1. Obtain a reasonable model.
2. Specify performance requirements — speed, accuracy, stability
3. Determine performance of system, as it is.
4. If not satisfactory, choose a design method.
5. Design controller.
6. Perform simulation study (linear & nonlinear models)
7. Test on actual system. ✓✓

control systems

Open loop

dynamic

~~linear~~

time-invariant

✓✓ continuous time

✓ deterministic

causal

closed loop

static

nonlinear

time varying

discrete time

stochastic

non-causal

$V = RI$

Superposition $\begin{cases} \text{Additivity} \\ \text{Homogeneity} \end{cases}$

If $u_1 \longrightarrow \boxed{\text{Sys}} \longrightarrow y_1$ & $u_2 \longrightarrow \boxed{\text{Sys}} \longrightarrow y_2$

then $u_1+u_2 \longrightarrow \boxed{\text{sys}} \longrightarrow y_1+y_2$

If $u \longrightarrow \boxed{\text{sys}} \longrightarrow y$

then $\alpha u \longrightarrow \boxed{\text{sys}} \longrightarrow \alpha y$

Linear

> If the o/ps of a system for inputs u_1 & u_2 are y_1 & y_2 respectively, then the output of the system for an input $\alpha_1 u_1 + \alpha_2 u_2$ is
>
> $$\alpha_1 y_1 + \alpha_2 y_2$$

1. $y = f(x) = 2x \quad , x \in \mathbb{R}$

static, linear system

$y_1 = 2x_1$

$y_2 = 2x_2$

$\hat{y} = 2(x_1 + x_2) = 2x_1 + 2x_2$

$= y_1 + y_2$

$y = 2x$

$\hat{y} = 2(\alpha x) = \alpha 2x = \alpha . y$

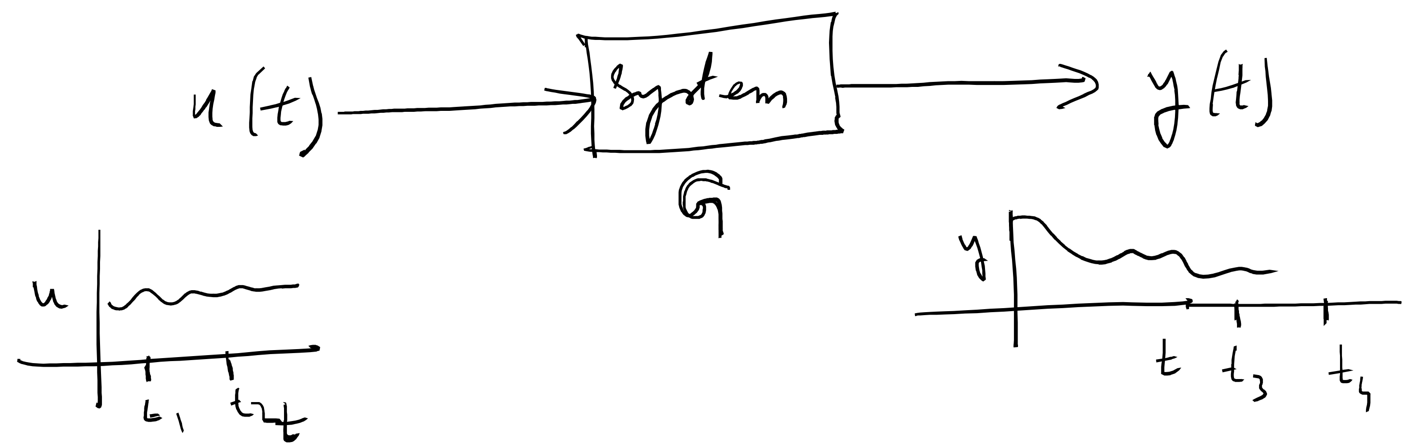
- $y = 2x + 1 \quad x \in \mathbb{R}$
- $y = x^2 + x, \quad x \in \mathbb{R}$
- $y = 3x, \quad x \in \{0, 2, 4, \ldots\}$
- $y = 2x, \quad x = \begin{bmatrix} a \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ b \end{bmatrix}$: $a, b \in \mathbb{R}$

~~Determine if these are linear systems~~

Determine if the following systems are static or dynamic.

