equation follows from the symmetry of $S(s)$.] Now the symmetry property of $S(s)$ implies (see Chapter 7) the existence of a symmetric nonsingular matrix A, uniquely defined by the minimal realization $\{F, G, H, J\}$ of $S(s)$, such that

$$\begin{aligned} AF &= F'A \\ AG &= -H \end{aligned} \tag{11.3.18}$$

One can then show that A possesses a decomposition

$$A = U'\Sigma U \tag{11.3.19}$$

in which U is an orthogonal matrix and

$$\Sigma = [I_{n_1} \dotplus (-I_{n_2})] \tag{11.3.20}$$

with $n_1 + n_2 = n = \delta[S(s)]$. The argument is very similar to one in Section 10.2; one shows, using (11.3.17), that A^{-1} satisfies the same equation as A, so that $A = A^{-1}$ or $A^2 = I$. The symmetry of A then yields (11.3.19).

Next we calculate another minimal realization of $S(s)$ given by

$$\{F_1, G_1, H_1, J\} = \{UFU', UG, UH, J\} \tag{11.3.21}$$

with U as defined above. It follows from (11.3.17) and (11.3.18) that $\{F_1, G_1, H_1, J\}$ satisfies the following properties:

$$\begin{aligned} F_1 + F_1' &= -H_1H_1' & -G_1 &= H_1J \\ I - J^2 &= 0 & \Sigma F_1 &= F_1'\Sigma \\ \Sigma G_1 &= -H_1 & J &= J' \end{aligned} \tag{11.3.22}$$

From the minimal realization $\{F_1, G_1, H_1, J\}$, a reciprocal synthesis for $S(s)$ is almost immediate. We derive the real constant matrix

$$S_a = \begin{bmatrix} J_w & H_w' \\ \Sigma G_w & \Sigma F_w \end{bmatrix} \tag{11.3.23}$$

where $\{F_w, G_w, H_w, J_w\}$ are computed from $\{F_1, G_1, H_1, J\}$ by the set of transformations of (11.3.10), viz.,

$$F_w = (F_1 + I)(F_1 - I)^{-1} \qquad G_w = \sqrt{2}(F_1 - I)^{-1}G_1$$
$$H_w = -\sqrt{2}(F_1' - I)^{-1}H_1 \qquad J_w = J - H_1'(F_1 - I)^{-1}G_1$$

Then it follows from (11.3.22) that

$$\begin{aligned} F_w'F_w - I &= -H_wH_w' & -F_w'G_w &= H_wJ_w \\ I - J_w'J_w &= G_w'G_w & \Sigma F_w &= F_w'\Sigma \\ \Sigma G_w &= H_w & J_w &= J_w' \end{aligned} \tag{11.3.24}$$

It is a trivial matter to check that (11.3.24) implies that S_a defined by (11.3.23) is *symmetric and orthogonal*; i.e., $S_a^2 = I$. The $(m + n) \times (m + n)$ real constant matrix S_a, being symmetric and orthogonal, is easily realizable with multiport transformers, open circuits, and short circuits. Thus, by the reactance-extraction technique, a synthesis of $S(s)$ follows from that for S_a. (We terminate the last n_2 ports of the frequency independent network synthesizing S_a in capacitors, and all but the first m ports in inductors.)

Example 11.3.1 Consider the bounded real scattering function

$$S(s) = \frac{3}{2s^2 + 2s + 5}$$

A minimal state-space realization of $S(s)$ satisfying (11.3.6) is

$$F = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad J = [0]$$

Then we calculate matrices F_w, G_w, H_w, and J_w in accordance with (11.3.10). Thus

$$F_w = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad G_w = \begin{bmatrix} -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} \end{bmatrix} \quad H_w = \begin{bmatrix} \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} \end{bmatrix} \quad J_w = [\tfrac{1}{3}]$$

These matrices of course readily yield a passive synthesis of $S(s)$, as the previous section shows; the synthesis however is a nonreciprocal one, as may be verified easily.

We compute next a symmetric nonsingular matrix P with $PF_w = F_w'P$ and $PG_w = H_w$. The result is

$$P = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

for which

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad V = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore

$$T = V'B^{1/2} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Then we obtain $\{F_w, G_w, H_w, J_w\} = \{TF_wT^{-1}, TG_w, (T^{-1})'H_w, J_w\}$ as follows:

$$F_{w_1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \quad G_{w_1} = \begin{bmatrix} 0 \\ -\frac{2}{3} \end{bmatrix} \quad H_{w_1} = \begin{bmatrix} 0 \\ \frac{2}{3} \end{bmatrix} \quad J_w = [\tfrac{1}{3}]$$

It can be readily checked that

$$S_a = \begin{bmatrix} J_{w_1} & H'_{w_1} \\ \Sigma G_{w_1} & \Sigma F_{w_1} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

is a constant scattering matrix satisfying both the reciprocity and the passivity conditions. From this point the synthesis procedure is straightforward and will be omitted.

Problem 11.3.1 Prove Lemma 11.3.1.

Problem 11.3.2 From a minimal realization $\{F, G, H, J\}$ of a symmetric bounded real $S(s)$ satisfying (11.3.6), derive a reciprocal passive reactance extraction synthesis of $S(s)$ under the condition that *no further state-space basis transformation is to be applied* to $\{F, G, H, J\}$ above. Note that the transformations of (11.3.10) used to generate $\{F_w, G_w, H_w, J_w\}$ may be used and that there is no restriction on the number of reactive elements required in the synthesis. [*Hint:* Use the identity

$$S(s) = J + \frac{H'}{\sqrt{2}}(sI - F)^{-1}\frac{G}{\sqrt{2}} + \frac{G'}{\sqrt{2}}(sI - F')^{-1}\frac{H}{\sqrt{2}}\Bigg]$$

Problem 11.3.3 Using the method outlined in this section, give reciprocal syntheses for

(a) The symmetric lossless bounded real scattering matrix

$$S(s) = \begin{bmatrix} \dfrac{s}{s^2 + 2s + 2} & \dfrac{2}{s(s^2 + 2s + 2)} \\ \dfrac{2}{s(s^2 + 2s + 2)} & \dfrac{-s}{s^2 + 2s + 2} \end{bmatrix}$$

(b) The bounded real scattering function

$$S(s) = \frac{s^2 + s + 1}{2s^2 + 2s + 1}$$

11.4 RESISTANCE EXTRACTION SCATTERING MATRIX SYNTHESIS

In this section we present a state-space version of the classical Belevitch resistance extraction synthesis of bounded real scattering matrices [7]. The key idea of the synthesis consists of augmenting a prescribed scattering matrix $S(s)$ to obtain a lossless scattering matrix $S_L(s)$ in such a manner that terminating a lossless network N_L which synthesizes $S_L(s)$ in unit resistors yields a synthesis of the prescribed $S(s)$ (see Fig. 11.4.1). This idea has been discussed briefly in Chapter 8.

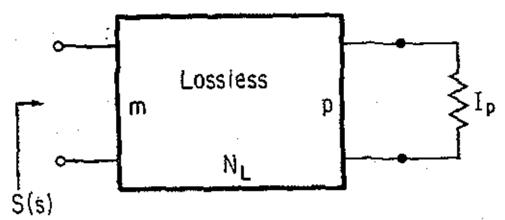

FIGURE 11.4.1. Resistance Extraction.

Suppose that the required lossless scattering matrix $S_L(s)$ is given by

$$S_L(s) = \begin{bmatrix} S(s) & S_{12}(s) \\ S_{21}(s) & S_{22}(s) \end{bmatrix} \tag{11.4.1}$$

where $S(s)$ denotes the $m \times m$ prescribed scattering matrix and the dimension of $S_{22}(s)$ is $p \times p$, say. Here what we have done is to augment $S(s)$ by p rows and columns to obtain a lossless $S_L(s)$. There is however one important constraint—a result stated in Theorem 8.3.1—which is that the number p must not be smaller than ρ = normal rank $[I - S'(-s)S(s)]$, the minimum number of resistors required in any synthesis of $S(s)$. Now the lossless constraint on $S_L(s)$, i.e.,

$$S'_L(-s)S_L(s) = S_L(s)S'_L(-s) = I$$

requires the following key equations to hold:

$$
\begin{aligned}
S'(-s)S(s) + S'_{21}(-s)S_{21}(s) &= I = S(s)S'(-s) + S_{12}(s)S'_{12}(-s) \\
S'_{12}(-s)S(s) + S'_{22}(-s)S_{21}(s) & \\
&= 0 = S_{21}(s)S'(-s) + S_{22}(s)S'_{12}(-s)
\end{aligned}
\tag{11.4.2}
$$

In the Belevitch synthesis and the later Oono and Yasuura synthesis [8], p is taken to be ρ; i.e., the syntheses use the minimum number of resistors. The crucial step in both the above syntheses lies in the factorization of $I - S'(-s)S(s)$ to obtain $S_{21}(s)$ and $I - S(s)S'(-s)$ to obtain $S_{12}(s)$ [see (11.4.2)]. The remaining matrix $S_{22}(s)$ is then given by $-S_{21}(s)S'(-s)[S'_{12}(-s)]^{-1}$ where precise meaning has to be given to the inverse. To obtain $S_{21}(s)$ and $S_{12}(s)$, Belevitch uses a Gauss factorization procedure, while Oono and Yasuura introduce a polynomial factorization scheme based on the theory of invariant factors. The method of Oono and Yasuura is far more complicated computationally than the Belevitch method, but it allows the equivalent network problem to be tackled.

An inherent property of both the Belevitch and the basic Oono and Yasuura procedures is that $\delta[S_L(s)] > \delta[S(s)]$ usually results; this means that the syntheses usually require more than the minimum number of reactive elements. However, the difficult Oono and Yasuura equivalent network theory does provide a technique for modifying the basic synthesis procedure to obtain a minimal reactive synthesis. In contrast, it is possible to achieve a minimal reactive synthesis *by state-space means* in a simple fashion.

Nonreciprocal Synthesis

Suppose that an $m \times m$ bounded real $S(s)$ is to be synthesized. We have learned from Chapter 8 that the underlying problem is one of finding constant real matrices F_L, G_L, H_L, and J_L—which constitute a minimal realization of the desired $S_L(s)$—such that

1. The equations of the lossless bounded real lemma hold:

$$
\begin{aligned}
PF_L + F'_L P &= -H_L H'_L \\
-PG_L &= H_L J_L \\
J'_L J_L &= I
\end{aligned}
\tag{11.4.3}
$$

   where P is symmetric positive definite.
2. With the partitioning

$$
G_L = [G_{L_1} \quad G_{L_2}] \qquad H_L = [H_{L_1} \quad H_{L_2}] \qquad J_L = \begin{bmatrix} J_{L_1} & J_{L_2} \\ J_{L_3} & J_{L_4} \end{bmatrix}
$$

   where G_{L_1} and H_{L_1} have m columns, and J_{L_1} has m columns and rows, $\{F_L, G_{L_1}, H_{L_1}, J_{L_1}\}$ is a realization of the prescribed $S(s)$.

If a lossless network N_L is found synthesizing the lossless scattering matrix $J_L + H_L'(sI - F_L)^{-1}G_L$, termination of all but the first m ports of this network N_L in unit resistances then yields a synthesis of $S(s)$.

Proceeding now with the synthesis procedure, suppose that $\{F, G, H, J\}$ is a *minimal* realization of $S(s)$, such that

$$\begin{aligned} F + F' &= -HH' - LL' \\ -G &= HJ + LW_0 \\ I - J'J &= W_0'W_0 \end{aligned} \tag{11.4.4}$$

with L' and W_0 having the minimum number of p rows. (Note that $p =$ normal rank $[I - S'(-s)S(s)]$.)

We shall now construct an $(m + p) \times (m + p)$ matrix $S_L(s)$, or, more precisely, constant matrices $\{F_L, G_L, H_L, J_L\}$, which represent a minimal realization of $S_L(s)$.

We start by computing the $(m + p) \times (m + p)$ real constant matrix

$$J_L = S_L(\infty) = \begin{bmatrix} J & J_{12} \\ W_0 & J_{22} \end{bmatrix} \tag{11.4.5}$$

The first m columns of J_L are orthonormal, by virtue of the relation $I - J'J = W_0'W_0$. It is easy to choose the remaining p columns so that the whole matrix J_L is orthogonal; i.e.,

$$J_L'J_L = J_LJ_L' = I \tag{11.4.6}$$

Next, we define real constant matrices F_L, G_L, and H_L by

$$F_L = F \qquad H_L = [H \quad L] \qquad G_L = [G \quad -HJ_{12} - LJ_{22}] \tag{11.4.7}$$

With these definitions, it is immediate from (11.4.4) that

$$-H_LH_L' = F_L + F_L' \qquad H_LJ_L = -G_L \tag{11.4.8}$$

hold, that is, Eqs. (11.4.3) hold with the matrix P being the identity matrix I. Also it is evident from the definitions of F_L, G_L, H_L, and J_L in (11.4.5) and (11.4.7) that the appropriate submatrices of G_L, H_L, and J_L constitute with F_L a realization, in fact, a minimal one, of the prescribed $S(s)$.

We conclude that the quadruple $\{F_L, G_L, H_L, J_L\}$ of (11.4.5) and (11.4.7) defines a lossless scattering matrix for the coupling network N_L. A synthesis of the lossless $S_L(s)$, and hence a synthesis of $S(s)$, may be obtained by available classical methods [9], the technique of Chapter 8, or the state-space approach discussed in Section 11.2. The latter method is preferable in this case, since one avoids computation of the lossless scattering matrix $S_L(s) =$

$J_L + H_L'(sI - F_L)^{-1}G_L$, and, more significantly, the matrices F_L, G_L, H_L, and J_L are such that a synthesis of the lossless network is immediately obtainable —*without any further need to change the state-space coordinate basis.* This is because $\{F_L, G_L, H_L, J_L\}$ satisfies the lossless bounded real lemma equation with $P = I$.

A quick study of the synthesis procedure will immediately reveal that

$$\delta[S_L(s)] = \text{dimension of } F_L = \delta[S(s)]$$

This implies that a synthesis of $S_L(s)$ is possible that uses $\delta[S(s)]$ reactive components. Therefore, the synthesis of $S(s)$ can be obtained with the minimum number of reactive elements. Moreover, the synthesis is also minimal resistive. This follows from the fact that ρ resistors are required in the synthesis, while also $\rho = \text{normal rank } [I - S'(-s)S(s)]$.

Example 11.4.1 Consider the bounded real scattering function

$$S(s) = \frac{s+3}{2s^2 + 3s + 5}$$

A minimal realization of $S(s)$ satisfying (11.4.4) is

$$F = \begin{bmatrix} -1 & -1 \\ 2 & -\frac{1}{2} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad H = \begin{bmatrix} \sqrt{2} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad J = [0]$$

with

$$L' = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad W_0 = [1]$$

Next, matrices $\{F_L, G_L, H_L, J_L\}$ are computed according to (11.4.5) and (11.4.7); the result is

$$F_L = \begin{bmatrix} -1 & -1 \\ 2 & -\frac{1}{2} \end{bmatrix} \qquad G_L = \begin{bmatrix} 0 & -\sqrt{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$H_L = \begin{bmatrix} 2 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad J_L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

It can be checked then that $S_L(s)$, found from the above $\{F_L, G_L, H_L, J_L\}$, is given by

$$S_L(s) = \frac{1}{2s^2 + 3s + 5}\begin{bmatrix} s+3 & 2s^2 - 2s + 4 \\ 2s^2 + 2s + 4 & s - 3 \end{bmatrix}$$

and is a lossless bounded real matrix; i.e., $I - S_L'(-s)S_L(s) = 0$. Furthermore, the (1, 1) principal submatrix is obviously $S(s)$. The remainder of the synthesis is straightforward and will be omitted.

Note that in the above example $S(s)$ is symmetric but $S_L(s)$ is not. This is true in general. Therefore, the above method gives in general a nonreciprocal synthesis. Let us now study how a reciprocal synthesis might be achieved.

Reciprocal Synthesis*

We consider now a prescribed $m \times m$ scattering matrix $S(s)$ that is bounded real and symmetric. The key idea in reciprocal synthesis of $S(s)$ is the same as that of the nonreciprocal synthesis just considered. Here we shall seek to construct a quadruple $\{F_L, G_L, H_L, J_L\}$, which represents a minimal realization of an $(m + r) \times (m + r)$ lossless $S_L(s)$, such that the same two conditions as for nonreciprocal synthesis are fulfilled, and also the third condition

$$J_L = J_L' \qquad H_L'(sI - F_L)^{-1}G_L = G_L'(sI - F_L')^{-1}H_L \tag{11.4.9}$$

The number of rows and columns r used to border $S(s)$ in forming $S_L(s)$, corresponding to the number of resistors used in the synthesis, is yet to be specified. This of course must not be less than ρ = normal rank $[I - S'(-s)S(s)]$. The third condition, which simply means that $S_L(s) = S_L'(s)$, is added here, since a reciprocal synthesis of $S(s)$ follows only if the lossless coupling network N_L synthesizing $S_L(s)$ is reciprocal.

As we have stated several times, it is not possible in general to obtain a reciprocal synthesis of $S(s)$ that is simultaneously minimal resistive and minimal reactive; with our goal of a minimal reactive reciprocal synthesis, the number of resistors r may therefore be greater than the normal rank ρ of $[I - S'(-s)S(s)]$. For this reason it is obvious that the method of nonreciprocal synthesis considered earlier needs to be modified. As also noted earlier, preliminary extractions can be used to guarantee that $I - S'(-\infty)S(\infty)$ is nonsingular. We shall assume this is true here. It then follows that normal rank $[I - S'(-s)S(s)] = m$.

Before we can compute a quadruple $\{F_L, G_L, H_L, J_L\}$, it is necessary to derive an appropriate minimal realization $\{F_1, G_1, H_1, J\}$ for the prescribed $S(s)$, so that from it $\{F_L, G_L, H_L, J_L\}$ may be computed.

The following lemma sets out the procedure for this.

Lemma 11.4.1. Let $\{F, G, H, J\}$ be a minimal realization of a symmetric bounded real $S(s)$ such that, for some L and W_0,

*The remainder of this section may be omitted at a first reading.

$$F + F' = -HH' - LL'$$
$$-G = HJ + LW_0 \qquad (11.4.4)$$
$$I - J'J = W_0'W_0$$

Let P be the unique symmetric matrix, whose existence is guaranteed by the symmetry of $S(s)$, that satisfies $PF = F'P$ and $PG = -H$, and let B be positive definite and V orthogonal such that $P = BV\Sigma V' = V\Sigma V'B$. (See Lemma 11.3.2.) Here $\Sigma = [I_{n_1} \dotplus (-1)I_{n_2}]$ with $n_1 + n_2 = n = \delta[S(s)]$. Define a coordinate basis change matrix T by

$$T = V'B^{1/2} \qquad (11.4.10)$$

and a new minimal realization of $S(s)$ by

$$\{F_1, G_1, H_1, J\} = \{TFT^{-1}, TG, (T^{-1})'H, J\} \qquad (11.4.11)$$

Then the following equations hold:

$$\Sigma F_1 = F_1'\Sigma \qquad \Sigma G_1 = -H_1 \qquad J = J' \qquad (11.4.12)$$

and

$$\begin{bmatrix} I - J^2 & -(G_1 + H_1J)' \\ -(G_1 + H_1J) & -(F_1 + F_1' + H_1H_1') \end{bmatrix} \geq 0 \qquad (11.4.13)$$

A complete proof will be omitted, but is requested in the problems. Note that the initial realization $\{F, G, H, J\}$ satisfies the special bounded real lemma equations, but in general will not satisfy (11.4.12). The coordinate basis change matrix T, because it satisfies $T'\Sigma T = P$ (as may easily be verified), leads to (11.4.12) holding. The choice of a particular T satisfying $T'\Sigma T = P$, rather than an arbitrary T, leads to the nonnegativity constraint (11.4.13). The argument establishing (11.4.13) is analogous to a number of earlier arguments in the proofs of similar results.

The construction of matrices of a minimal realization of $S_L(s)$ is our prime task. For the moment however, we need to study a consequence of (11.4.12) and (11.4.13).

Lemma 11.4.2. Let $\{F_1, G_1, H_1, J\}$ and Σ be defined as in Lemma 11.4.1. Then there exists an $n_1 \times n_1$ nonnegative symmetric Q_1 and an $n_2 \times n_2$ nonnegative symmetric Q_2 such that

$$-(F_1 + F_1' + H_1H_1') - (G_1 + H_1J)(I - J^2)^{-1}(G_1 + H_1J)' = Q_1 \dotplus Q_2 \qquad (11.4.14)$$

Proof. That the left side of (11.4.14) is nonnegative definite follows from (11.4.13). Nonsingularity of $I - S'(-\infty)S(\infty)$ implies nonsingularity of $I - J^2$. Now set

$$-(F_1 + F_1' + H_1H_1') - (G_1 + H_1J)(I - J^2)^{-1}(G_1 + H_1J)' = Q$$

Simple manipulations lead to

$$\begin{aligned} Q = -(F_1 + F_1') - G_1(I - J^2)^{-1}G_1' - H_1(I - J^2)^{-1}H_1' \\ - G_1J(I - J^2)^{-1}H_1' - H_1J(I - J^2)^{-1}G_1' \end{aligned}$$

Now premultiply and postmultiply by Σ. The right-hand side is seen to be unaltered, using (11.4.12). Therefore

$$\Sigma Q \Sigma = Q$$

which means that $Q = Q_1 \dotplus Q_2$ as required. ∇ ∇ ∇

With Lemmas 11.4.1 and 11.4.2 in hand, it is now possible to define a minimal realization of $S_L(s)$ of dimension $(2m + n) \times (2m + n)$, which serves to yield a resistance-extraction synthesis. Subsequent to the proof of the theorem containing the main result, we shall indicate how the dimensions of $S_L(s)$ may sometimes be reduced.

Theorem 11.4.1. Let $\{F_1, G_1, H_1, J\}$ and Σ be defined as in Lemma 11.4.1 and Q_1 and Q_2 be defined as in Lemma 11.4.2. Define also

$$\begin{aligned} F_L &= F_1 \\ G_L &= [G_1 \mid -H_1(I - J^2)^{1/2} \\ &\quad - (G_1 + H_1J)(I - J^2)^{-1/2}J \mid (-Q_1^{1/2}) \dotplus Q_2^{1/2}] \\ H_L &= [H_1 \mid -(G_1 + H_1J)(I - J^2)^{-1/2} \mid Q_1^{1/2} \dotplus Q_2^{1/2}] \\ J_L &= \begin{bmatrix} J & (I - J^2)^{1/2} & 0 \\ (I - J^2)^{1/2} & -J & 0 \\ 0 & 0 & \Sigma \end{bmatrix} \end{aligned} \tag{11.4.15}$$

Then

1. The top left $m \times m$ submatrix of $S_L(s)$ is $S(s)$.
2. $S_L(s)$ is lossless bounded real; in fact,

$$\begin{aligned} F_L + F_L' &= -H_LH_L' \\ -G_L &= H_LJ_L \\ I - J_L'J_L &= 0 \end{aligned} \tag{11.4.16}$$

3. $S_L(s)$ is symmetric; in fact,

$$\Sigma F_L = F_L'\Sigma \qquad \Sigma G_L = -H_L \qquad J_L = J_L' \tag{11.4.17}$$

Proof. The first claim is trivial to verify, and we examine now (11.4.16). Using the definitions in (11.4.15), we have

$$\begin{aligned}F_L + F_L' + H_L H_L' = F_1 + F_1' + H_1 H_1' \\ + (G_1 + H_1 J)(I - J^2)^{-1}(G_1 + H_1 J)' \\ + (Q_1 + Q_2)\end{aligned}$$

which is zero, by (11.4.14). Also,

$$\begin{aligned}H_L J_L &= [H_1 J - (G_1 + H_1 J) \mid H_1(I - J^2)^{1/2} \\ &\quad + (G_1 + H_1 J)(I - J^2)^{-1/2} J \mid Q_1^{1/2} + (-Q_2^{1/2})] \\ &= -G_L\end{aligned}$$

Verification of the third equation in (11.4.16) is immediate. Finally, (11.4.17) must be considered. The first equation in (11.4.17) is immediate from the definition of F_L and from (11.4.12). The third equation is immediate from the definition of J_L. Finally,

$$\begin{aligned}\Sigma G_L &= [\Sigma G_1 \mid -\Sigma H_1(I - J^2)^{1/2} \\ &\quad - (\Sigma G_1 + \Sigma H_1 J)(I - J^2)^{-1/2} J \mid (-Q_1^{1/2}) + (-Q_2^{1/2})] \\ &= [-H_1 \mid G_1(I - J^2)^{1/2} \\ &\quad + (H_1 + G_1 J)(I - J^2)^{-1/2} J \mid (-Q_1^{1/2}) + (-Q_2^{1/2})] \\ &= [-H_1 \mid (G_1 + H_1 J)(I - J^2)^{-1/2} \mid (-Q_1^{1/2}) + (-Q_2^{1/2})] \\ &= -H_L \qquad \nabla\nabla\nabla\end{aligned}$$

The above theorem implicitly contains a procedure for synthesizing $S(s)$ with $r = (m + n)$ resistors. If Q is singular, having rank $n^* < n$, say, it becomes possible to use $r = (m + n^*)$ resistors. Define R_1 and R_2 as matrices having a number of rows equal to their rank and satisfying

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} R_1' & 0 \\ 0 & R_2' \end{bmatrix}\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \tag{11.4.18}$$

Between them, R_1 and R_2 will have n^* rows. Let R_1 have n_1^* rows and R_2 have n_2^* rows. Define $\Sigma^* = I_{n_1^*} \dotplus (-1)I_{n_2^*}$. Then instead of the definitions of (11.4.15) for G_L, H_L, and J_L, we have

$$\begin{aligned} G_L &= \left[G_1 \,\vdots\, -H_1(I - J^2)^{1/2} - (G_1 + H_1J)(I - J^2)^{-1/2}J \,\vdots \begin{bmatrix} -R_1' & 0 \\ 0 & R_2' \end{bmatrix} \right] \\ H_L &= \left[H_1 \,\vdots\, -(G_1 + H_1J)(I - J^2)^{-1/2} \,\vdots \begin{bmatrix} R_1' & 0 \\ 0 & R_2' \end{bmatrix} \right] \\ J_L &= \begin{bmatrix} J & (I - J^2)^{1/2} & 0 \\ (I - J^2)^{1/2} & -J & 0 \\ 0 & 0 & \Sigma^* \end{bmatrix} \end{aligned} \tag{11.4.19}$$

These claims may be verified by working through the proof of Theorem 11.4.1 with but minor adjustments.

Let us now summarize the synthesis procedure.

1. From a minimal realization $\{F, G, H, J\}$ of $S(s)$ that satisfies (11.4.4), calculate the symmetric nonsingular matrix P satisfying $PF = F'P$ and $PG = -H$.
2. Express P in the form $BV\Sigma V'$, with B positive definite, V orthogonal, and $\Sigma = [I_{n_1} \dotplus (-1)I_{n_2}]$. Then with $T = V'B^{1/2}$, form another minimal realization $\{F_1, G_1, H_1, J\}$ of $S(s)$ given by $\{TFT^{-1}, TG, (T^{-1})'H, J\}$.
3. With Q_1 and Q_2 defined as in (11.4.14), define F_L, G_L, H_L, and J_L as in (11.4.15), or by (11.4.18) and (11.4.19).
4. Synthesize $S_L(s)$ by, for example, application of the preliminary extraction procedures of Chapter 8 or via the reactance-extraction procedure, and terminate in unit resistors all but the first m ports of the network synthesizing $S_L(s)$. Note that because the quadruple $\{F_L, G_L, H_L, J_L\}$ satisfies (11.4.16) and (11.4.17), reactance-extraction synthesis is particularly straightforward. No coordinate basis change is necessary, and $S_L(s)$ itself does not have to be calculated.

Example 11.4.2 Consider the same bounded real scattering function

$$S(s) = \frac{s + 3}{2s^2 + 3s + 5}$$

as in Example 11.4.1. A minimal realization of $S(s)$ that satisfies (11.4.4) has been found to be

$$F = \begin{bmatrix} -1 & -1 \\ 2 & -\frac{1}{2} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad H = \begin{bmatrix} \sqrt{2} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \quad J = [0]$$

Then the symmetric nonsingular matrix P, satisfying $PF = F'P$ and $PG = -H$, is computed to be

$$P = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$$

from which $\Sigma = [1 \dotplus -1]$ and

$$T = V'B^{1/2} = \begin{bmatrix} 1.81 & 0.75 \\ 0.52 & -1.25 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 0.47 & 0.28 \\ 0.20 & -0.68 \end{bmatrix}$$

Thus, another minimal realization of $S(s)$ is

$$F_1 = \begin{bmatrix} -0.57 & 1.40 \\ -1.40 & -0.93 \end{bmatrix} \quad G_1 = \begin{bmatrix} -0.53 \\ 0.88 \end{bmatrix} \quad H_1 = \begin{bmatrix} 0.53 \\ 0.88 \end{bmatrix} \quad J = [0]$$

Using (11.4.14), we obtain $Q_1 = 0.58$ and $Q_2 = 0.32$. The formulas (11.4.15) then yield

$$F_L = \begin{bmatrix} -0.57 & 1.40 \\ -1.40 & -0.93 \end{bmatrix} \qquad G_L = \begin{bmatrix} -0.53 & -0.53 & -0.76 & 0 \\ 0.88 & -0.88 & 0 & 0.57 \end{bmatrix}$$

$$H_L = \begin{bmatrix} 0.53 & 0.53 & 0.76 & 0 \\ 0.88 & -0.88 & 0 & 0.57 \end{bmatrix} \qquad J_L = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Synthesis from this point is straightforward.

Problem 11.4.1 Prove Lemma 11.4.1.

Problem 11.4.2 Given the bounded real scattering function

$$S(s) = \frac{-s^2 + 1}{3s^2 + 2s + 3}$$

give a nonreciprocal and a reciprocal synthesis.

Problem 11.4.3 Synthesize the symmetric bounded real scattering matrix

$$S(s) = \frac{1}{7s + 8}\begin{bmatrix} 3s & 2s + 4 \\ 2s + 4 & -s - 2 \end{bmatrix}$$

using the reciprocal synthesis procedure of this section.

Problem 11.4.4 Show that Lemma 11.4.2 and Theorem 11.4.1 will cover the case when $I - J^2$ is singular, if $(I - J^2)^{-1}$ is replaced when it occurs by $(I - J^2)^{\#}$, i.e., the pseudo inverse of $I - J^2$. [The pseudo inverse of a symmetric matrix A is defined as $A^{\#} = V'(B^{-1} \dotplus 0)V$, where V is an orthogonal matrix such that $A = V'(B \dotplus 0)V$ with B nonsingular.]

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12

Transfer-Function Synthesis

12.1 INTRODUCTION

Probably the most practical application of passive network synthesis lies in the synthesis of transfer functions or transfer-function matrices rather than immittances, though, to be sure, immittance synthesis is often a concomitant part of transfer-function synthesis. For example, in matching an antenna to a transmitter output stage, a passive coupling network is required, the design of which may be based on impedance matching at the two ports of the coupling network.

As we shall see, state-space transfer-function synthesis is rather less developed than immittance or scattering synthesis. Even the approaches we shall present are somewhat disjoint, though the same could probably be said of the many classical approaches to transfer-function synthesis. There is probably scope therefore for great advancement, with practical payoffs, in the use of state-space procedures for transfer-function synthesis. To give some idea for the basis of this statement, we might note that much of classical theory demands particular structures, for example, the lattice and ladder structures. In any given situation, it may well be that neither of these structures offers especially good sensitivity characteristics; i.e., the performance of a filter using one of these structures may deteriorate severely in the face of variation of one or more of the element values. Now workers in control systems have found that the state-space description of systems offers a convenient vehicle for analysis and synthesis of prescribed sensitivity characteristics. Thus, presumably, one could expect low sensitivity network designs to follow via state-space procedures.

We now describe the basic idea behind the transfer-function (and transfer-function matrix) syntheses of this chapter. For convenience in describing the approach, but with no loss of generality, suppose that the input variables are currents (or a single current in the nonmatrix situation) and the output variables are all voltages. We shall also suppose that any port at which there is an exciting input variable is not a port at which we measure an output variable. Figure 12.1.1 illustrates the situation; here, I_1 is a vector of currents,

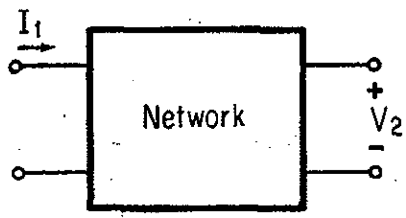

FIGURE 12.1.1. Response and Excitation Variables Associated with Different Ports.

the inputs, and V_2 a vector of voltages, the outputs. We are given the transfer-function matrix relating I_1 to V_2, and we seek a network that synthesizes this matrix. Generally, the transfer-function matrix is in state-space form.

The key step in the synthesis is to conceive of the prescribed transfer-function matrix as being a submatrix of a passive impedance matrix. Thus if $T(s)$ is the transfer-function matrix, we conceive of a positive real $Z(s)$ such that

$$Z(s) = \begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ T(s) & Z_{22}(s) \end{bmatrix} \tag{12.1.1}$$

In synthesizing $Z(s)$ with a network, we thereby obtain a synthesis of $T(s)$. For observe that

$$\begin{bmatrix} V_1(s) \\ V_2(s) \end{bmatrix} = \begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ T(s) & Z_{22}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ I_2(s) \end{bmatrix} \tag{12.1.2}$$

where the port voltage and current vectors have been partitioned to conform with the partition of $Z(s)$. If now $I_2(s) = 0$, i.e., the second set of ports are open circuit, we have

$$V_2(s) = T(s)I_1(s) \tag{12.1.3}$$

This is, of course the relation required.

As will be shown, passage from $T(s)$ to an appropriate positive real $Z(s)$ is not especially difficult. In fact, we can make life easy for ourselves in the following way. First, instead of passing from the frequency-domain quantity $T(s)$ to a frequency-domain quantity $Z(s)$, we shall pass from a state-space description of $T(s)$ to a state-space description of $Z(s)$. Second, we shall ensure that the matrices appearing in the state-space description of $Z(s)$ satisfy the equations of the positive real lemma with the matrix P of that lemma equal to the identity matrix; i.e., if $\{F_z, G_z, H_z, J_z\}$ is the state-space realization of $Z(s)$ formed from a state-space realization of $T(s)$, then

$$\begin{aligned} F_z + F_z' &= -L_z L_z' \\ G_z &= H_z - L_z W_{0z} \\ J_z + J_z' &= W_{0z}' W_{0z} \end{aligned} \tag{12.1.4}$$

for some L_z and W_{0z}. As we know, $Z(s)$ may easily be synthesized given a realization satisfying (12.1.4). Therefore, we shall regard the transfer-function-matrix synthesis problem as solved if a $Z(s)$ can be found with a realization satisfying (12.1.4), and we shall not carry the synthesis procedure further.

The above discussion has been based on the assumption that all input variables have been taken to be currents and all output variables as voltages. This assumption can be removed by embedding the transfer-function matrix $T(s)$ in a hybrid matrix $\mathcal{H}(s)$ rather than an impedance matrix. Consider, for example, the case when $T(s)$ is a scalar current-to-current transfer function. Then $\mathcal{H}(s)$ would be found as the hybrid matrix of a two port, with equations

$$\begin{bmatrix} V_1(s) \\ I_2(s) \end{bmatrix} = \begin{bmatrix} \mathcal{H}_{11}(s) & \mathcal{H}_{12}(s) \\ T(s) & \mathcal{H}_{22}(s) \end{bmatrix} \begin{bmatrix} I_1(s) \\ V_2(s) \end{bmatrix} \tag{12.1.5}$$

Because of the practical importance of gyratorless circuits, we can seek to ensure satisfaction of a symmetry constraint on $Z(s)$, in addition to the other constraints. In view of our earlier discussion of reciprocal synthesis, we may and shall regard the problem of reciprocal transfer function synthesis as solved if we can embed the prescribed $T(s)$ in a symmetric $Z(s)$, the latter with a state-space realization $\{F_z, G_z, H_z, J_z\}$ satisfying (12.1.4).

In all the synthesis procedures, as we have remarked, we begin with a state-space realization, actually a minimal one, of the prescribed transfer-function matrix $T(s)$. Call this realization $\{F_T, G_T, H_T, J_T\}$. We shall generally require the additional constraint

$$F_T + F_T' \leq 0 \tag{12.1.6}$$

or even

$$F_T + F_T' < 0 \tag{12.1.7}$$

If the constraint is not satisfied for an arbitrary realization, and it may well not be, we can easily carry out a coordinate-basis change to ensure that it is satisfied, provided that the following assumption holds.

> **Assumption 12.1.1.** The prescribed transfer-function matrix $T(s)$ is such that the poles of every element lie in $\text{Re}[s] < 0$, or are simple on $\text{Re}[s] = 0$.

(The physical meaning of this restriction is obvious.) To see that (12.1.6) can be achieved starting with an arbitrary realization, we proceed as follows. Let $\{F_{T_1}, G_{T_1}, H_{T_1}, J_{T_1}\}$ be an arbitrary realization. If F_{T_1} has all eigenvalues in $\text{Re}[s] < 0$, take Q as an arbitrary positive definite matrix, and define P as the positive definite solution of

$$PF_{T_1} + F'_{T_1}P = -Q \tag{12.1.8}$$

With R any matrix such that $R'R = P$, take

$$F_T = RF_{T_1}R^{-1} \qquad G_T = RG_{T_1} \qquad H_T = (R^{-1})'H_{T_1} \tag{12.1.9}$$

Then (12.1.8) and (12.1.9) imply that

$$F_T + F'_T = -(R')^{-1}QR^{-1} < 0 \tag{12.1.10}$$

Notice that the computation of P and R is straightforward (see, e.g., [1]).

If F_{T_1} has eigenvalues with zero real part, we must first find a matrix R_1 such that

$$R_1F_{T_1}R_1^{-1} = \begin{bmatrix} \mathfrak{F}_{T_2} & 0 \\ 0 & \mathfrak{F}_{T_3} \end{bmatrix}$$

where $\mathfrak{F}_{T_2}$ has all eigenvalues with negative real parts, and $\mathfrak{F}_{T_3}$ is skew. A further transformation R_2 can be found so that

$$R_2(R_1F_{T_1}R_1^{-1})R_2^{-1} = \begin{bmatrix} \mathfrak{F}_{T_4} & 0 \\ 0 & \mathfrak{F}_{T_3} \end{bmatrix}$$

with $\mathfrak{F}_{T_4} + \mathfrak{F}'_{T_4} < 0$. [This can be done by the technique described in Eqs. (12.1.8) through (12.1.10) and the associated remarks.] Now set

$$R = R_2R_1 \tag{12.1.11}$$

and define F_T, G_T, and H_T by (12.1.9). It follows from the properties of $\mathfrak{F}_{T_3}$ and $\mathfrak{F}_{T_4}$ that

$$F_T + F'_T \leq 0 \tag{12.1.6}$$

Note that if F_T has zero real part eigenvalues, it is impossible to have

$$F_T + F'_T < 0 \tag{12.1.7}$$

(The lemma of Lyapunov would be contradicted in this instance.)

We shall now give a brief summary of the remainder of this chapter. In Section 12.2 we consider two techniques for nonreciprocal synthesis, the first being based on a technique due to Silverman [2]. In Section 12.3 reciprocal synthesis is considered. In Section 12.4 we consider synthesis with prescribed load terminations, discussing results of [3].