- $y(t) = x(3t)$
- $y(t) = 5x(t)$
- $y(t) = x(\cos t)$.

<u>Qn</u> Does $y(t)$ depend solely on $x(t)$ $\forall t$?



$$G(u(t)) = y(t)$$

- Do <u>not</u> look for linearity w.r.t. time
- The input could be defined for an interval $[t_1, t_2]$ while the output could be for some other interval $[t_3, t_4]$

If $\quad u_1(t) \longrightarrow \boxed{\quad G \quad} \longrightarrow y_1(t) \qquad u_2(t) \longrightarrow \boxed{G} \longrightarrow y_2(t)$

then $\quad \alpha_1 u_1(t) + \alpha_2 u_2(t) \longrightarrow \boxed{\quad G \quad} \longrightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t).$

$$y' = \frac{dy}{dt} \; \ldots$$

Ex

$$a_0 y''' + a_1 y'' + a_2 y' + a_3 y = b_0 u'' + b_1 u' + b_2 u$$

$u \longrightarrow$ input

$y \longrightarrow$ output.

check if this system is linear.

$$a_0 y_1''' + a_1 y_1'' + a_2 y_1' + a_3 y_1 = b_0 u_1'' + b_1 u_1' + b_2 u_1 \quad \text{--- (1)}$$

$$a_0 y_2''' + a_1 y_2'' + a_2 y_2' + a_3 y_2 = b_0 u_2'' + b_1 u_2' + b_2 u_2 \quad \text{--- (2)}$$

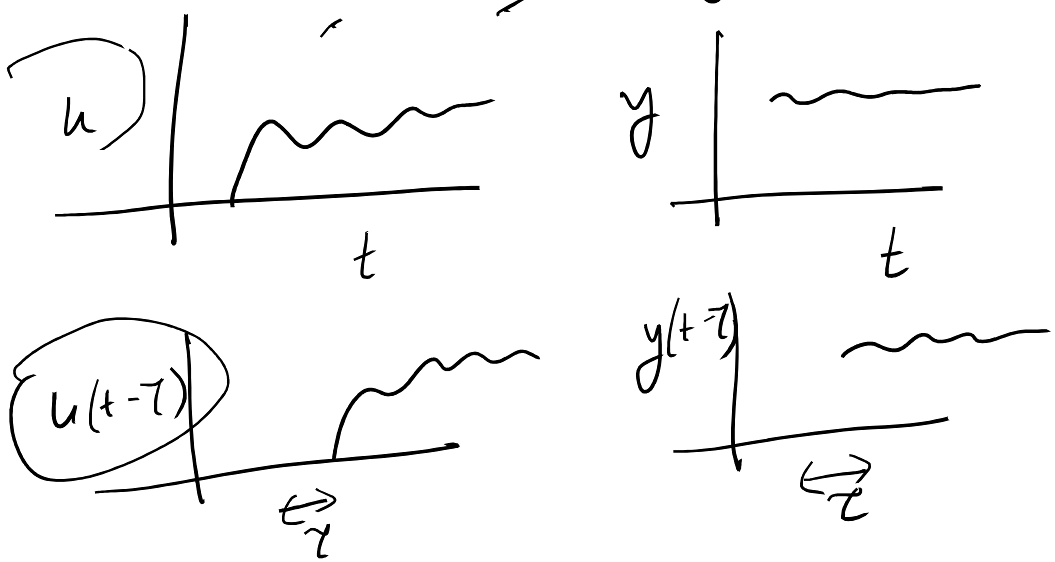
$$\begin{aligned}
a_0 \hat{y}''' + \cdots + a_3 \hat{y} &= b_0 (u_1 + u_2)'' + b_1 (u_1 + u_2)' + b_2 (u_1 + u_2) \\
&= (b_0 u_1'' + b_1 u_1 + b_2 u_1) + (b_0 u_2'' + \cdots + b_2 u_2) \\
&= (a_0 y_1''' + \cdots + a_3 y_1) + (a_0 y_2''' + \cdots a_3 y_2) \\
&= a_0 (y_1 + y_2)''' + \cdots + a_3 (y_1 + y_2)
\end{aligned}$$

Clearly $\hat{y} = y_1 + y_2$ satisfies the eqn

Ex Verify the homogeneity prop

$$G(u(t)) = y(t)$$

define σ_τ as the shift operation



$$\boxed{\sigma_\tau G = G \sigma_\tau}$$

$$y = Gu \qquad \sigma_\tau y = \sigma_\tau G u$$

$$\sigma_\tau y = G \sigma_\tau u$$

Check for time inv.