One approach to state-space transfer-function synthesis that we do not discuss in detail is that based on ladder network synthesis [4–7]. As was noted in Section 4.2, the network of Fig. 12.1.2 has the state equations

$$\dot{x} = \begin{bmatrix} -\frac{1}{C_1R_2} & \frac{1}{\sqrt{L_1C_1}} & 0 & \cdots & & & \\ -\frac{1}{\sqrt{L_1C_1}} & 0 & \frac{1}{\sqrt{L_1C_2}} & & & & \\ 0 & -\frac{1}{\sqrt{L_1C_2}} & 0 & & & & \\ \vdots & & & \ddots & & & \\ & & & & 0 & \frac{1}{\sqrt{L_{n-1}C_{n-1}}} & 0 \\ & & & & -\frac{1}{\sqrt{L_{n-1}C_{n-1}}} & 0 & \frac{1}{\sqrt{L_{n-1}C_n}} \\ & & & & 0 & -\frac{1}{\sqrt{L_{n-1}C_n}} & -\frac{1}{C_nR_1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \\ \frac{1}{\sqrt{C_nR_1}} \end{bmatrix} V \tag{12.1.12}$$

$$I_{R_1} = \begin{bmatrix} \frac{1}{\sqrt{C_1R_2}} & 0 & \cdots & 0 \end{bmatrix} x$$

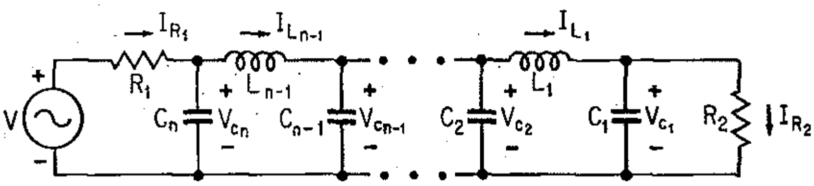

FIGURE 12.1.2. Ladder Network with State Equation (12.1.12).

The transfer function relating $V(s)$ to $I_{R_2}(s)$ is of the form

$$\frac{I_{R_2}(s)}{V(s)} = T(s) = \frac{a_0}{s^n + a_n s^{n-1} + \cdots + a_1} \tag{12.1.13}$$

and the a_i in (12.1.13) are related to the element values of the circuit of Fig. 12.1.2 by straightforward formulas set out in the references. Thus, hardly with intervention of state-space techniques, (12.1.13) can be synthesized by a ladder network. Synthesis of transfer admittances with numerators other than a constant can also be achieved, as discussed in [7], by using the same basic network, but adjusting the points at which excitations are applied or responses are measured.

Problem 12.1.1 Show that procedures for the synthesis of a transfer-function matrix $T(s)$ with $T(\infty)$ infinite, but with $\lim_{s\to\infty} T(s)/s$ finite, can be derived from procedures for synthesizing transfer-function matrices $T(s)$ for which $T(\infty)$ is finite.

12.2 NONRECIPROCAL TRANSFER-FUNCTION SYNTHESIS

In this section we shall translate into more precise terms the transfer-function synthesis technique outlined in the previous section. We suppose that we are given a transfer-function matrix $T(s)$ with a minimal realization $\{F_T, G_T, H_T, J_T\}$. We suppose moreover that

$$F_T + F_T' \leq 0 \tag{12.2.1}$$

Without loss of generality, we shall assume that $T(s)$ is a current-to-voltage transfer-function matrix. We shall suppose too that F_T is $n \times n$, G_T is $n \times p$, H_T is $n \times m$, and J_T is $m \times p$.

We now define quantities F_z, G_z, H_z, and J_z by

$$F_z = F_T \qquad G_z = [G_T \quad H_T] \qquad H_z = [G_T \quad H_T]$$
$$J_z = \begin{bmatrix} 0 & -J_T' \\ J_T & 0 \end{bmatrix} \tag{12.2.2}$$

The quadruple $\{F_z, G_z, H_z, J_z\}$ is a realization, actually a minimal one, of a matrix $Z(s)$ that is $(p + m) \times (p + m)$. If $Z(s)$ is partitioned as

$$Z(s) = \begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \end{bmatrix} \tag{12.2.3}$$

with $Z_{11}(s)$ a $p \times p$ matrix, it is easy to verify that

$$Z_{21}(s) = T(s) \tag{12.2.4}$$

Observe also, using (12.2.1) and (12.2.2), that

$$\begin{aligned} F_z + F'_z &= -L_z L'_z \\ G_z &= H_z \\ J_z + J'_z &= 0 \end{aligned} \tag{12.2.5}$$

where the existence of L_z follows from (12.2.1) and the fact that $F_z = F_T$.

Equations (12.2.5) are the positive real lemma equations with the matrix P equal to the identity matrix. Therefore, the synthesis problem is essentially solved.

Notice that the computations required for transfer-function synthesis are much simpler than for immittance synthesis. The positive real lemma equations never have to be solved.

Notice also that if the synthesis is completed by, say, the reactance-extraction technique, the number of reactive elements used will be n, where n is the dimension of F_z and F_T. Since F_T is of minimal dimension, the synthesis therefore uses a minimal number of reactive elements. It does not necessarily use a minimal number of resistive elements, however. Problem 12.2.1 develops a synthesis for scalar transfer functions using the minimal number of resistors, and also asks the student to look at such syntheses for transfer-function matrices.

Example 12.2.1 Consider the Butterworth function

$$T(s) = \frac{\frac{1}{2}}{s^2 + \sqrt{2}\,s + 1}$$

A minimal realization for $T(s)$ is easily derived as

$$F_T = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix} \quad G_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H_T = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \quad J_T = 0$$

Notice that F_T has nonpositive-definite symmetric part. Next, we construct

$$F_z = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix} \qquad G_z = H_z = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \qquad J_z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

To synthesize the impedance of which these matrices are a minimal realization, we form

$$M = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -1 \\ 1 & 0 & 1 & \sqrt{2} \end{bmatrix}$$

and find a nondynamic network of which this is the impedance matrix. We note that

$$\tfrac{1}{2}(M + M') = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sqrt[4]{2} \end{bmatrix} [1][0 \quad 0 \quad 0 \quad \sqrt[4]{2}]$$

and

$$\tfrac{1}{2}(M - M') = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2^{-1/2} \\ 1.5 & 0 & 2^{-1/2} & 0 \\ 0 & -1 & 0 & 2^{-1/2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2^{-1/2} & 0 \\ 0 & 2^{-1/2} & 0 & 2^{-1/2} \end{bmatrix}$$

Thus the nondynamic network of impedance matrix M is that of Fig. 12.2.1, and the two-port network having the prescribed transfer function is obtained by terminating the network above at its last two ports in unit inductances, as shown in Fig. 12.2.2.

As the reader will appreciate, the synthesis of a transfer-function matrix has a high degree of nonuniqueness about it. The procedure we have given has the merit of being simple, but ensures that $T(s)$ is embedded in a positive real $Z(s)$ that is uniquely determined by $T(s)$. Of course, there are an infinity of positive real $Z(s)$ for which $T(s)$ is a submatrix. So it may well be that

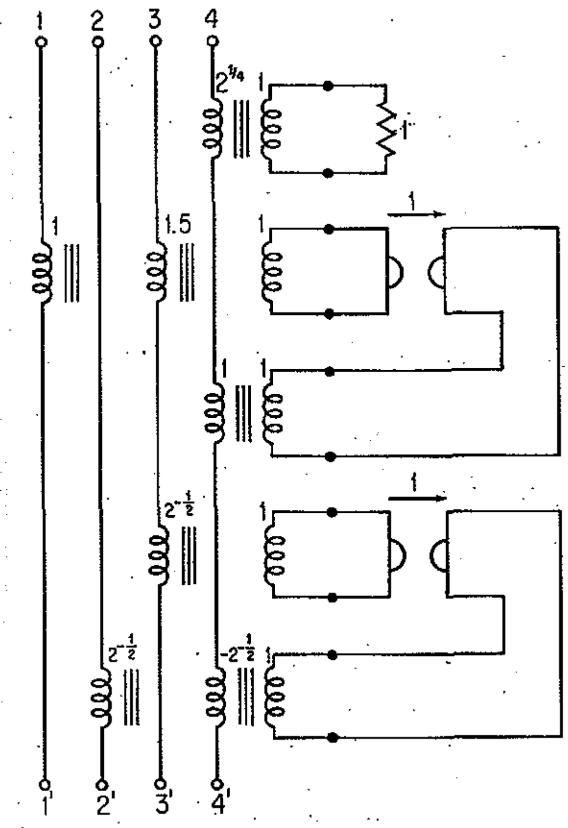

FIGURE 12.2.1. Nondynamic Network of Example 12.2.1.

there are other synthesis procedures almost, if not equally, as convenient, which depend on $T(s)$ being embedded in a $Z(s)$ different from that used above. We shall examine one such procedure here.

In the procedure now to be presented we shall restrict consideration to those $T(s)$ for which all eigenvalues of the matrix F_T in a minimal realization of $T(s)$ lie in $\text{Re}\,[s] < 0$. Thus we can ensure that

$$F_T + F_T' < 0 \tag{12.2.6}$$

Without loss of generality, we shall take $T(s)$ to be an $m \times p$ current-to-voltage transfer-function matrix, and we will embed it in a $(p + m) \times (p + m)$ $Z(s)$ of the form

$$Z(s) = \begin{bmatrix} J_{11} & 0 \\ T(s) & J_{22} \end{bmatrix} \tag{12.2.7}$$

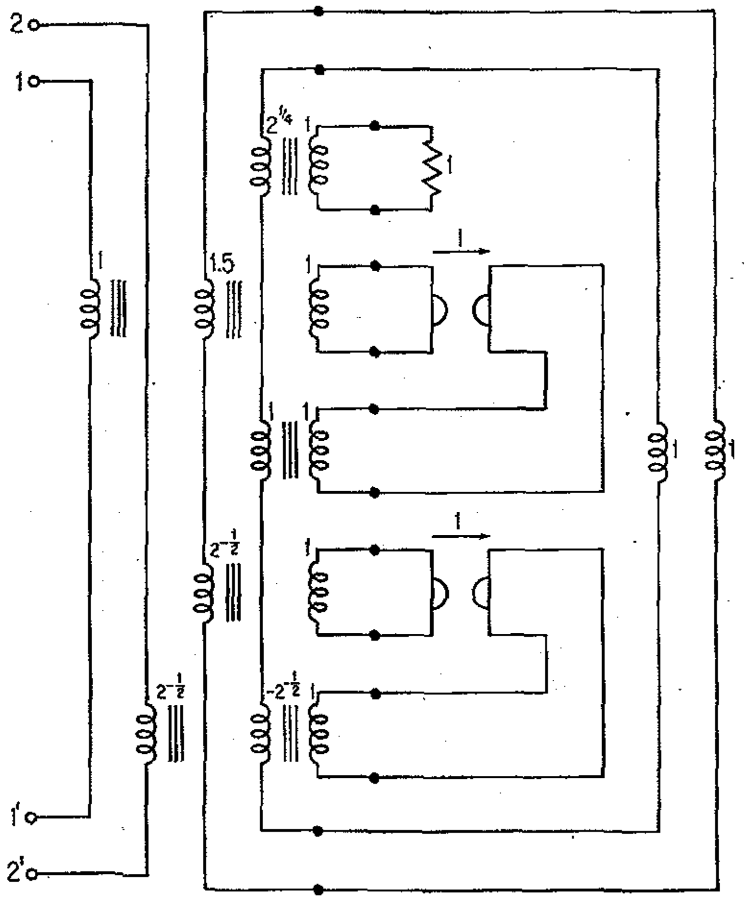

FIGURE 12.2.2. Synthesis of Transfer Function $\frac{1}{2}(s^2 + \sqrt{2}s + 1)^{-1}$.

where J_{11} and J_{22} are constant. *Notice that this implies that there is zero transmission from the second set of ports to the first set;* the voltage at the first set of ports depends purely on the current at those ports and is independent of the current at the second set of ports. (It is easy to envisage practical situations where this may be helpful.) Extension to the case when the zero block is replaced by a nonzero block is also possible (see Problem 12.2.6).

We define a realization of $Z(s)$ in the following way:

$$F_z = F_T \qquad G_z = [G_T \quad 0] \qquad H_z = [0 \quad H_T] \tag{12.2.8}$$

The matrix J_z will be defined shortly. First, we shall define L_z and W_{0z}. With

$T(s)$ an $m \times p$ matrix, let $s = \min\{m, p\}$. With F_T an $n \times n$ matrix, let S be any nonsingular $n \times n$ matrix such that

$$F_T + F_T' = -SS' \tag{12.2.9}$$

(A triangular S is readily computed.) Then we define

$$L_z = [0_{n\times s} \quad S] \tag{12.2.10}$$

Notice that

$$F_z + F_z' = -L_z L_z' \tag{12.2.11}$$

The matrix L_z has n rows and $s + n$ columns; the matrix W_{0z} will have $s + n$ rows and $p + m$ columns. Let us partition it as

$$W_{0z} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{12.2.12}$$

where W_{11} is $s \times p$. The dimensions of the remaining submatrices are then automatically defined. Next, temporarily postponing definition of W_{11} and W_{12}, we define

$$W_{21} = -S^{-1}G_T \qquad W_{22} = S^{-1}H_T \tag{12.2.13}$$

Now observe that

$$\begin{aligned} L_z W_{0z} &= [SW_{21} \quad SW_{22}] \\ &= [-G_T \quad H_T] \\ &= -G_z + H_z \end{aligned} \tag{12.2.14}$$

It remains to define J_z, W_{11}, and W_{12}. We let W_{11} and W_{12} be any matrices such that

$$W_{12}'W_{11} = J_T - W_{22}'W_{21} \tag{12.2.15}$$

Notice that J_T is $m \times p$, while W_{11} is $s \times p$ and W_{12} is $s \times m$. Since $s = \min\{m, p\}$, there certainly exist W_{11} and W_{12} satisfying (12.2.15). Finally, define

$$J_z = \begin{bmatrix} \frac{1}{2}(W_{11}'W_{11} + W_{21}'W_{21}) & 0 \\ J_T & \frac{1}{2}(W_{12}'W_{12} + W_{22}'W_{22}) \end{bmatrix} \tag{12.2.16}$$

Equations (12.2.12), (12.2.15), and (12.2.16) imply that

$$J_z + J_z' = W_{0z}'W_{0z} \tag{12.2.17}$$

Equations (12.2.8) and (12.2.16) define the minimal realization of $Z(s)$, with the matrices W_{ij} in (12.2.16) defined by (12.2.9) through (12.2.15). Equations (12.2.11), (12.2.14), and (12.2.17) show that the realization $\{F_z, G_z, H_z, J_z\}$ satisfies the positive real lemma equations with the P matrix equal to the identity matrix. Equations (12.2.8) imply that

$$H'_z(sI - F_z)^{-1}G_z = \begin{bmatrix} 0 & 0 \\ H'_T(sI - F_T)^{-1}G_T & 0 \end{bmatrix}$$

and, taken in conjunction with (12.2.16), this means that $Z(s)$ has the desired form of (12.2.7):

$$Z(s) = \begin{bmatrix} J_{11} & 0 \\ T(s) & J_{22} \end{bmatrix} \tag{12.2.7}$$

The synthesis procedure is therefore essentially complete. Notice that, again, a minimal number of reactive elements is used. The number of resistive elements will, however, be large (see Problem 12.2.4).

As a summary of the above synthesis procedure, the following steps are noted:

1. Check that the prescribed current-to-voltage transfer function $T(s)$ fulfills the condition that poles of each element all lie in the strict left half-plane, $\text{Re}[s] < 0$.
2. Obtain a minimal realization $\{F_T, G_T, H_T, J_T\}$ of $T(s)$ for which $F_T + F'_T < 0$.
3. Calculate any nonsingular matrix S such that $F_T + F'_T = -SS'$.
4. Compute matrices W_{21}, W_{22} using (12.2.13) and W_{11}, W_{12} for which (12.2.15) holds.
5. Obtain F_z, G_z, H_z simply via (12.2.8) and J_z from (12.2.16). Give a synthesis by any known procedure for the positive real impedance $Z(s)$ for which $\{F_z, G_z, H_z, J_z\}$ is a minimal realization. The network so obtained synthesizes the prescribed $T(s)$.

Example 12.2.2 We take again

$$T(s) = \frac{\frac{1}{2}}{s^2 + \sqrt{2}s + 1}$$

For a minimal realization, we adopt

$$F_T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad G_T = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad H_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad J_T = 0$$

so that, following (12.2.8),

$$F_z = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad G_z = \begin{bmatrix} 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad H_z = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A matrix S satisfying (12.2.9) is given by

$$S = \begin{bmatrix} \sqrt[4]{2} & 0 \\ 0 & \sqrt[4]{2} \end{bmatrix}$$

and so, with $s = 1$,

$$L_z = \begin{bmatrix} 0 & \sqrt[4]{2} & 0 \\ 0 & 0 & \sqrt[4]{2} \end{bmatrix}$$

Next, W_{21} and W_{22} are given from (12.2.13) by

$$W_{21} = \begin{bmatrix} 0 \\ -2^{-3/4} \end{bmatrix} \qquad W_{22} = \begin{bmatrix} 2^{-1/4} \\ 0 \end{bmatrix}$$

Matrices W_{12} and W_{11} satisfying (12.2.15) are

$$W_{11} = W_{12} = 0$$

From (12.2.16) this leads to

$$J_z = \begin{bmatrix} \frac{1}{4\sqrt{2}} & 0 \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix}$$

With the quantities F_z, G_z, H_z, and J_z known, a synthesis of $Z(s)$ can be achieved by, for example, the reactance-extraction approach. We form the constant impedance matrix

$$M = \begin{bmatrix} \frac{1}{4\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & -1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We have then

$$\frac{1}{2}(M+M')=\begin{bmatrix} \frac{1}{4\sqrt{2}} & 0 & 0 & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\sqrt{2} \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{4\sqrt{2}} & 0 \\ 0 & \frac{1}{2\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -\sqrt{2} & 0 \end{bmatrix}$$

and

$$\frac{1}{2}(M-M')=\begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{2} \\ \frac{5}{2} & 0 & 1 & 0 \\ 0 & -\frac{1}{2\sqrt{2}} & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4\sqrt{2}} \\ 0 & 0 & -\frac{1}{4\sqrt{2}} & 0 \end{bmatrix}$$

$$\times\begin{bmatrix} 1 & 0 & \frac{5}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & 2\sqrt{2} & 0 & 1 \end{bmatrix}$$

A synthesis for the impedance matrix M is shown in Fig. 12.2.3, while a final circuit realizing the prescribed transfer function is given by Fig. 12.2.4.

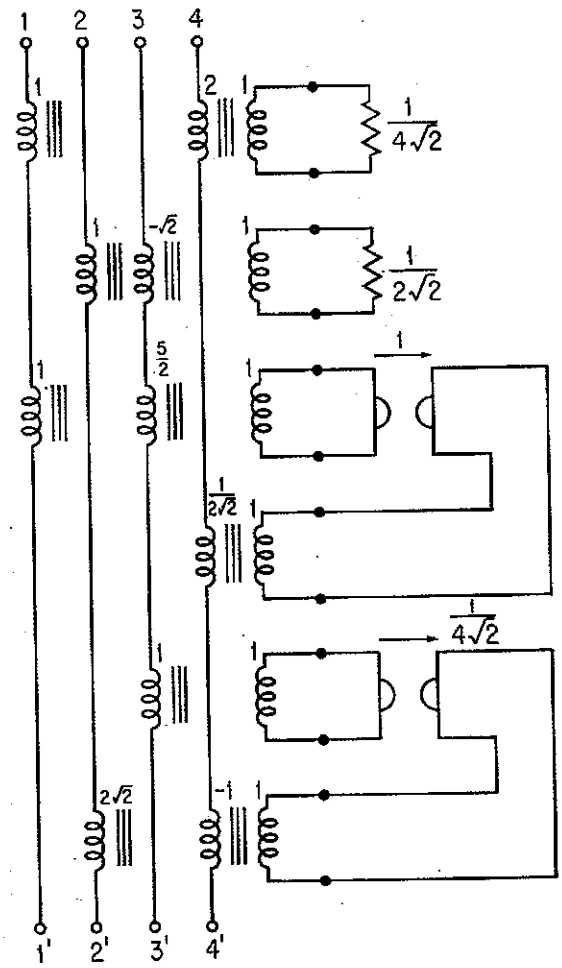

FIGURE 12.2.3. Nondynamic Network of Example 12.2.2.

Problem 12.2.1 This problem investigates synthesis using a minimal number of resistive elements. Suppose that $T(s)$ is a scalar transfer function, with minimal realization $\{F_{T_1}, g_{T_1}, h_{T_1}, j_{T_1}\}$ and with all eigenvalues of F_{T_1} in $\text{Re}[s] < 0$. Because $[F_{T_1}, h_{T_1}]$ is a completely observable pair, there exists a positive definite symmetric P such that

$$PF_{T_1} + F'_{T_1}P = -h_{T_1}h'_{T_1}$$

Carry out a coordinate-basis change, derived from P, and show that $T(s)$ can be embedded in a 2×2 positive real matrix $Z(s)$ with minimal reali-

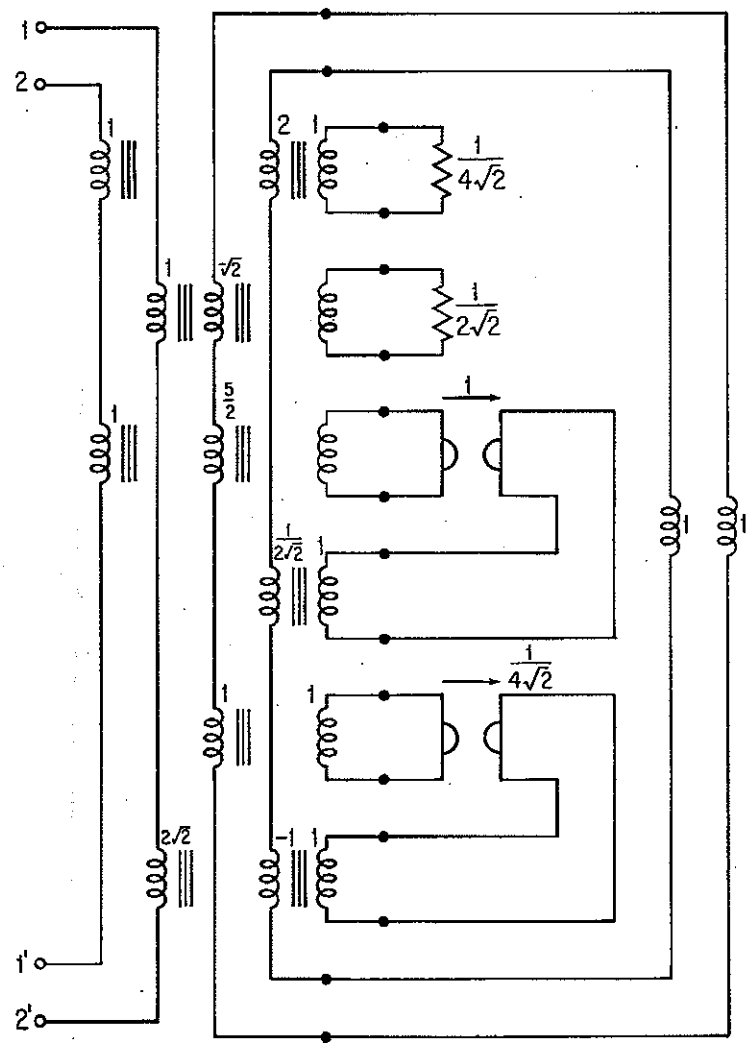

FIGURE 12.2.4. Synthesis of Transfer Function $\frac{1}{2}(s^2 + \sqrt{2}s + 1)^{-1}$.

zation $\{F_z, G_z, H_z, J_z\}$ and such that (12.2.5) is satisfied with L_z a vector. Conclude the existence of a synthesis using only one resistor. Show also that there is no synthesis containing no resistors. What happens for matrix $T(s)$?

Problem 12.2.2 Show that if $T(s)$ is a transfer-function matrix such that each element has poles that are always simple and with zero real part, then $T(s)$ can be synthesized by a lossless network.

Problem 12.2.3 Synthesize the scalar voltage-to-voltage transfer function

$$T(s) = \frac{1}{(s+1)(s^2+s+1)}$$

Problem 12.2.4 Compute the number of resistors resulting from the second synthesis procedure presented.

Problem 12.2.5 Synthesize the scalar voltage-to-voltage transfer function

$$T(s) = \frac{\frac{1}{2}}{s^2+\sqrt{2}\,s+1}$$

using the second method outlined in the text.

Problem 12.2.6 Suppose that a passive synthesis is required of two current-to-voltage transfer-function matrices $T_{12}(s)$ and $T_{21}(s)$. These transfer-function matrices relate variables $I_1(s)$, $I_2(s)$, $V_1(s)$, and $V_2(s)$ according to the equations

$$V_1(s) = T_{12}(s)I_2(s) \qquad V_2(s) = T_{21}(s)I_1(s)$$

with $I_1(s)$ and $V_1(s)$ being associated with the same set of ports, and likewise for $I_2(s)$ and $V_2(s)$. Develop a synthesis procedure following ideas like those used in the second procedure discussed in the text. By taking $T_{21}(s) = T'_{12}(s)$, can one embed $T_{12}(s)$ in a symmetric $Z(s)$?

12.3 RECIPROCAL TRANSFER-FUNCTION SYNTHESIS*

In this section we demand that the network synthesizing a prescribed transfer function be reciprocal. We shall start with a minimal realization F_T, G_T, H_T, J_T of a prescribed $m \times p$ transfer-function matrix $T(s)$, assumed for convenience to be current to voltage. We shall moreover suppose that

$$F_T + F'_T < 0 \tag{12.3.1}$$

The strict inequality is equivalent to a requirement that the poles of each element of $T(s)$ should all lie in Re $[s] < 0$, rather than some poles possibly lying on Re $[s] = 0$. A straightforward way to remove this restriction is not known.

As the first step in the synthesis we shall change the coordinate basis so that in addition to (12.3.1), we have

$$\Sigma F_T = F'_T \Sigma \tag{12.3.2}$$

*This section may be omitted at a first reading.

where Σ is a diagonal matrix of the form $I_{n_1} \dotplus (-I_{n_2})$. Then we shall construct a minimal realization $\{F_z, G_z, H_z, J_z\}$ of a $(p+m) \times (p+m)$ positive real impedance matrix $Z(s)$ satisfying the equations

$$\begin{aligned} F_z + F_z' &= -L_z L_z' \\ G_z &= H_z - L_z W_{0z} \\ J_z + J_z' &= W_{0z}' W_{0z} \end{aligned} \tag{12.3.3}$$

and

$$\begin{aligned} \Sigma F_z &= F_z' \Sigma \\ \Sigma G_z &= -H_z \\ J_z &= J_z' \end{aligned} \tag{12.3.4}$$

Equations (12.3.3) imply that a passive synthesis of $Z(s)$ may easily be found, while (12.3.4) imply that this synthesis may be made reciprocal. Of course, $Z(s)$ has $T(s)$ as a submatrix.

Naturally, the calculations required for reciprocal synthesis are more involved than those for nonreciprocal synthesis. However, at no stage do the positive real lemma equations have to be solved.

We now proceed with the first part of the synthesis. Suppose that we have a minimal realization $\{F_{T_1}, G_{T_1}, H_{T_1}, J_{T_1}\}$ of $T(s)$ such that

$$F_{T_1} + F_{T_1}' < 0 \tag{12.3.5}$$

while $\Sigma F_{T_1} \neq F_{T_1}' \Sigma$. Lemma 12.3.1 explains how a coordinate-basis change may be used to derive F_T satisfying (12.3.1) and (12.3.2).

> **Lemma 12.3.1.** Suppose that F_{T_1} satisfies Eq. (12.3.5). With A any matrix such that $AF_{T_1} = F_{T_1}'A$, there exist a positive definite symmetric B, an orthogonal V, and a matrix $\Sigma = I_{n_1} \dotplus (-I_{n_2})$ such that
>
> $$A + A' = BV\Sigma V' = V\Sigma V'B \tag{12.3.6}$$
>
> Further, with
>
> $$R = V'B^{1/2} \tag{12.3.7}$$
>
> and
>
> $$F_T = RF_{T_1}R^{-1} \tag{12.3.8}$$
>
> the matrix F_T satisfies (12.3.1) and (12.3.2).

This lemma is a special case of Theorem 10.3.1, and its proof will not be included.

With Lemma 12.3.1 in hand, we can now derive a minimal realization of an appropriate impedance $Z(s)$, such that $T(s)$ is a submatrix and such that Eqs. (12.3.3) and (12.3.4) are satisfied by the matrices of the state-space realization of $Z(s)$.

The matrices F_z, G_z, and H_z are defined by

$$F_z = F_T \qquad G_z = [G_T \quad -\Sigma H_T] \qquad H_z = [-\Sigma G_T \quad H_T] \tag{12.3.9}$$

We shall define J_z shortly. Notice that, except for the equation $J_z = J_z'$, the reciprocity equations (12.3.4) hold.

To define J_z, we need first to define L_z and W_{0z}. Recalling that $F_T + F_T'$ is negative definite, let S be an $n \times n$ matrix (where F_T is $n \times n$) such that

$$F_T + F_T' = -SS' \tag{12.3.10}$$

(It is easy to compute such an S, especially a triangular S.) Let $s = \min \{p, m\}$, and set, similarly to the procedure of Section 12.2,

$$L_z = [0_{n \times s} \quad S] \tag{12.3.11}$$

Observe that

$$F_z + F_z' = -L_z L_z' \tag{12.3.12}$$

as required. Next, again following Section 12.2, we form W_{0z}, which is a matrix of $(s + n)$ rows and $(p + m)$ columns. We shall identify submatrices of W_{0z} according to the partitioning

$$W_{0z} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \tag{12.3.13}$$

where W_{11} is $s \times p$ and the dimensions of the other submatrices are automatically defined. Now we set (in contrast to Section 12.2)

$$W_{21} = -S^{-1}(\Sigma + I)G_T \tag{12.3.14}$$

$$W_{22} = S^{-1}(\Sigma + I)H_T \tag{12.3.15}$$

These definitions are made so that the equation

$$G_z = H_z - L_z W_{0z} \tag{12.3.16}$$

holds. To check this, observe from (12.3.11) and (12.3.13) that

$$\begin{aligned} L_z W_{0z} &= [SW_{21} \quad SW_{22}] \\ &= [-(\Sigma + I)G_T \quad (\Sigma + I)H_T] \qquad \text{by (12.3.14) and (12.3.15)} \\ &= H_z - G_z \qquad \text{by (12.3.9)} \end{aligned}$$

Next, as in Section 12.2, we define W_{11} and W_{12} as any pair of matrices satisfying

$$W'_{12}W_{11} = 2J_T - W'_{22}W_{21} \tag{12.3.17}$$

Notice that J is $m \times p$, while W_{11} and W_{12} are, respectively, $s \times p$ and $s \times m$. With $s = \min\{p, m\}$, it is always possible to find W_{11} and W_{12} satisfying this equation. Finally, we define

$$J_z = \frac{1}{2}\begin{bmatrix} W'_{11}W_{11} + W'_{21}W_{21} & W'_{11}W_{12} + W'_{21}W_{22} \\ W'_{12}W_{11} + W'_{22}W_{21} & W'_{12}W_{12} + W'_{22}W_{22} \end{bmatrix} \tag{12.3.18}$$

This ensures that J_z is symmetric, that

$$J_z + J'_z = W'_{0z}W_{0z} \tag{12.3.19}$$

and finally that the 2–1 entry of J_z is, by (12.3.17), precisely J_T. This fact, together with Eqs. (12.3.9), *implies that $T(s)$ is the $m \times p$ lower left submatrix* of the $(p + m) \times (p + m)$ matrix $Z(s)$ with realization $\{F_z, G_z, H_z, J_z\}$. Equations (12.3.12), (12.3.16), and (12.3.19) jointly amount to (12.3.3), while the reciprocity equations (12.3.4) have also been demonstrated. This completes a statement of the synthesis procedure.

Notice that if the remainder of the synthesis proceeds by, say, the reactance-extraction approach, n reactive elements will be used, where n is the size of F_z and F_T. Since the dimension of F_T is minimal, this means that $T(s)$ is synthesized with the *minimum* number of reactive elements.

To summarize, the synthesis procedure for a current-to-voltage transfer-function matrix is now given:

1. Examine whether poles of each element of the prescribed $T(s)$ lie in the left half-plane $\text{Re}[s] < 0$.
2. Compute a minimal realization $\{F_{T_1}, G_{T_1}, H_{T_1}, J_{T_1}\}$ of $T(s)$ with $F_{T_1} + F'_{T_1} < 0$.
3. With the coordinate-basis change R found according to Lemma 12.3.1, another minimal realization $\{F_T, G_T, H_T, J_T\}$ of $T(s)$ is computed.
4. Calculate a nonsingular matrix S such that $F_T + F'_T = -SS'$.
5. From (12.3.14) and (12.3.15) calculate matrices W_{21} and W_{22}, respectively; then obtain any pair of matrices W_{11} and W_{12} for which (12.3.17) holds.
6. Calculate J_z from (12.3.18) and F_z, G_z, and H_z from (12.3.9).
7. Give a synthesis for the positive real impedance matrix $Z(s)$ for which $\{F_z, G_z, H_z, J_z\}$ is a minimal realization.

Example 12.3.1 We shall synthesize the current-to-voltage transfer function

$$T(s) = \frac{\frac{1}{2}}{s^2 + \sqrt{2}s + 1}$$

As a minimal realization, we take

$$F_T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad G_T = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad H_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad J_T = 0$$

Observe that $\Sigma F_T = F_T'\Sigma$, where

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Following the procedure specified, we take

$$F_z = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad G_z = \begin{bmatrix} 0 & -1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \qquad H_z = \begin{bmatrix} 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

A matrix S satisfying $F_T + F_T' = -SS'$ is given by

$$S = \begin{bmatrix} \sqrt[4]{2} & 0 \\ 0 & \sqrt[4]{2} \end{bmatrix}$$

whence

$$L_z = \begin{bmatrix} 0 & \sqrt[4]{2} & 0 \\ 0 & 0 & \sqrt[4]{2} \end{bmatrix}$$

Also

$$\begin{aligned} W_{21} &= -S^{-1}(\Sigma + I)G_T \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} W_{22} &= S^{-1}(\Sigma + I)H_T \\ &= \begin{bmatrix} 2^{3/4} \\ 0 \end{bmatrix} \end{aligned}$$

Using (12.3.17), we see that $W_{11} = W_{12} = 0$ is adequate, so that by (12.3.18),

$$J_z = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

with

$$W_{0z} = \begin{bmatrix} 0 & 0 \\ 0 & 2^{3/4} \\ 0 & 0 \end{bmatrix}$$

Now we synthesize $Z(s)$ with minimal realization $\{F_z, G_z, H_z, J_z\}$. This requires us to form the hybrid matrix of a nondynamic network as

$$M = \begin{bmatrix} J_z & -H_z' \\ G_z & -F_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \sqrt{2} & -1 & 0 \\ 0 & -1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The upper 3×3 diagonal submatrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & -1 \\ 0 & -1 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt[4]{2} \\ -2^{-1/4} \end{bmatrix} [1] [0 \quad \sqrt[4]{2} \quad -2^{-1/4}]$$

corresponds to an impedance matrix, while the lower diagonal submatrix $[1/\sqrt{2}]$ is an admittance matrix. The network of Fig. 12.3.1 gives a synthesis of the hybrid matrix M. Terminating this network at port 3 in a unit inductor and at port 4 in a unit capacitor then yields a synthesis of $Z(s)$, as shown in Fig. 12.3.2.

Problem 12.3.1 Give a reciprocal synthesis of the voltage-to-voltage transfer function

$$\frac{1}{(s+1)(s^2+s+1)}$$

Problem 12.3.2 Can you extend the synthesis technique of this section to cover the case when F_T may have pure imaginary eigenvalues? (This is an unsolved problem.)

Problem 12.3.3 Show that if F_T has all real eigenvalues, and that if $T(s)$ is a current-to-voltage transfer-function matrix, then there exists a synthesis using only inductors, resistors, and transformers.

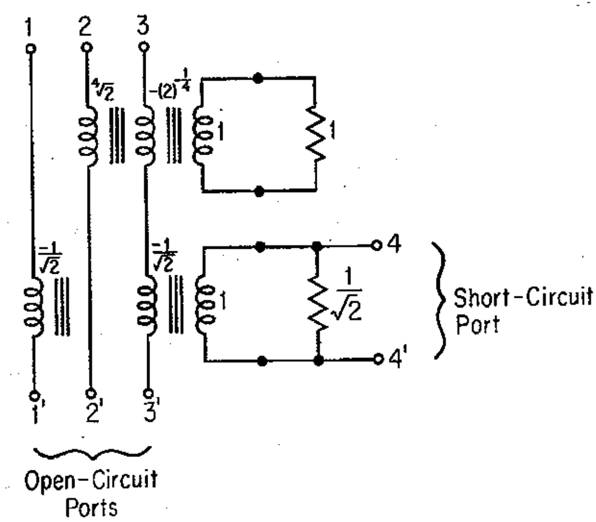

FIGURE 12.3.1. Nondynamic Network of Example 12.3.1.

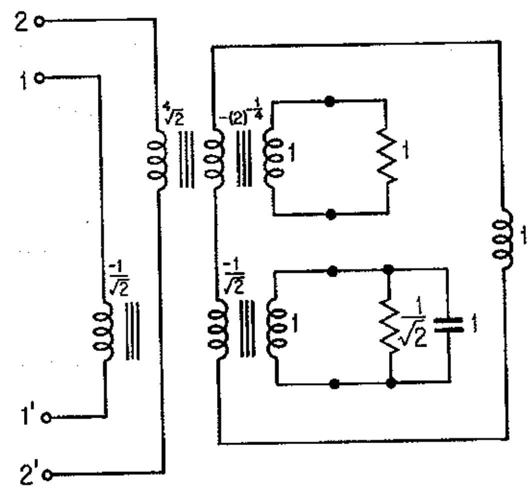

FIGURE 12.3.2. Reciprocal Synthesis of the Transfer Function $\frac{1}{2}(s^2 + \sqrt{2}s + 1)^{-1}$.

Problem 12.3.4 Give a reciprocal synthesis of the current-to-voltage transfer function

$$\frac{1}{(s+1)(s+2)}$$

using no capacitors. Use the procedure obtained in Problem 12.3.3.

12.4 TRANSFER-FUNCTION SYNTHESIS WITH PRESCRIBED LOADING

We shall restrict our remarks in this section to the synthesis of current-to-voltage transfer-function matrices. Extension to the voltage-to-voltage and other cases is not difficult.

The general situation we face can be described with the aid of Fig. 12.4.1. (If we were dealing with, for example, voltage-to-current transfer functions,

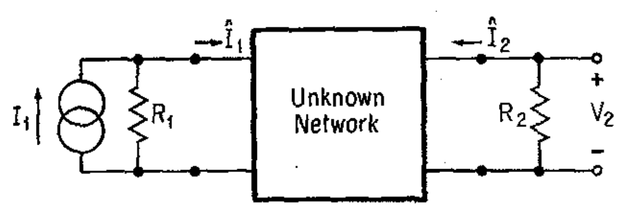

FIGURE 12.4.1. Synthesis Arrangement with R_1 and R_2 Known.