- $\boxed{y(t) = x(3t)}$

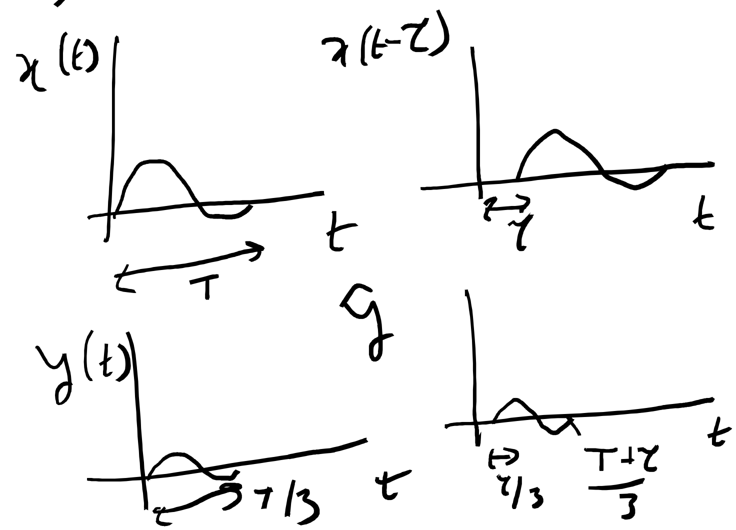
- $y(t) = \int_{t-3}^{t} u^2(t_1)\,dt_1$

- $y(t) = \int_0^t \sqrt{u(t_1)}\,dt_1$

$$\sigma_\tau y(t) = y(t-\tau) = x(3(t-\tau))$$

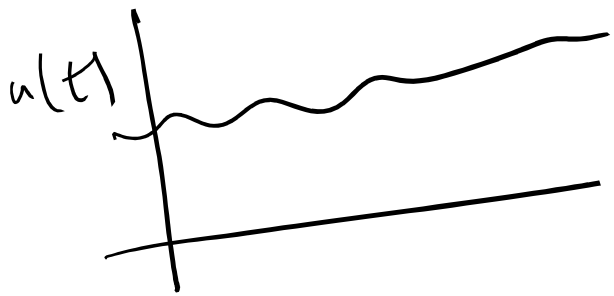
$$\sigma_\tau x(t) = x(t-\tau)$$

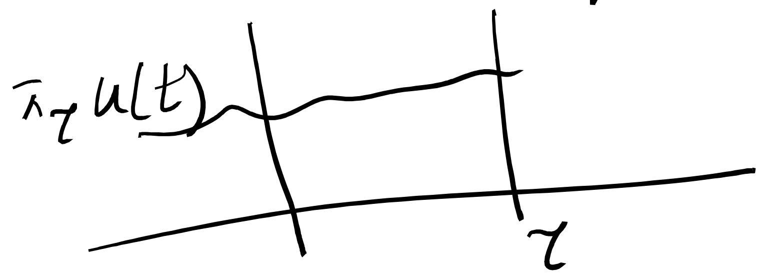
$$\hat{y}(t) = x(3t-\tau)$$



Causality

Π_τ : Define as the truncation operation.