Fig. 12.4.2 would apply.) The new feature introduced in Fig. 12.4.1 is the loading at the input and output sets of ports. (Note that I_1 will be a vector and R_1 a diagonal matrix if the number of ports at the left-hand side of the unknown network exceeds one.)

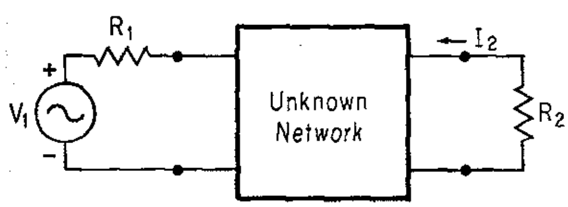

FIGURE 12.4.2. Synthesis Arrangement for Voltage-to-Current Transfer Function.

The synthesis problem is to pass from a prescribed $T(s)$, R_1, and R_2 to a network N such that the arrangement of Fig. 12.4.1, with N replacing the unknown network, synthesizes $T(s)$, in the sense that

$$V_2(s) = T(s)I_1(s) \tag{12.4.1}$$

As soon as dissipation is introduced into the picture, it makes sense not to consider $T(s)$ with elements possessing imaginary poles on physical grounds. Therefore, we shall suppose that every entry of $T(s)$ is such that all its poles lie in $\mathrm{Re}\,[s] < 0$.

Because of the way we have defined the loading elements—as resistors—we rule out the possibility of R_1 or R_2 being singular. (This avoids pointless short circuiting of the input current generators or the output.) We do permit R_1^{-1} and R_2^{-1} to become singular, in the sense that diagonal entries of R_1 and R_2 may become infinite; reference to Fig. 12.4.1 will show the obvious physical significance.

As we shall see, if either of R_1^{-1} or R_2^{-1} is zero, synthesis is always possible, while if both quantities are nonzero, synthesis is not always possible. A simple argument can be used to illustrate the latter point for the single-input, single-output case. Observe that the maximum average power flow into the coupling network obtainable from a sinusoidal source is $R_1 I_1^2/4$, where I_1 is the root-mean-square current associated with the source. Clearly, the power dissipated in R_2 cannot exceed this quantity, and so if V_2 is the root-mean-square voltage at the output,

$$\frac{V_2^2}{R_2} \leq \frac{R_1 I_1^2}{4}$$

It follows that

$$|T(j\omega)|^2 \leq \frac{R_1 R_2}{4} \tag{12.4.2}$$

for all ω. If a transfer function is specified together with values for R_1 and R_2 such that this inequality fails for some ω, then synthesis will be impossible.

Because of the importance of the cases $R_1^{-1} = 0$ and $R_2^{-1} = 0$, we shall consider these cases separately. Of course, the treatment in earlier sections has been based on $R_1^{-1} = R_2^{-1} = 0$. Here we shall demand that one of R_1^{-1} and R_2^{-1} be nonzero.

Absence of Input Loading

First, we shall suppose that $R_1^{-1} = 0$ (see Fig. 12.4.3). Before explaining a synthesis procedure, it proves advantageous to do some analysis. Let us suppose that the unknown network has an impedance-matrix description identical to the impedance-matrix description we derived in Section 12.2 in the first synthesis procedure. In other words, we suppose that for some $\{\hat{F}_T, \hat{G}_T, \hat{H}_T, \hat{J}_T\}$ we have

$$\hat{F}_T + \hat{F}'_T \leq 0 \tag{12.4.3}$$

and the unknown network is described by the state-space equations

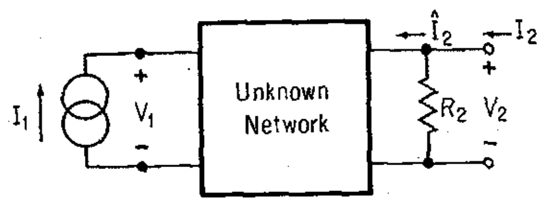

FIGURE 12.4.3. Arrangement for Output Loading Only.

$$\begin{aligned}\dot{x} &= \hat{F}_T x + [\hat{G}_T \quad \hat{H}_T]\begin{bmatrix} I_1 \\ \hat{I}_2 \end{bmatrix} \\ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} &= \begin{bmatrix} \hat{G}'_T \\ \hat{H}'_T \end{bmatrix} x + \begin{bmatrix} 0 & -\hat{J}'_T \\ \hat{J}_T & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ \hat{I}_2 \end{bmatrix}\end{aligned} \tag{12.4.4}$$

If R_2^{-1} were zero, and if also the quadruple $\{\hat{F}_T, \hat{G}_T, \hat{H}_T, \hat{J}_T\}$ were a realization of $T(s)$, then (12.4.4) would define state-space equations for a readily synthesizable positive real $Z(s)$, the synthesis of which would also yield a synthesis of $T(s)$.

From Eqs. (12.4.4) and the relation

$$\hat{I}_2 = I_2 - R_2^{-1} V_2 \tag{12.4.5}$$

it is not difficult to derive state-space equations for the impedance matrix of the unknown network when terminated with R_2. Substitution of (12.4.5) into (12.4.4) yields these equations as

$$\begin{aligned}\dot{x} &= (\hat{F}_T - \hat{H}_T R_2^{-1} \hat{H}'_T)x + [(\hat{G}_T - \hat{H}_T R_2^{-1} \hat{J}_T) \quad \hat{H}_T]\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \\ \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} &= \begin{bmatrix} \hat{G}'_T + \hat{J}'_T R_2^{-1} \hat{H}'_T \\ \hat{H}'_T \end{bmatrix} x + \begin{bmatrix} \hat{J}' R_2^{-1} \hat{J} & -\hat{J}' \\ \hat{J} & 0 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}\end{aligned} \tag{12.4.6}$$

The transfer-function matrix relating I_1 to V_2 (with $I_2 = 0$) is seen by inspection to be

$$T(s) = \hat{J}_T + \hat{H}'_T[sI - (\hat{F}_T - \hat{H}_T R_2^{-1} \hat{H}'_T)]^{-1}(\hat{G}_T - \hat{H}_T R_2^{-1} \hat{J}_T) \tag{12.4.7}$$

This concludes our analysis, and we turn now to synthesis. In the synthesis problem we are given $T(s)$, and we require the unknown network. The analysis problem suggests one way to do this. We could start with a *minimal* realization $\{F_T, G_T, H_T, J_T\}$ of $T(s)$. Then we could try to find quantities $\{\hat{F}_T, \hat{G}_T, \hat{H}_T, \hat{J}_T\}$ such that (12.4.7) holds. One obvious choice of such quantities that we might try is

$$\hat{J}_T = J_T \qquad \hat{H}_T = H_T \qquad \hat{F}_T = F_T + H_T R_2^{-1} H_T' \\ \hat{G}_T = G_T + H_T R_2^{-1} J_T \tag{12.4.8}$$

Then, *provided $\hat{F}_T$ has nonpositive definite symmetric part*, we could synthesize a network defined by the state-space equations (12.4.4), appropriately terminate its right-hand ports in R_2, and thereby achieve a synthesis of the desired $T(s)$ with an appropriately loaded network.

The question remains as to whether $\hat{F}_T$ will have nonpositive definite symmetric part. Using (12.4.8), we see that we need to know if there is a minimal realization $\{F_T, G_T, H_T, J_T\}$ of the prescribed $T(s)$ such that

$$F_T + F_T' + 2H_T R_2^{-1} H_T' \leq 0 \tag{12.4.9}$$

If (12.4.9) can be satisfied, synthesis is easy. Let us now show how (12.4.9) can be achieved. Let $\{F_{T_1}, G_{T_1}, H_{T_1}, J_{T_1}\}$ be an arbitrary minimal realization of $T(s)$ for which (12.4.9) does not hold. Form the equation

$$PF_{T_1} + F_{T_1}'P = -2H_{T_1} R_2^{-1} H_{T_1}' - Q \tag{12.4.10}$$

where Q is nonnegative definite, and such that this equation is guaranteed to have a positive definite solution. [*Note:* Conditions on Q are easy to find; if R_2^{-1} is nonsingular, the complete observability of (F_{T_1}, H_{T_1}) guarantees by the lemma of Lyapunov that any nonnegative definite Q will work. In any case, any positive definite Q will work. See also Problem 12.4.1.]

Let S be any matrix such that

$$S'S = P \tag{12.4.11}$$

and set

$$F_T = SF_{T_1}S^{-1} \qquad G_T = SG_{T_1} \qquad H_T = (S^{-1})'H_{T_1} \tag{12.4.12}$$

With these definitions, Eq. (12.4.10) implies

$$\begin{aligned} F_T + F_T' &= -2H_T R_2^{-1} H_T' - (S^{-1})'QS^{-1} \\ &\leq -2H_T R_2^{-1} H_T' \end{aligned}$$

This inequality is the same as (12.4.9).

Example 12.4.1 We take

$$T(s) = \frac{\frac{1}{2}}{s^2 + \sqrt{2}s + 1}$$

with

$$F_T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad G_T = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad H_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad J_T = 0$$

and suppose that $R_2 = 2$. Observe that

$$F_T + F_T' + 2H_T R_2^{-1} H_T' = \begin{bmatrix} -\sqrt{2}+1 & 0 \\ 0 & -\sqrt{2} \end{bmatrix} \leq 0$$

Therefore, no coordinate-basis transformation is necessary. Using (12.4.8), we have

$$\hat{J}_T = 0 \quad \hat{H}_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \hat{F}_T = \begin{bmatrix} \frac{1-\sqrt{2}}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad \hat{G}_T = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Using Eqs. (12.4.4), we see that a state-space realization of the impedance of the "unknown network" of Fig. 12.4.3 is $\{F_z, G_z, H_z, J_z\}$, where

$$F_z = \begin{bmatrix} \frac{1-\sqrt{2}}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad G_z = H_z = \begin{bmatrix} 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \quad J_z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The impedance can be synthesized in any of the standard ways.

Absence of Output Loading

Now we suppose that R_1^{-1} is nonzero, but $R_2^{-1} = 0$. We shall see that the synthesis procedure is very little different from the case we have just considered, where $R_1^{-1} = 0$ but R_2^{-1} is nonzero. Figure 12.4.4 now applies.

As before, we analyze before we synthesize. Thus we suppose the unknown

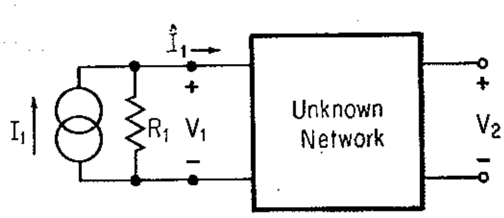

FIGURE 12.4.4. Arrangement for Input Loading Only.

network has state-space equations

$$\begin{aligned}\dot{x} &= \hat{F}_T x + [\hat{G}_T \quad \hat{H}_T]\begin{bmatrix}\hat{I}_1 \\ I_2\end{bmatrix} \\ \begin{bmatrix}V_1 \\ V_2\end{bmatrix} &= \begin{bmatrix}\hat{G}'_T \\ \hat{H}'_T\end{bmatrix}x + \begin{bmatrix}0 & -\hat{J}'_T \\ \hat{J}_T & 0\end{bmatrix}\begin{bmatrix}\hat{I}_1 \\ I_2\end{bmatrix}\end{aligned} \tag{12.4.13}$$

Also, we require

$$\hat{F}_T + \hat{F}'_T \leq 0 \tag{12.4.3}$$

We now have

$$\hat{I}_1 = I_1 - R_1^{-1}V_1 \tag{12.4.14}$$

which leads to the following state-space equations relating the input variables I_1 and I_2 to the output variables V_1 and V_2:

$$\begin{aligned}\dot{x} &= (\hat{F}_T - \hat{G}_T R_1^{-1}\hat{G}'_T)x + [\hat{G}_T \quad \hat{H}_T + \hat{G}_T R_1^{-1}\hat{J}'_T]\begin{bmatrix}I_1 \\ I_2\end{bmatrix} \\ \begin{bmatrix}V_1 \\ V_2\end{bmatrix} &= \begin{bmatrix}\hat{G}'_T \\ \hat{H}'_T - \hat{J}_T R_1^{-1}\hat{G}'_T\end{bmatrix}x + \begin{bmatrix}0 & -\hat{J}'_T \\ \hat{J}_T & \hat{J}_T R_1^{-1}\hat{J}'_T\end{bmatrix}\begin{bmatrix}I_1 \\ I_2\end{bmatrix}\end{aligned} \tag{12.4.15}$$

The transfer function relating I_1 to V_2 is obtained by setting $I_2 = 0$, and is

$$T(s) = \hat{J}_T + (\hat{H}_T - \hat{G}_T R_1^{-1}\hat{J}'_T)'[sI - (\hat{F}_T - \hat{G}_T R_1^{-1}\hat{G}'_T)]^{-1}\hat{G}_T \tag{12.4.16}$$

For the synthesis problem, we shall be given a *minimal realization* $\{F_T, G_T, H_T, J_T\}$ of $T(s)$. A synthesis of $T(s)$ will follow from this minimal realization if we set

$$\hat{J}_T = J_T \quad \hat{G}_T = G_T \quad \hat{H}_T = H_T + G_T R_1^{-1}J_T \quad \hat{F}_T = F_T + G_T R_1^{-1}G'_T \tag{12.4.17}$$

and synthesize (12.4.13) as the "unknown network," with the procedure working *only if* (12.4.3) holds. From (12.4.17) it follows that (12.4.3) is equivalent to

$$F_T + F'_T + 2G_T R_1^{-1}G'_T \leq 0 \tag{12.4.18}$$

Though this condition is not necessarily met by an arbitrary *minimal* realization of $T(s)$, the means of changing the coordinate basis to ensure satisfaction of (12.4.18) should be almost clear from the previous discussion dealing with the case of output loading. The equation replacing (12.4.10) is

$$F_{T_1}P + PF'_{T_1} = -2G_{T_1}R_1^{-1}G'_{T_1} - Q \tag{12.4.19}$$

With S satisfying $SS' = P$, we set $F_T = S^{-1}F_{T_1}S$, $G_T = S^{-1}G_{T_1}$, and $H_T = S'H_{T_1}$. Essentially, the remainder of the calculations are the same, and we will not detail them further.

Input and Output Loading

Now we suppose that both R_1^{-1} and R_2^{-1} are nonzero. We shall also suppose that $T(s)$ is such that $T(\infty) = 0$, because the calculations otherwise become very awkward. This is not a significant restriction in practice, since most transfer-function matrices whose syntheses are desired will probably meet this restriction.

Our method for tackling the problem is largely unchanged. With reference to Fig. 12.4.1, we assume that we have available a state-space description of the impedance of the unknown network of the form

$$\dot{x} = \hat{F}_T x + [\hat{G}_T \quad \hat{H}_T]\begin{bmatrix}\hat{I}_1 \\ \hat{I}_2\end{bmatrix}$$
$$\begin{bmatrix}V_1 \\ V_2\end{bmatrix} = \begin{bmatrix}\hat{G}'_T \\ \hat{H}'_T\end{bmatrix}x \tag{12.4.20}$$

Observe that in (12.4.20) there is no direct feedthrough term; i.e., V_1 and V_2 are linearly dependent directly on x, rather than x, $\hat{I}_1$, and $\hat{I}_2$. As we shall see, this represents no real restriction. Of course, as before, $\hat{F}_T$ is constrained by

$$\hat{F}_T + \hat{F}'_T \leq 0 \tag{12.4.21}$$

Using the equations

$$\hat{I}_1 = I_1 - R_1^{-1}V_1 \qquad \hat{I}_2 = I_2 - R_2^{-1}V_2 \tag{12.4.22}$$

it is not hard to set up state-space equations in which the input variables are I_1 and I_2. These will of course be state-space equations of the impedance of the network comprising the unknown network and its terminations. The actual equations are

$$\dot{x} = (\hat{F}_T - \hat{G}_T R_1^{-1}\hat{G}'_T - \hat{H}_T R_2^{-1}\hat{H}'_T)x + [\hat{G}_T \quad \hat{H}_T]\begin{bmatrix}I_1 \\ I_2\end{bmatrix}$$
$$\begin{bmatrix}V_1 \\ V_2\end{bmatrix} = \begin{bmatrix}\hat{G}'_T \\ \hat{H}'_T\end{bmatrix}x \tag{12.4.23}$$

We see from these equations that the transfer-function matrix $T(s)$ relating $I_1(s)$ to $V_2(s)$ is

$$T(s) = \hat{H}'_T[sI - (\hat{F}_T - \hat{G}_T R_1^{-1}\hat{G}'_T - \hat{H}_T R_2^{-1}\hat{H}'_T)]^{-1}\hat{G}_T \tag{12.4.24}$$

The synthesis problem requires us to proceed from a minimal realization $\{F_T, G_T, H_T\}$ of $T(s)$ to a network synthesizing $T(s)$. Study of (12.4.24) shows that this will be possible by reversing the above procedure if we define

$$\hat{G}_T = G_T \qquad \hat{H}_T = H_T \qquad \hat{F}_T = F_T + G_T R_1^{-1} G'_T + H_T R_2^{-1} H'_T \quad (12.4.25)$$

and if the resulting $\hat{F}_T$ satisfies (12.4.21), i.e., if

$$F_T + F'_T + 2G_T R_1^{-1} G'_T + 2H_T R_2^{-1} H'_T \leq 0 \qquad (12.4.26)$$

In general, (12.4.26) will not be satisfied by the matrices of an arbitrary minimal realization. So, again, we seek to compute a coordinate-basis change that will take an arbitrary minimal realization $\{F_{T_1}, G_{T_1}, H_{T_1}\}$ to this form. If such a coordinate-basis change can be found, then the synthesis is straightforward. The following lemma is relevant:

Lemma 12.4.1. Let $\{F_{T_1}, G_{T_1}, H_{T_1}\}$ be an arbitrary minimal realization of $T(s)$. Then there exists a *minimal realization* $\{F_T, G_T, H_T\}$ satisfying (12.4.26) if and only if there exists a positive-definite symmetric P such that

$$PF_{T_1} + F'_{T_1}P + 2PG_{T_1}R_1^{-1}G'_{T_1}P + 2H_{T_1}R_2^{-1}H'_{T_1} \leq 0 \quad (12.4.27)$$

Proof. Here we shall only prove that (12.4.27) implies the existence of $\{F_T, G_T, H_T\}$ satisfying (12.4.26). Proof of the converse is demanded in a problem. Let $P = S'S$, where S is square; because P is nonsingular, so is S. Then set

$$F_T = SF_{T_1}S^{-1} \qquad G_T = SG_{T_1} \qquad H_T = (S^{-1})'H_{T_1} \quad (12.4.28)$$

Equation (12.4.26) follows by multiplying both sides of (12.4.27) on the left by $(S^{-1})'$ and on the right by S^{-1}. $\nabla\nabla\nabla$

The significance of the lemma is as follows. If, given an arbitrary minimal realization $\{F_{T_1}, G_{T_1}, H_{T_1}\}$ of $T(s)$, a nonsingular matrix P can be found satisfying (12.4.27), then the synthesis problem is, in essence, solved. For from P we can compute S, and then F_T, G_T, and H_T according to (12.4.28). Next, $\hat{F}_T$, $\hat{G}_T$, and $\hat{H}_T$ follow from (12.4.25), with the fundamental restriction (12.4.21) holding. We can then synthesize the unknown network using its impedance description in state-space terms of (12.4.20).

Lemma 12.4.1 raises two questions: (1) when can (12.4.27) be solved, and (2) how can a solution of (12.4.27) be computed? To answer these questions, let us define the transfer-function matrix $\Sigma(s)$ by

$$\Sigma(s) = (\sqrt{2}H_{T_1}R_2^{-1/2})'(sI - F_{T_1})^{-1}(\sqrt{2}G_{T_1}R_1^{-1/2}) \qquad (12.4.29)$$

Problem 12.4.4 requests a proof of the following result.

> **Lemma 12.4.2.** With $\Sigma(s)$ defined as in (12.4.29), there exists a positive definite P satisfying (12.4.27) if and only if $\Sigma(s)$ is a *bounded real (scattering) matrix.*

The matrix P in the bounded real lemma equation for $\Sigma(s)$ is the same as the matrix P in (12.4.27). Thus Lemma 12.4.2 answers question 1, while earlier work on the calculation of solutions to the positive and bounded real lemma equations answers question 2.

It is worthwhile rephrasing the scattering matrix condition slightly. In view of the fact that F_{T_1} has all its poles in $\text{Re}\,[s] < 0$, it follows that $\Sigma(s)$ is a bounded real scattering matrix if and only if

$$I - \Sigma'(-j\omega)\Sigma(j\omega) \geq 0 \qquad \text{for all real } \omega$$

or

$$I - 4R_1^{-1/2}G'_{T_1}(-j\omega I - F'_{T_1})^{-1}H_{T_1}R_2^{-1}H'_{T_1}(j\omega I - F_{T_1})^{-1}G_{T_1}R_1^{-1/2} \geq 0 \qquad \text{for all real } \omega$$

or

$$\frac{R_1}{4} \geq T'(-j\omega)R_2^{-1}T(j\omega) \qquad \text{for all real } \omega \tag{12.4.30}$$

This is a remarkable condition, for it is the matrix generalization of condition (12.4.2) that we derived by simple energy considerations. Therefore, even though we have assumed a specialized sort of unknown network (by demanding that its impedance matrix have a certain structure), we do not sacrifice the ability to synthesize any transfer-function matrix that can be synthesized. Further, we see that the class of synthesizable transfer-function matrices is readily described by simple physical reasoning.

In contrast to synthesis procedures presented earlier in this chapter, the procedure we have just given demands as much computation as an immittance- or scattering matrix synthesis. Notice though that a minimal number of reactive elements is used, just as for the earlier transfer-function syntheses. Except in isolated situations, the network resulting from use of this synthesis procedure will be nonreciprocal. A reciprocal synthesis procedure has not yet been developed.

Example 12.4.2 We take again

$$T(s) = \frac{\frac{1}{2}}{s^2 + \sqrt{2}s + 1}$$

with minimal realization

$$F_{T_1} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix} \qquad G_{T_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad H_{T_1} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

Suppose that a current-to-voltage synthesis is desired with $R_1 = R_2 = 1$. Notice that

$$\begin{aligned}\frac{R_1}{4} - \frac{|T(j\omega)|^2}{R_2} &= \frac{1}{4}\left[1 - \frac{1}{|(1-\omega^2) + \sqrt{2}\,j\omega|^2}\right] \\ &= \frac{1}{4}\left[1 - \frac{1}{(1+\omega^4)}\right] \\ &\geq 0 \qquad \text{for all real } \omega\end{aligned}$$

Therefore, synthesis is possible.

Because of the low dimensionality of $T(s)$, it is easy to solve (12.4.27) directly. If $P = (p_{ij})$, and the equality sign is adopted in (12.4.27), we obtain—by equating to zero components of the matrix on the left side of (12.4.27)—the equations

$$\begin{aligned}-2p_{12} + 2p_{12}^2 + \tfrac{1}{2} &= 0 \\ p_{11} - p_{22} - \sqrt{2}\,p_{12} + 2p_{12}p_{22} &= 0 \\ 2p_{12} - 2\sqrt{2}\,p_{22} + 2p_{22}^2 &= 0\end{aligned}$$

from which

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Next, a matrix S satisfying $S'S = P$ is readily found as

$$S = \begin{bmatrix} 2^{-1/4} & 2^{-3/4} \\ 0 & 2^{-3/4} \end{bmatrix}$$

Using (12.4.28), we define a minimal realization for $T(s)$ as

$$F_T = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad G_T = \begin{bmatrix} 2^{-3/4} \\ 2^{-3/4} \end{bmatrix} \qquad H_T = \begin{bmatrix} 2^{-3/4} \\ -2^{-3/4} \end{bmatrix}$$

Finally, we have $\hat{G}_T$ and $\hat{H}_T$ identical with G_T and H_T, while

$$\hat{F}_T = F_T + G_T R^{-1} G_T' + H_T R_2^{-1} H_T'$$
$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

The unknown network therefore has impedance matrix $Z(s)$ with state-space realization

$$F_z = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \qquad G_z = H_z = \begin{bmatrix} 2^{-3/4} & 2^{-3/4} \\ 2^{-3/4} & -2^{-3/4} \end{bmatrix} \qquad J_z = 0$$

and synthesis follows in the usual way.

Problem 12.4.1 Consider the equation

$$PF_{T_1} + F_{T_1}'P = -2H_{T_1}R_2^{-1}H_{T_1}' - Q$$

with all eigenvalues of F_{T_1} possessing negative real parts. Show that P is positive definite if and only if $\{F_{T_1}, [H_{T_1}R_2^{-1/2} \quad D]\}$ is completely observable, where D is any matrix such that $DD' = Q$. (*Hint:* Vary the standard proof of the lemma of Lyapunov.)

Problem 12.4.2 Synthesize the voltage-to-voltage transfer function

$$T(s) = \frac{\frac{1}{2}}{s^2 + \sqrt{2}s + 1}$$

for the following two cases:

(a) There is a resistor in series with the source of 2 Ω and an open-circuit load.

(b) There is a load resistor of 2 Ω and no source resistance.

Problem 12.4.3 Complete the proof of Lemma 12.4.1.

Problem 12.4.4 Let $\{F_{T_1}, G_{T_1}, H_{T_1}\}$ be a minimal realization of $T(s)$, and let $\{F_{T_1}, \sqrt{2}\,G_{T_1}R_1^{-1/2}, \sqrt{2}\,H_{T_1}R_2^{-1/2}\}$ be a minimal realization of a matrix transfer function $\Sigma(s)$. Using the bounded real lemma, show that $\Sigma(s)$ is a bounded real scattering matrix if and only if there exists a positive definite P such that

$$PF_{T_1} + F_{T_1}'P + 2PG_{T_1}R_1^{-1}G_{T_1}'P + 2H_{T_1}R_2^{-1}H_{T_1}' \leq 0$$

Problem 12.4.5 Attempt to obtain reciprocal synthesis techniques along the lines of the nonreciprocal procedures of this section.

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Part VI

ACTIVE *RC* NETWORKS

Modern technologies have brought us to the point where network synthesis using resistors, capacitors, and active elements, rather than all passive elements, may be preferred in some situations. In this part we study the synthesis of transfer functions using active elements, with strong emphasis on state-space methods.

13

Active State-Space Synthesis

13.1 INTRODUCTION TO ACTIVE *RC* SYNTHESIS

In this chapter we aim to introduce the reader to the ideas of active *RC* synthesis, that is, synthesis using resistors, capacitors, and active elements. The trend to circuit miniaturization and the advent of solid-state devices have led to a rethinking of the aims of circuit design, with the conclusion that often the use of active devices is to be preferred to the use of inductors and transformers.

Here we shall be principally concerned with the use of state-space methods in active *RC* synthesis, and this leads to the use of *operational amplifiers* as active elements. Other active devices can be and are used in active *RC* synthesis, such as the negative resistor, controlled source, and negative impedance converter. A few brief remarks concerning these devices will be offered later in this chapter, but for any extensive account the reader should consult any of a number of books, e.g., [1–4].

In Section 13.2 we introduce the operational amplifier in some detail, and illustrate in Section 13.3 its application to transfer-function-matrix synthesis, paying special attention to scalar transfer function synthesis. Section 13.4 carries the ideas of Section 13.3 further by describing a circuit, termed the *biquad*, that will synthesize an almost arbitrary second-order transfer function. Finally, in Section 13.5 we discuss briefly other approaches to synthesis, including immittance synthesis as distinct from *transfer-function synthesis.* In the remainder of this section, meanwhile, we offer brief comments on various practical aspects of active synthesis.

In our earlier work on passive-circuit synthesis we paid some attention to seeking circuits containing a minimal number of elements, especially a minimal number of reactive elements. Likewise, early active *RC* syntheses were concerned with minimizing the number of reactive elements; however, there was also concern to minimize the number of active elements, and many syntheses only contained one active element. But the decreasing cost of active elements, either in discrete or integrated form, coupled with advantages to be derived from using more than one, has reduced the emphasis on minimizing the number in a circuit. At the same time, the desire to integrate circuits has led to the requirement to keep the total value of resistance down. Therefore, notions of minimality require rethinking in the context of active *RC* synthesis. For further discussion, see especially [1–3].

One of the major problems that has had to be overcome in active-circuit synthesis (and which is not a major problem in passive synthesis) is the sensitivity problem. It has been found in practice that the input–output performance of some active circuits is very sensitive to variations in absolute values of the components and to parasitic effects, including phase shift. These problems tend to be particularly acute in circuits synthesising high Q transfer functions. To a certain extent the problems are unavoidable, being due simply to the noninclusion of inductors in the circuits (see, e.g., [1, 5]). Many techniques have, however, been developed for coping with the sensitivity problem; a common one in transfer-function synthesis is to synthesize a transfer function as a cascade of syntheses of first-order and second-order transfer functions. Whether a circuit is discrete or integrated also has a bearing on the sensitivity problem; in an integrated circuit there is likely to be correlation between element variations to a greater degree than in a discrete-component circuit. Although this may be a disadvantage in some situations, it is often possible to use this correlation to an advantage. Integrated differential amplifiers, for example, have extremely low sensitivity to parameter variations, precisely because any gain fluctuation owing to a deviation in one element value tends to be canceled by a corresponding variation in another. Again, when the element variations are due to temperature fluctuation, it is often possible to arrange the temperature dependence of integrated resistors and capacitors to be such as to cause compensating variations.

Another practical problem arising with active *RC* synthesis, and not associated with passive-circuit synthesis, is that of circuit instability. One class of instability can arise simply through element variation away from nominal values, which may cause a left half-plane transfer-function pole to move into the right half-plane—a phenomenon not observed in passive circuits. This phenomenon is essentially a manifestation of the sensitivity problem, and techniques aimed at combatting the sensitivity problem will serve here. More awkward are instabilities arising from parasitics contained within the active elements; a controlled source may be modeled as having a gain K, when the gain is actually $Ka/(s + a)$ for some large a. Designs may neglect the presence of the pole and not predict an oscillation, which would be predicted were the pole taken into account. Techniques for combatting this problem generally revolve around the addition of compensating resistor and capacitor elements to the basic active element (see, e.g., [3] for a discussion of some of these techniques).

13.2 OPERATIONAL AMPLIFIERS AS CIRCUIT ELEMENTS

In this section we examine the operational amplifier, which is the basic active device to be used in most of the state-space synthesis procedures to be presented. We are interested particularly in its use to form finite gain, summing and integrating elements, both inverting and noninverting.

Basically, an operational amplifier is simply an amplifier of very high gain, usually with two inputs, called noninverting and inverting. A signal applied at the inverting input appears at the output both amplified and reversed in polarity. The usual representation of the differential-input operational amplifier is shown in Fig. 13.2.1a; for many applications the noninverting input terminal will be grounded, and the resulting single-ended operational ampli-

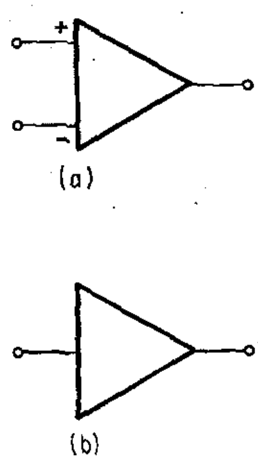


FIGURE 13.2.1. (a) Differential-Input Operational Amplifier; (b) Single-Ended Operational Amplifier.

fier is depicted as shown in Fig. 13.2.1b. Inversion of the signal polarity is implicitly assumed.

The DC gain of an actual operational amplifier may be anywhere between 10^4 to 10^6 with fall-off at comparatively low frequencies; the unity gain frequency is typically a few megahertz or more. Ideally, the input impedance of an operational amplifier is infinite, as is its output admittance. In practice, a figure of 100 kilohms (kΩ) is more likely to be attained for the input impedance and 50 Ω for the output impedance. Ideally, amplifiers will be linear for arbitrary input levels, show no dependence on temperature, offer equal input impedance and gains at the input terminals with no cross coupling, and be noiseless. In practice again, none of these idealizations is totally valid. Further, it is sometimes necessary to add additional components externally to the operational amplifier to combat instability problems associated with phase shift in the amplifier. All these points are generally dealt with in manufacturers' literature.

Achieving a Finite Gain Using an Operational Amplifier

Consider the arrangement of Fig. 13.2.2. Assuming that the amplifier has a gain K and infinite input impedance and output admittance, it is readily established that

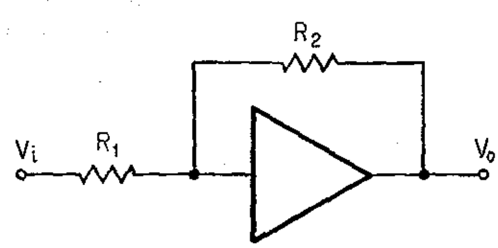


FIGURE 13.2.2. Single-Ended Operational Amplifier Generating Gain Element of Gain $-(R_2/R_1)$.

$$\frac{V_o}{V_i} = -\frac{R_2}{R_1[1 + (1/K)(1 + (R_2/R_1))]}$$

so that for very large K, we can write

$$\frac{V_o}{V_i} = -\frac{R_2}{R_1} \tag{13.2.1}$$

In this way we can build constant negative-voltage-gain amplifiers. Note that by taking $R_1 = R_2$, the gain is -1, so that an inverting amplifier is obtained. A positive voltage gain is then achievable by cascading an inverting amplifier with a negative-gain amplifier. Alternatively, a positive-voltage-gain amplifier is achievable from a differential-input amplifier. Rather than discussing this separately, we can regard it as a special case of the differential-input summer, considered below.

Summing Using an Operational Amplifier

Consider the arrangement of Fig. 13.2.3, which generalizes that of Fig. 13.2.2. Straightforward analysis will show that for infinite gain, input impedance, and output admittance of the amplifier, one has

$$V_o = -\sum_{i=1}^{m} \left(\frac{R}{R_i}\right) V_i \tag{13.2.2}$$

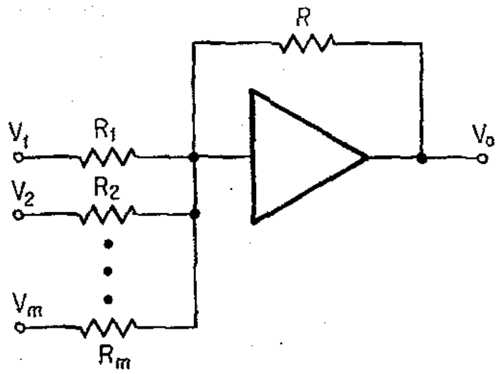


FIGURE 13.2.3. Single-Ended Operational Amplifier Generating a Summer.

This equation shows that, in effect, *weighted* summation is possible, with the gain coefficients arbitrary except for their negative sign. Introduction of inverting amplifiers or, alternatively, use of the differential-input operational amplifier allows the achieving of positive gains in a summation. Consider the arrangement of Fig. 13.2.4. In this case, again assuming an ideal amplifier, we have

$$V_o = -\sum_{i=1}^{n} \frac{R}{R_i^-} V_i^- + \sum_{i=1}^{m} \frac{R\left[\sum_{i=0}^{n} (1/R_i^-)\right] + 1}{R_i^+\left[\sum_{i=0}^{m} (1/R_i^+)\right]} V_i^+ \tag{13.2.3}$$

Integrating Using an Operational Amplifier

The single-ended operational amplifier circuit for integrating is reasonably well known and is illustrated in Fig. 13.2.5. It is easy to show

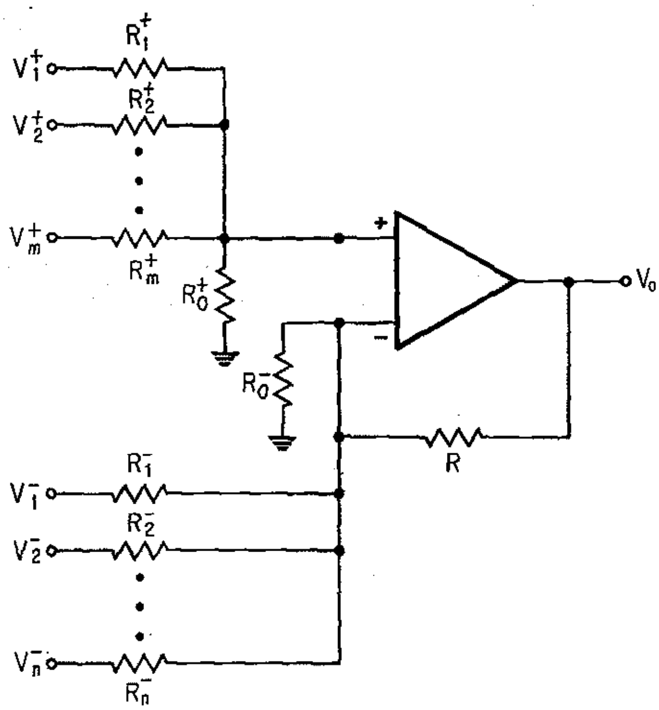


FIGURE 13.2.4. Differential-Input Operational Amplifier Generating a Summer.

that the transfer function relating V_i to V_o under the usual idealization assumptions is

$$\frac{V_o}{V_i} = -\frac{1}{sRC} \tag{13.2.4}$$

The noninverting integrator circuits are perhaps not as familiar; two of them are shown in Fig. 13.2.6. The resistance bridge integrator of Fig. 13.2.6a has a transfer function

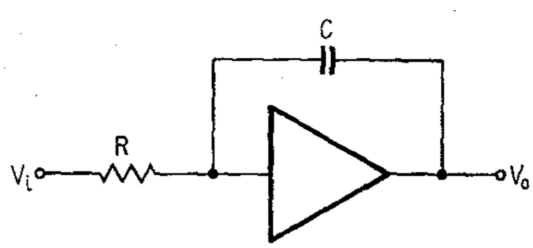
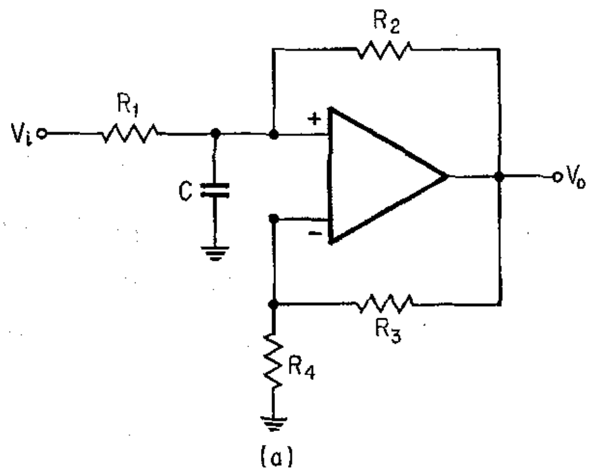


FIGURE 13.2.5. Inverting Integrator Circuit.