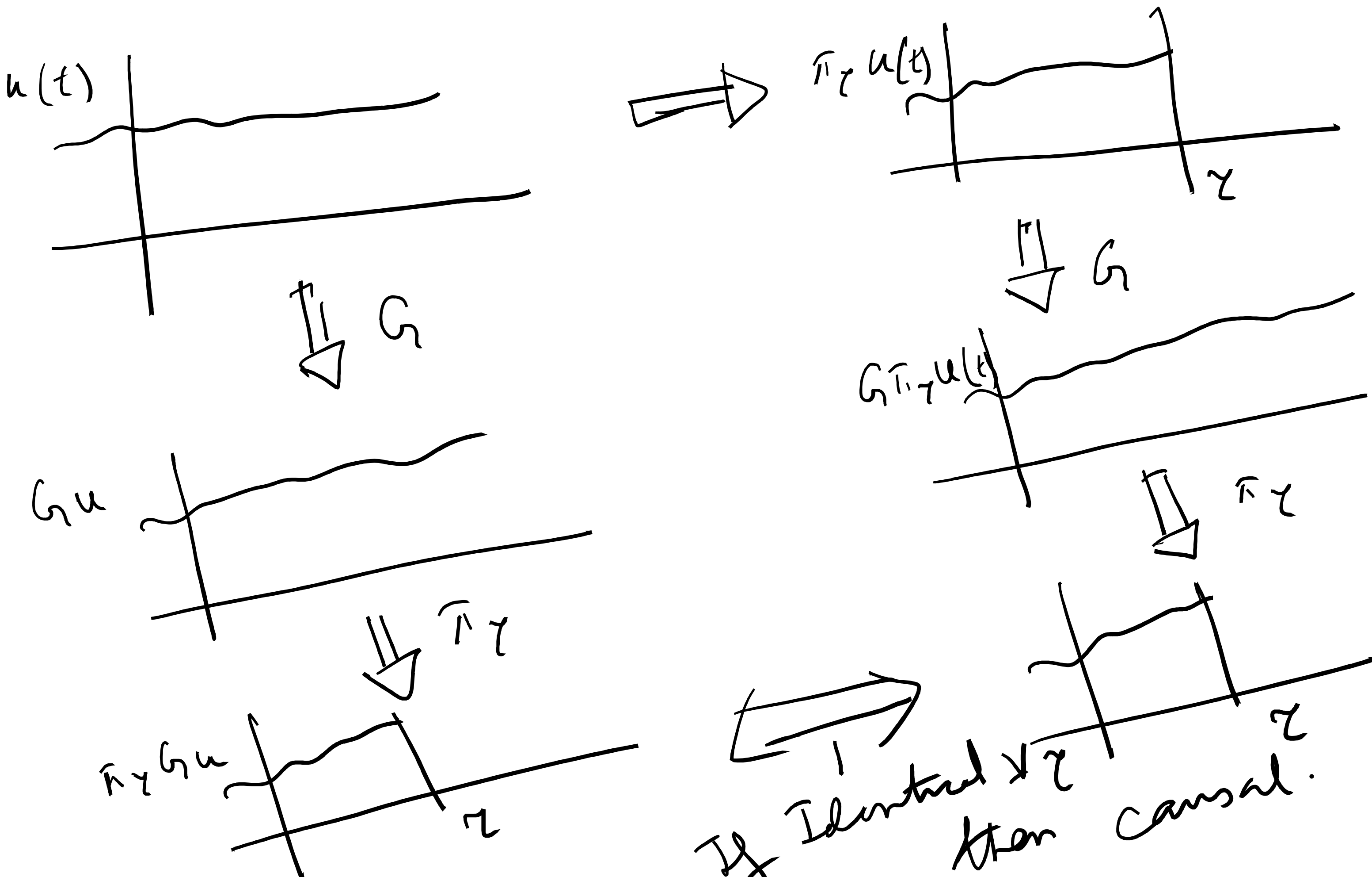




Suppose a system is described by the i/p o/p relation $y = Gu$

Causality implies $\boxed{\Pi_\tau G \Pi_\tau = \Pi_\tau G \quad \forall \tau}$.

$$\pi_\tau G \pi_\tau u = \pi_\tau G u \quad \forall \tau$$



If Identical ∀ τ then causal.