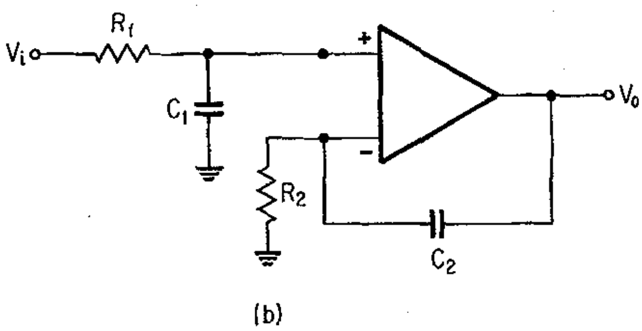


FIGURE 13.2.6. (a) Resistance Bridge Non-Inverting Integrator; (b) Balanced Time Constant Integrator.

$$\frac{V_o}{V_i} = \frac{(R_2/R_1) + 1}{sCR_1} \tag{13.2.5}$$

provided that $R_1R_3 = R_2R_4$. The balanced time-constant integrator of Fig. 13.2.6b has a transfer function

$$\frac{V_o}{V_i} = \frac{1}{sC_1R_1} \tag{13.2.6}$$

provided that $R_1C_1 = R_2C_2$.

It may be thought that in all cases one should use differential-input amplifiers if positive gains are required in an active circuit, rather than single-ended amplifiers followed or preceded by an inverting amplifier. However, alignment or tuning of the circuit is usually easier when inverting amplifiers are used, and there is lower sensitivity to variations in the passive elements. Less compensation to avoid instability is also required, since when a single-ended amplifier is used to provide a gain element, a summer, or an integrator, there is no positive feedback applied around the amplifier. This is in contrast to the situation prevailing when differential-input amplifiers are used with feedback to the positive terminal, as in the circuit of Fig. 13.2.6a, for example.

Combinations of Integration and Summation

It is straightforward to combine the operations of integration and summation; Fig. 13.2.7 illustrates the relation

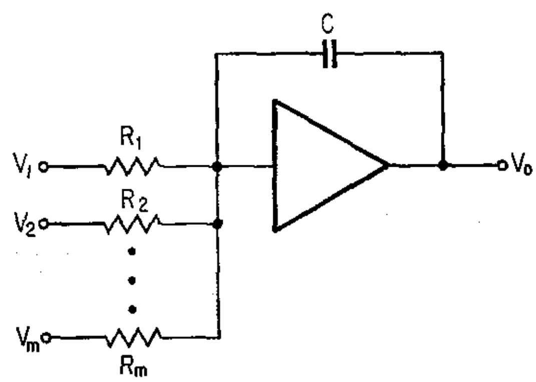


FIGURE 13.2.7. Summation and Integration.

$$V_o = -\sum_{i=1}^{m} \frac{V_i}{sCR_i} \tag{13.2.7}$$

A further important arrangement is that of Fig. 13.2.8, which illustrates the equation

$$V_o = -\sum_{i=1}^{m} \frac{R_0}{(sCR_0 + 1)} \frac{V_i}{R_i} \tag{13.2.8}$$

This equation may also be written

$$sV_o = -\sum_{i=0}^{m} \frac{V_i}{CR_i} \tag{13.2.9}$$

One can think of the R_0 feedback as turning an ideal integrator into a lossy integrator—a notion better described by (13.2.8)—or as providing negative feedback of V_o in conjunction with integration—a notion described by (13.2.9).

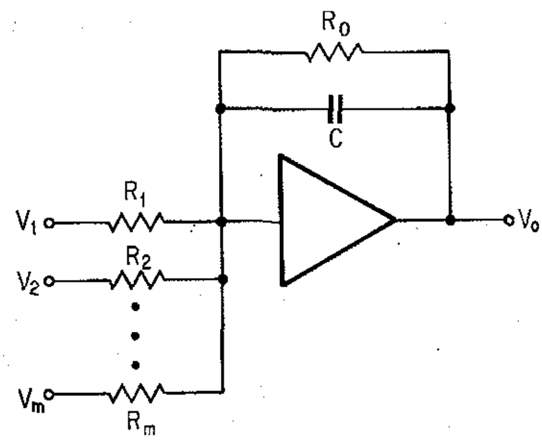


FIGURE 13.2.8. Summation with Lossy Integration.

Problem 13.2.1 Verify Eq. 13.2.3.

Problem 13.2.2 Verify that Eqs. (13.2.5) and (13.2.6) hold, subject to the resistance bridge and time-constant balance conditions.

Problem 13.2.3 In this section the fact that operational amplifiers can be used to provide constant gain elements has been established. Show also that they can be used to provide gyrators [6], negative resistors, and negative impedance converters [7]. (A negative resistor is a one port defined by $v = -Ri$ for a positive constant R. A negative impedance converter is a two port defined by equations $v_1 = kv_2$, $i_1 = ki_2$ for some constant k.)

13.3 TRANSFER-FUNCTION SYNTHESIS USING SUMMERS AND INTEGRATORS

As we shall see in this section, state-space synthesis of transfer-function matrices is particularly straightforward when the available elements are resistors, capacitors, and operational amplifiers. We shall begin our discussions by considering the synthesis of transfer-function matrices, and then specialize first to scalar transfer functions, and then to degree 1 and degree 2 transfer functions. The synthesis of degree 2 transfer functions will be covered at greater length in the next section.

Suppose that $T(s)$ is an $m \times p$ transfer function; for convenience, we shall suppose that no input is to be observed at the same port as any output. We shall suppose too that $T(\infty) < \infty$, and that a state-space set of equations is known for $T(s)$:

$$\begin{aligned} \dot{x} &= Fx + Gu \\ y &= H'x + Ju \end{aligned} \tag{13.3.1}$$

Consider the arrangement of Fig. 13.3.1, which constitutes a diagrammatic representation of (13.3.1). Observe further that practical implementation of the scheme of Fig. 13.3.1 requires constant gain elements, summers, and integrators, and these are precisely what operational amplifiers can provide. In other words, Eq. (13.3.1) are immediately translatable into an active RC circuit, using the arrangement of Fig. 13.3.1 in conjunction with the schemes discussed in the last section for the use of operational amplifiers.

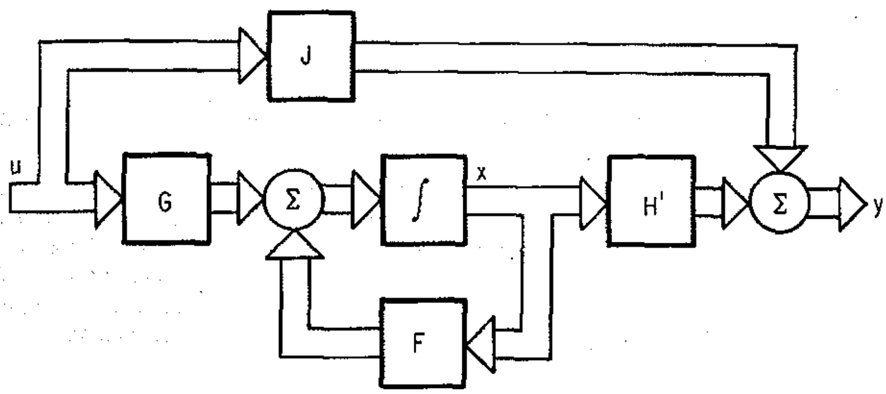


FIGURE 13.3.1. Block Diagram Representation of State-Space Equations.

Example 13.3.1 Consider the state equations

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} u$$
$$y = [1 \quad 2]x$$

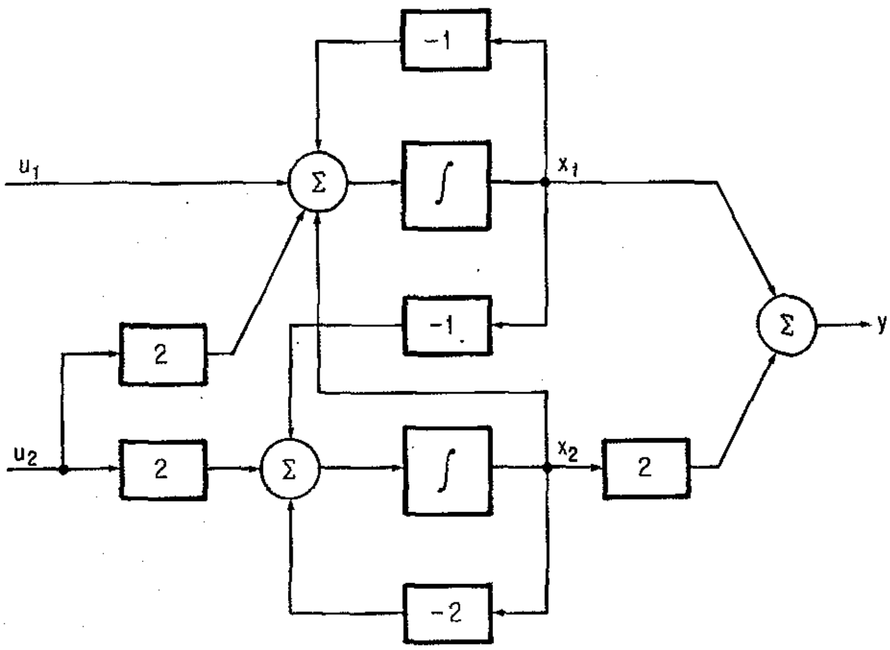
Figure 13.3.2a shows a block diagram containing integrators, gain blocks, and summers that possesses the requisite state-space equations relating u_1 and u_2 to y. Figure 13.3.2b shows the translation of this into a scheme involving operational amplifiers, resistors, and capacitors. Notice that the outputs of amplifiers A_1 and A_2 are $-x_1$ and $-x_2$, respectively; it would be pointless to use two inverting amplifiers to attain x_1 and x_2, since another inverting amplifier would be needed to obtain y from x_1 and x_2.

Notice that it is straightforward to obtain an infinite number of syntheses of a prescribed $T(s)$. Change of the state-space coordinate basis will cause changes in F, G, and H in (13.3.1) with resultant changes in the gain values, but the general scheme of Fig. 13.3.1 will of course remain unaltered.

Suppose now that $T(s)$ is a scalar, given by

$$T(s) = \frac{b_n s^{n-1} + b_{n-1}s^{n-2} + \cdots + b_1}{s^n + a_n s^{n-1} + \cdots + a_1} + c \tag{13.3.2}$$

For this scalar $T(s)$ we can write down canonical forms, and study the structure of the associated circuit synthesizing $T(s)$ a little more closely. As we know, a quadruple $\{F, g, h, j\}$ realizing $T(s)$ is given by



(a)

FIGURE 13.3.2. (a) Synthesis for Example 12.3.1, Showing Integrators, Gain Blocks and Summers.

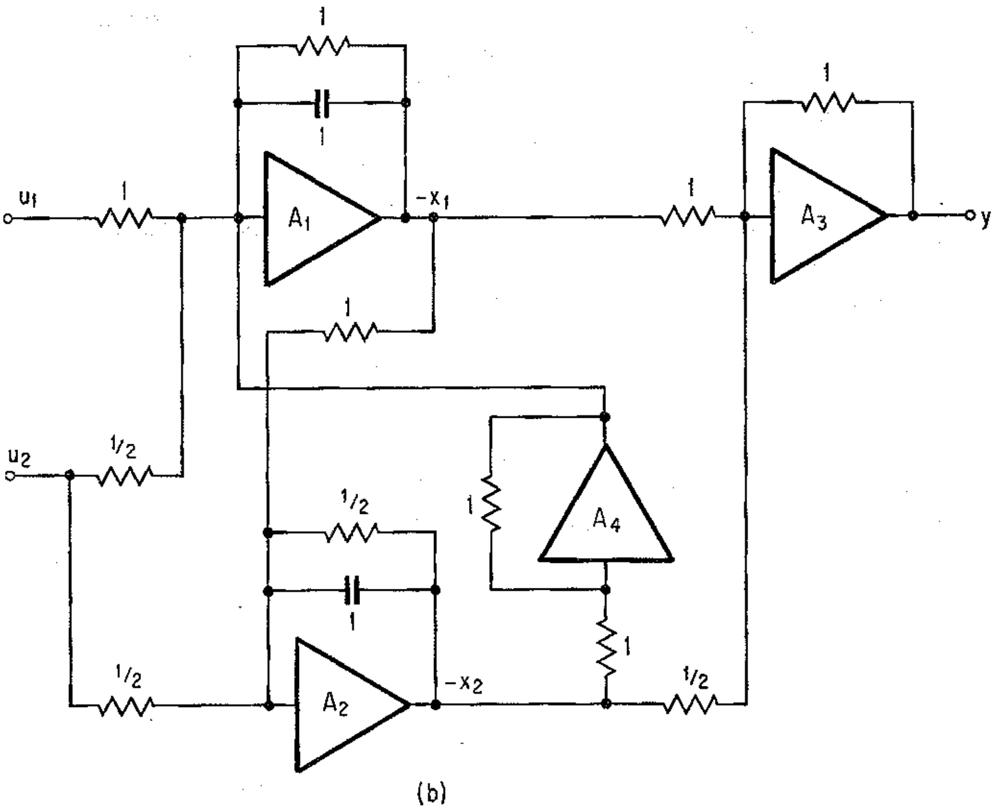


FIGURE 13.3.2. (b) Synthesis of Example 12.3.1, Showing Operational Amplifiers, Resistors and Capacitors.

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} \quad g = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \quad h = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_{n-1} \\ b_n \end{bmatrix} \quad j = c \tag{13.3.3}$$

A scheme for synthesizing $T(s)$ based on (13.3.3) is shown in Fig. 13.3.3. Notice that, in contrast to the scheme of Fig. 13.3.1, all variables depicted are scalar. The fact that so many of the entries of the F matrix and g vector are zero means that the scheme of Fig. 13.3.3 is economic in terms of the

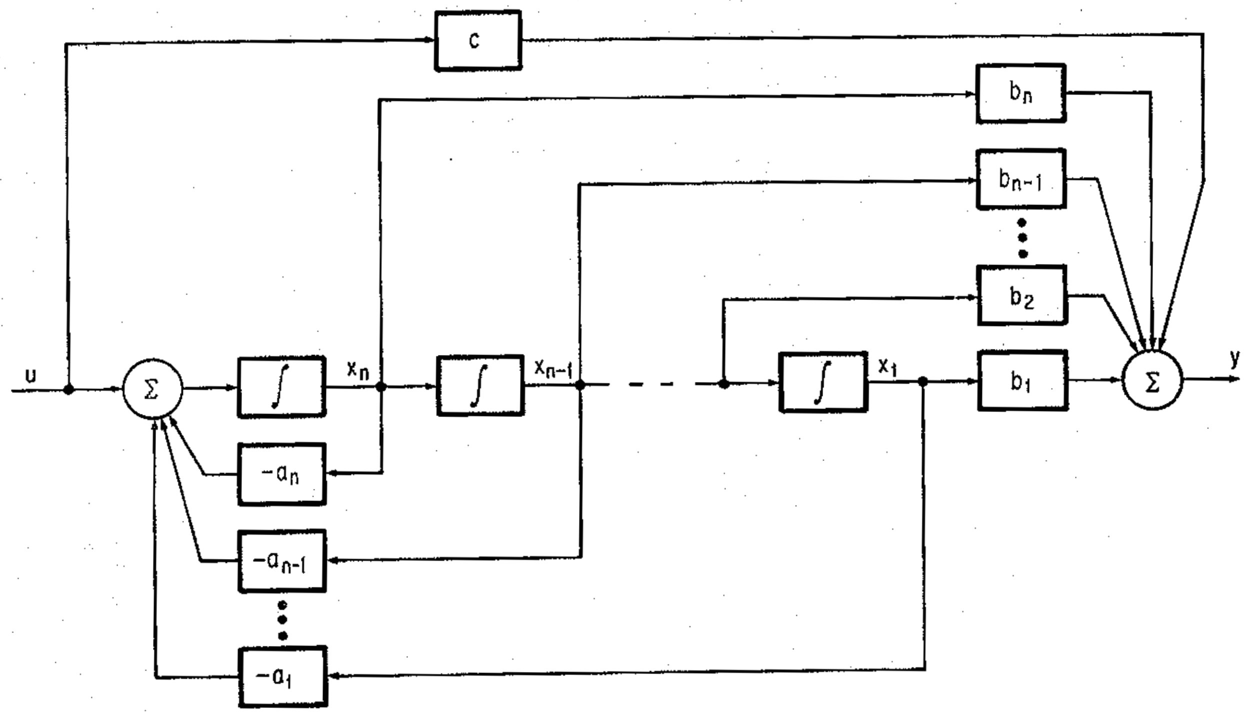


FIGURE 13.3.3. Block Diagram Representation of a State-Space Realization of a Scalar Transfer Function.

number of gain and summing elements required, and this may be an attractive feature. On the other hand, there may be reasons against use of the scheme. First, the dynamic range of some of the variables may exceed that achievable with the active components; second, the performance of the circuit may be very sensitive to variations in the individual components. In rough terms, the basic reason for this is that the zeros of a polynomial may vary greatly when *a coefficient of the polynomial undergoes small variation*. Therefore, small variation in a component synthesizing a_i in (13.3.3) and (13.3.2) may cause large variation in the zeros of the denominator of $T(s)$; consequently, if such a zero is near the $j\omega$ axis, $T(j\omega)$ for $j\omega$ near this zero will vary substantially.

The dynamic-range problem can sometimes be tackled by scaling. Let Λ be a diagonal matrix of positive entries, chosen so that the entries of the state vector $\hat{x} = \Lambda x$ take values that are more acceptable than those of the state vector x. Then we simply build a realization based on $\Lambda F \Lambda^{-1}$, Λg, $\Lambda^{-1} h$; notice that $\Lambda F \Lambda^{-1}$ and Λg have the same sparsity of elements as do F and g, and almost the same resultant economy of realization. (Some unity entries in F and g may become nonunity entries in $\Lambda F \Lambda^{-1}$ and Λg. Extra elements may then be required in the circuit simply as a result of this change.) Application of the scaling idea is of course not restricted to the simulation of scalar transfer functions.

A basis for tackling the sensitivity problem is the observation that the *problem is not particularly significant* in the case of transfer functions of degree 1 and degree 2. This is borne out by practical experience and by extensive calculations made on the performance of the biquad circuit, which is a circuit synthesizing a degree 2 transfer function to be discussed in the next section. An arbitrary degree n transfer function $T(s)$ can always be expressed as the product (or the sum) of a set of degree 1 and degree 2 transfer functions. Expression as a product requires factoring the numerator and the denominator, and pairing numerator terms with denominator terms. We allow pairing of constant, linear, or quadratic numerator terms with a quadratic denominator term, and pairing of a constant or linear numerator term with a linear denominator term. Expression of $T(s)$ as a sum of degree 1 and degree 2 transfer functions requires factoring of the denominator of $T(s)$ and the derivation of a partial fraction expansion. Care must be taken in obtaining the factors if $T(s)$ is not already in factored form; it may be quite difficult to obtain accurate coefficients in the factors if high-order polynomials need to be factored.

When $T(s)$ is expressed in product form, each component of the product is synthesized individually, and the collection is cascaded. When $T(s)$ is expressed in sum form, each summand is still synthesized individually, but the collection is put in parallel in an obvious fashion. From the viewpoint of eliminating sensitivity problems, the product form is preferred. Problem 13.3.4 asks for an argument justifying this conclusion.

As already noted, synthesis of scalar, degree 2 transfer functions will be studied in the next section. Synthesis of degree 1 transfer functions is almost trivial, but it is helpful to note that frequently no active element will be required at all. The circuit of Fig. 13.3.4 has a (voltage-to-voltage) transfer function

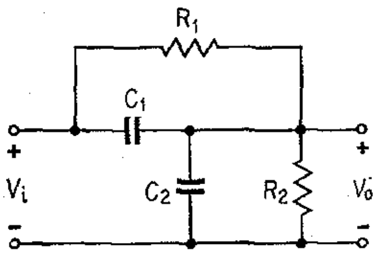


FIGURE 13.3.4. Realization of a First Order Transfer Function Using Passive Components Only.

$$T(s) = \frac{c_2 s + c_1}{s + a_1} \tag{13.3.4}$$

where $C_1 = c_2$, $C_2 = (1 - c_2)$, $R_1 = c_1^{-1}$, and $R_2 = (a_1 - c_1)^{-1}$. These equations show that it is always possible to set C_1 or C_2 equal to zero; they also show that the coefficients in (13.3.4) are constrained by $c_2 \le 1$ and $c_1/a_1 \le 1$. Inclusion of a constant gain element at the output (such as might be needed for buffering anyway) enables these constraints to be overcome.