Qn

For ~~both~~ t. inv & causal can the requirement be simplified?

$$\Pi_\tau G \bar{\Pi}_\tau = \bar{\Pi}_\tau G \quad \forall \tau \text{ ——— (I)}$$

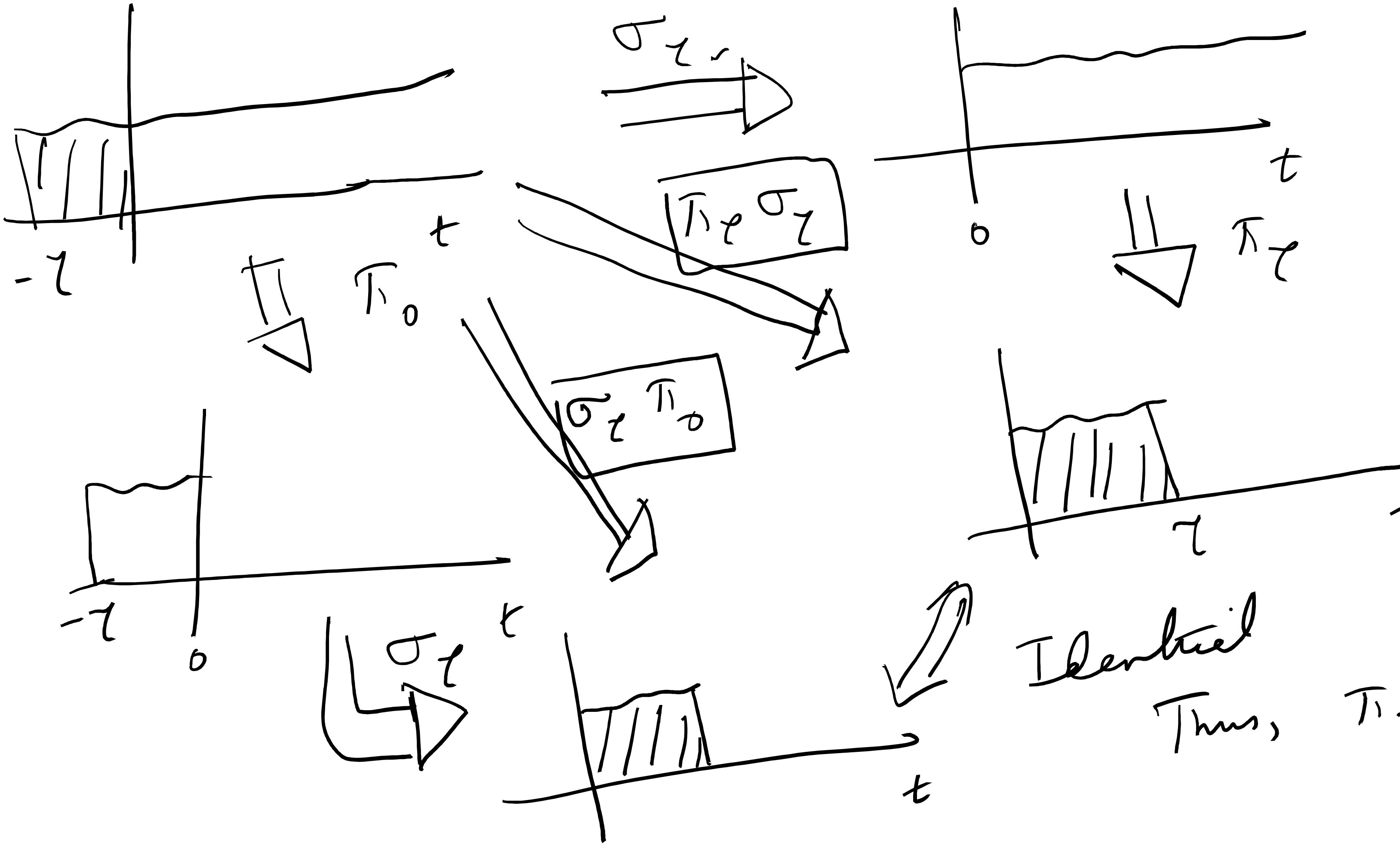
Let $\tau = 0$ w.l.g.

$$\Pi_0 G \bar{\Pi}_0 = \Pi_0 G$$

$$\Rightarrow \left(\Pi_0 G \bar{\Pi}_0 - \bar{\Pi}_0 G \right) = 0 \text{ ——— (1)}$$

From t. inv. $\sigma_\tau G = G \sigma_\tau$ ——— (2)

Claim $\boxed{\Pi_\tau \sigma_\tau = \sigma_\tau \Pi_0}$ ——— (3)