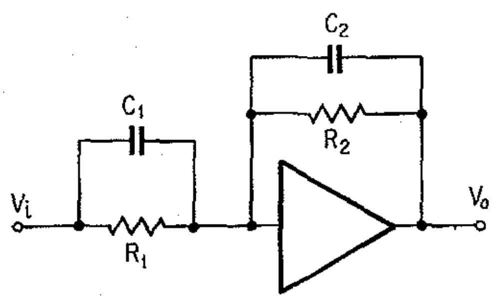


FIGURE 13.3.5. Realization of a First Order Transfer Function, Using Resistors, Capacitors and an Operational Amplifier.

Degree 1 *transfer functions* can also be synthesized with the circuit shown in Fig. 13.3.5. The associated transfer function is

$$T(s) = -\frac{R_2}{R_1}\frac{sR_1C_1 + 1}{sR_2C_2 + 1}$$

A right-half-plane zero can be obtained with an extra amplifier.

Note that it is never possible to synthesize a degree 2 transfer function possessing complex poles with resistor and capacitor elements only. This fact is proved in many classical texts (see, e.g., [8]).

Problem 13.3.1 Draw three block diagrams to illustrate three different syntheses of the voltage-to-voltage third-order Butterwoth *transfer functions* $T(s) = 1/(s^3 + 2s^2 + 2s + 1)$ obtained by not factoring the denominator, and by expressing $T(s)$ both as a sum and a product.

Problem 13.3.2 Draw circuits showing resistor and capacitor values corresponding to each block diagram obtained in Problem 13.3.1.

Problem 13.3.3 How may a transfer function *matrix* synthesis problem be broken up into elemental degree 1 and degree 2 synthesis problems?

Problem 13.3.4 Argue why, from the *sensitivity point of view*, it is better to express $T(s)$ as a product than a sum and to synthesize accordingly.

13.4 THE BIQUAD

The general transfer function synthesis problem is, as we have seen, reducible to the problem of first-order and second-order transfer-function synthesis. We have already dealt with first-order synthesis. Here we study second-order synthesis, exposing a most important circuit, originating with [9] and developed further in [10–12]. In [11] it was named the biquad. Our discussion will first cover a general description of the circuit; then we shall discuss questions of *sensitivity*, following this by consideration of a number of other practical points.

Our starting point is the general second-order *transfer function*

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{c_3s^2 + c_2s + c_1}{s^2 + a_2s + a_1} \qquad (13.4.1)$$

We shall assume throughout this section that $T(s)$ has its poles in Re $[s] < 0$. Further, we shall only discuss in detail the case when $T(s)$ has its zeros in Re $[s] < 0$. This latter restriction is however inessential.

Provided that $c_3a_2 - c_2$ is nonzero, $T(s)$ has a minimal realization

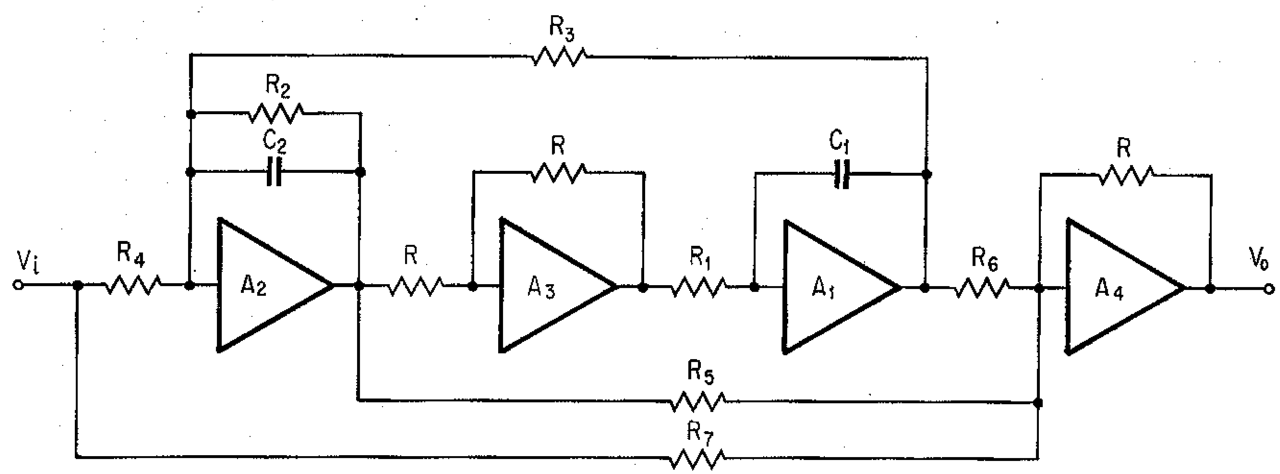


FIGURE 13.4.1. Biquad Circuit.

$$F = \begin{bmatrix} 0 & \dfrac{\sqrt{a_1}}{\lambda_1} \\ -\lambda_1\sqrt{a_1} & -a_2 \end{bmatrix} \qquad g = \begin{bmatrix} 0 \\ -\lambda_2 |c_3 a_2 - c_2| \end{bmatrix}$$

$$h = \begin{bmatrix} \dfrac{\lambda_1}{\lambda_2} \dfrac{c_3 a_1 - c_1}{|c_3 a_2 - c_2| \sqrt{a_1}} \\ \dfrac{1}{\lambda_2} \operatorname{sgn}(c_3 a_2 - c_2) \end{bmatrix} \qquad j = c_3 \tag{13.4.2}$$

where λ_1 and λ_2 are arbitrary positive numbers. [*Problem 13.4.1* requests verification that (13.4.2) defines a minimal realization of $T(s)$.]

A circuit synthesis suggested by (13.4.2) is shown in Fig. 13.4.1 for the case $c_3 a_2 > c_2$ and $c_3 a_1 > c_1$. (Straightforward variations apply if these inequalities do not hold.) Element values are

R, C_1, and C_2 arbitrary

$$R_1 = \frac{\lambda_1}{\sqrt{a_1} C_1} \qquad R_2 = \frac{1}{a_2 C_2} \qquad R_3 = \frac{1}{\lambda_1 \sqrt{a_1} C_2}$$

$$R_4 = \frac{1}{\lambda_2 (c_3 a_2 - c_2) C_2} \qquad R_5 = \lambda_2 R \qquad R_6 = \frac{\lambda_2}{\lambda_1} \frac{c_3 a_2 - c_2}{c_3 a_1 - c_1} \sqrt{a_1} R \tag{13.4.3}$$

$$R_7 = \frac{R}{c_3}$$

The entry x_1 of the state vector $x = [x_1 \quad x_2]'$ will be observed at the output of the operational amplifier A_1, and x_2 at the output of the operational amplifier A_2.

The amplifiers A_1, A_2, and A_3 constitute the main loop, with amplifier A_3 serving simply as an inverting amplifier. Note that one of the first two amplifiers could be replaced by a noninverting integrator, but at the expense of heightening any potential sensitivity and instability problem associated with the nonideal nature of the amplifiers. Generally, the gain around the loop should be equally distributed between the two integrating amplifiers, so that $R_3 C_2 = R_1 C_1$, or $\lambda_1 = 1$.

The amplifier A_4 serves simply to construct the output as a weighted sum of the input and the components of the state vector. With the signs assumed, this amplifier is used in a single-ended configuration, though it may well have to be used in a differential-input configuration (or in conjunction with an inverting amplifier) for certain parameter values.

Sensitivity

The critical aspects of input-output performance of the biquad are the inverse damping factor Q and resonant frequency ω_0 associated with the denominator of the transfer function. In our case, $Q = \sqrt{a_1/a_2}$ and $\omega_0 = \sqrt{a_1}$. The quantities of interest are the sensitivity coefficients S_x^Q and $S_x^{\omega_0}$, where x is an arbitrary circuit parameter, and

$$S_x^Q = \frac{\partial Q}{\partial x}\frac{x}{Q} \qquad S_x^{\omega_0} = \frac{\partial \omega_0}{\partial x}\frac{x}{\omega_0} \tag{13.4.4}$$

The sensitivity coefficients for x identified with any passive component in the biquad circuit come out to be never greater than 1 in magnitude; some are zero, others are $\frac{1}{2}$. These sorts of sensitivity coefficients are of the same order as apply when purely passive syntheses of a transfer function (including inductor elements) are considered. Sensitivities with respect to operational amplifier gains are at the most in magnitude approximately $2Q/K_i$, where K_i is the gain. It follows that a variation of 50 percent in an operational amplifier gain—as may well occur in practice—will cause a variation no greater than approximately 1 percent in Q or ω_0 if $K_i \geq 100Q$; the conclusion is that a Q of several hundred can be obtained with high precision.

Particularly when integrated circuits are used, sensitivity problems associated with temperature variation can often be dealt with by arranging that the effect of some components cancels that of others. References [10] and [11] contain additional details.

Miscellaneous Practical Points

A number of points are made in [10–12], which should be studied in detail by anyone interested in constructing biquad circuits. Some of these points will be outlined here.

Standardization. Subject to restrictions such as $c_3a_2 > c_2$, etc., an identical form of circuit is used for all second-order transfer functions; only the resistor values need be changed. Adjustment of key circuit parameters can be effected by adjusting one resistor per parameter (with no interaction between the adjustments). This means that the design of variable filters is simplified.

Cascading. Because the output is inherently low impedance (and the input can be made to be a moderately high impedance), cascading to build up syntheses of higher-order transfer functions is straightforward.

Absolute Stability. The natural frequencies remain in the left half-plane irrespective of passive component values. In this restricted sense, the circuit possesses absolute stability. However, stability difficulties can arise on account of nonideal behavior of the operational amplifiers.

Integrability. The full circuit can be integrated if desired. Trimming may be carried out solely with resistor adjustment, and technology allows such trimming to be effected through anodizing of thin-film resistors. It is also possible to largely overcome the problem of limitation of the total resistance in any integrated version of the circuit with an ingenious resistance multiplication scheme [12].

Q Enhancement. The finite bandwidth of the operational amplifiers causes the actual Q to exceed the design Q, even to the point of oscillation. Reference [11] includes formulas for Q enhancement and indicates compensation techniques for minimizing the problem.

Noise. Satisfactory output signal-to-noise ratios are achievable, e.g., 80 dB, with, if necessary in the case of low signal level, downward adjustment of the gain-determining resistor R_4 of Fig. 13.4.1. Note that, in effect, such an adjustment is equivalent to scaling.

Distortion. It is claimed in [11] that better distortion figures can often be obtained with the biquad than with some passive circuits, particularly when R_4 is adjusted.

Problem 13.4.1 Verify that the state-space realization defined as (13.4.2) is a realization of $T(s)$, as given in (13.4.1).

Problem 13.4.2 Suggest changes to the design procedure for the case when $c_3a_2 - c_2 = 0$.

Problem 13.4.3 Which resistors may most easily be varied in the biquad circuit to alter Q, ω_0, and the voltage gain at resonance (see [11])?

13.5 OTHER APPROACHES TO ACTIVE *RC* SYNTHESIS

Before leaving the subject of active-circuit synthesis, we mention briefly two additional topics. First, we shall comment on the use of gyrators, negative resistors, negative impedance converters, and controlled sources in active synthesis. Then we conclude the section by pointing out how a dynamic synthesis problem, that is, a problem involving synthesis of a nonconstant transfer-function matrix or even hybrid matrix, can be replaced in a straightforward way by a problem requiring synthesis of a constant matrix.

Synthesis with Gyrators

Though gyrators are constructible at microwave frequencies and can be operated with no external power source, this is not the case at audio or even RF frequencies. In order to build a gyrator, either a Hall-effect device can be used, or a circuit comprising transistors and resistors. This means though that any gyrator in a circuit is replaceable by active devices and resistors, and since, as we know, any inductor can be replaced by a gyrator terminated in a capacitor, a mechanism is available for replacing inductors by a set of active *RC* elements. Further, as we noted in Chapter 2, a transformer may be replaced by a gyrator cascade, so even transformers may be replaced by a set of active *RC* elements. Consequently, *any passive synthesis of an impedance or transfer-function matrix can be replaced by an active RC synthesis, with the aid of active realizations of the gyrator.*

The resulting structure may be inefficient or awkward to actually construct; for it does not follow that a structure developed for passive synthesis should be satisfactory for active synthesis. Nevertheless, the method does have some appealing features. As pointed out by Orchard [13], active *RC* circuits derived this way can be expected to enjoy very good sensitivity properties when passive element variation is considered. Variations within active elements, including the active circuits providing a gyrator, may, however, cause a problem. For example, phase shift within the gyrators can cause instability.

Reference [1] gives one of the most extensive treatments of the use of gyrators in active-circuit design. Active circuits realizing gyrators will be found there, together with variations on passive-circuit-synthesis procedures designed to accommodate the peculiarities of practical gyrators.

Negative Resistors, Negative Impedance Converters, and Controlled Sources

Historically, these three classes of elements were the first to be considered and used in active *RC* synthesis. Negative resistors are probably the least interesting and were the least used. Negative resistors do not exist in practice, and so they must be obtained with active devices. Two very straightforward ways of obtaining a negative resistor are through the use of a negative impedance converter (dealt with below), and through the use of a controlled source. In a sense then, active *RC* synthesis with negative resistors is subsumed by active *RC* synthesis using negative impedance converters or controlled sources.

The negative impedance converter (NIC) is a two-port device with the property that if an impedance $Z(s)$ is used to terminate one port, an impedance $-Z(s)$ is observed at the other port. Very many of the earliest attempts at active synthesis relied on use of negative impedance converters with one NIC being permitted per circuit (see, e.g., [14]). Most of the early circuits were particularly beset with sensitivity problems.

The controlled source as an active *RC* circuit element has enjoyed greater popularity than the negative resistance or the NIC. In ideal terms, it is a constant–gain voltage amplifier, with zero input admittance and output impedance. Departures from the ideal prove less embarrassing and are more easily dealt with than in the case of the negative resistor and the NIC; these factors, combined with ease of construction, perhaps account for its popularity since the early suggestion of [15]. Important refinements are still being made, which make the circuits more suitable for integrating (see, e.g., [1] where unpublished work of W. J. Kerwin is discussed). The controlled source would appear to have greater sensitivity problems associated with its use than the operational amplifier, at least in most situations. The fact that a lower gain is used than in the operational amplifier is however a point in its favor.

Conversion of a Dynamic Problem to a Nondynamic Problem

In view of the above discussion, it should be clear that any technique that may be used for passive synthesis may also be used as a basis of active RC synthesis; if such a technique requires the use of gyrators, inductors, or transformers, the active equivalents of these components are available. In fact, the effort involved in such techniques can be greatly reduced, since passivity is never really needed at any point in the synthesis. We now illustrate this point in detail.

We shall consider the question of hybrid-matrix synthesis; this, as we know, includes immittance synthesis and transfer-function-matrix synthesis. In case we aim to synthesize a transfer-function matrix, we conceive of its being embedded within a hybrid matrix.

Let $\mathcal{H}(s)$ be an $m \times m$ hybrid matrix—not necessarily passive—possessing an n-dimensional state-space realization $\{F, G, H, J\}$. Let us conceive of $\mathcal{H}(s)$ being synthesized by a nondynamic $m + n$ port N_r terminated in capacitors (see Fig. 13.5.1). For convenience in the succeeding calculations

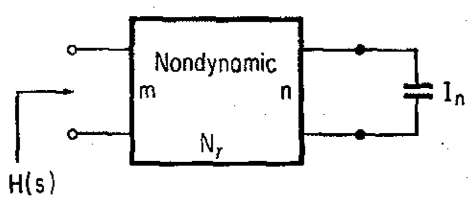


FIGURE 13.5.1. Idea Behind Synthesis of $\mathcal{H}(s)$.

we shall assume the capacitors to all have unit value; however, this is not essential to the argument.

As we know, $\mathcal{H}(s)$ will be observed as required if N_r is chosen to have a hybrid matrix

$$M = \begin{bmatrix} J & -H' \\ G & -F' \end{bmatrix} \tag{13.5.1}$$

In this hybrid description the last n ports of N_r are assumed to be current excited, while excitation at the first m ports coincides with that used in defining exciting variables for $\mathcal{H}(s)$.

Evidently, the problem of synthesizing the hybrid matrix $\mathcal{H}(s)$ is equivalent to the problem of synthesizing the constant hybrid matrix M. This conversion of a dynamic synthesis problem to a nondynamic one has been achieved, it should be noted, at the expense of having to compute matrices of a state-space realization, which is hardly a severe problem.

How may M be synthesized? There are a number of methods, but the simplest would appear to be a method based on the use of operational amplifiers as summers, together with voltage-to-current and current-to-voltage converters, as required. Problem 13.5.4 asks for details. As far as is known, this technique has yet to be used for practical synthesis, and it is not known how it would compare with the earlier techniques described.

Problem 13.5.1 In using gyrators comprised of active devices and resistors, it is particularly helpful to be able to ground two of the gyrator terminals. (One reason is that connections to a power supply can be achieved more straightforwardly.) Demonstrate the equivalence of Fig. 13.5.2, which is of practical benefit in converting classical ladder structures to active *RC* structures (see [16]). Demonstrate also the equivalence of Fig. 13.5.3

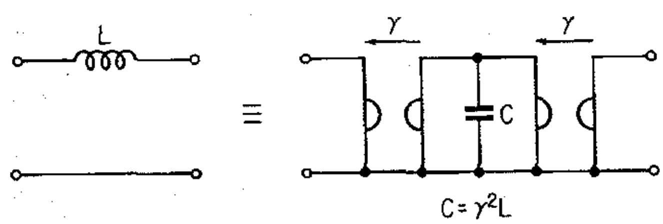


FIGURE 13.5.2. Inductor Equivalence Involving Grounded Capacitor and Gyrators.

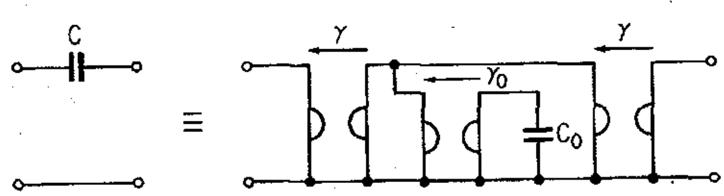


FIGURE 13.5.3. Replacement of Ungrounded Capacitor by Grounded Elements.

(see [17]); this equivalence can be useful when a capacitor is to be realized in an integrated circuit.

Problem 13.5.2 Verify the circuit equivalence of Fig. 13.5.4, allowing replacement of coupled coils by a capacitor–gyrator circuit. Obtain relations between the element values [18].

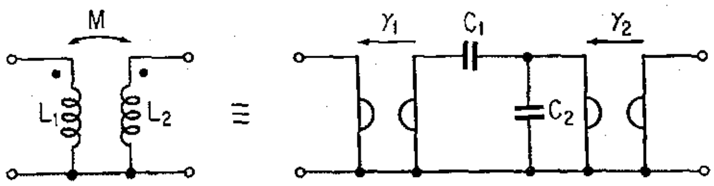


FIGURE 13.5.4. Replacement of Coupled Coils by Capacitors and Gyrators.

Problem 13.5.3 Show how a negative resistance can be obtained using a controlled source, such as an ideal voltage gain amplifier, and a positive resistor.

Problem 13.5.4 Discuss details of the synthesis of the constant hybrid matrix M of (13.5.1).

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EPILOGUE:

What of the Future?

An epilogue can only be long enough to make one main point, and our main point is this. Network analysis and synthesis via state-space methods are comparatively new notions, and few would deny that they are capable of significant development. We believe these developments will be exciting and will increase the area of practical applicability of the ideas many times.

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