Thus, $\pi_\tau \sigma_\tau = \sigma_\tau \pi_0$

$$(\Pi_0 G - \Pi_0 G \Pi_0) = 0$$

$$\Rightarrow \Pi_0 G (I - \Pi_0) = 0$$

$$\Rightarrow \underbrace{\sigma_\tau \Pi_0}_{} G (I - \Pi_0) = 0$$

$$\Rightarrow \Pi_\tau \sigma_\tau G (I - \Pi_0) = 0$$

$$\Rightarrow \Pi_\tau G \sigma_\tau (I - \Pi_0) = 0 \quad \text{(time inv)}$$

$$\Rightarrow \Pi_\tau G (\sigma_\tau - \underbrace{\sigma_\tau \Pi_0}_{}) = 0$$

$$\Rightarrow \Pi_\tau G (\sigma_\tau - \Pi_\tau \sigma_\tau) = 0$$

$$\Rightarrow \Pi_\tau G (I - \Pi_\tau) \boxed{\sigma_\tau} = 0$$

↳ invertible.

<u>Note</u>: Order of operations matter in general..

$$\Rightarrow \Pi_\tau G (I - \Pi_\tau) = 0$$

<u>Note</u>: All these steps are reversible. (since σ_τ is invertible operation)

$$\boxed{\Pi_\tau G(I-\Pi_\tau) = 0 \quad \forall \tau}$$ if $\Pi_0 G(I-\Pi_0) = 0$ &

system G is time invariant.

$M \times N$

$\text{Antlog}\left(\underline{\log M} \pm \underline{\log N} \right)$

ODE

$\Downarrow$

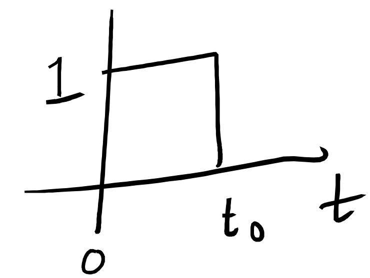
$\alpha \cdot T$

Check for causality of the following systems.

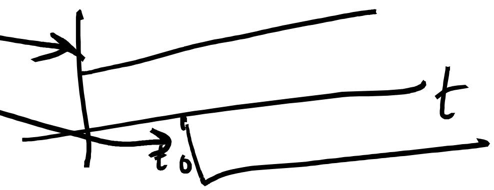
- $y(t) = 2u(t) + 3u(t-3)$
- $y(t) = 2u(t) + 3u(t-3) + 6u(t+3)$

Unit pulse function

$$f(t) = 1(t) - 1(t - t_0)$$



$$= \begin{cases} 0 & t < 0 \\ 1 & 0 \leq t \leq t_0 \\ 0 & t > t_0 \end{cases}$$

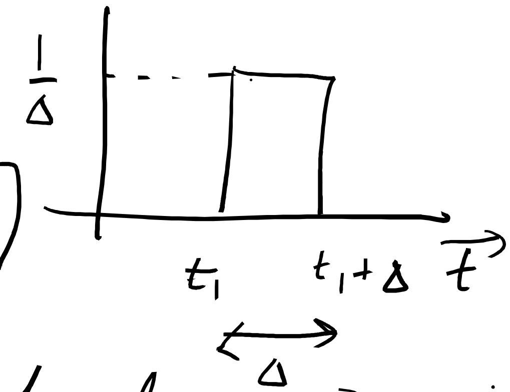


$$\mathcal{L}(f(t)) = \mathcal{L}[1(t)] - \mathcal{L}[1(t - t_0)]$$

$$= \frac{1}{s} - \frac{e^{-st_0}}{s}$$

$$= \frac{1 - e^{-st_0}}{s}$$

Delta function



$$\delta_\Delta(t-t_1) = \frac{1}{\Delta}\left[1(t-t_1) - 1(t-t_1-\Delta)\right]$$

As $\Delta \to 0$, this is said to approach the Dirac delta

(*) In the limit $\Delta \to 0$, the L.T. of $\delta_\Delta(t-t_1)$ is e^{-st_1} (Exercise)

Sifting Property

$$\boxed{\int_{-\infty}^{\infty} f(t)\,\delta(t-t_1)\,dt = f(t_1)}$$

$$\mathcal{L}[f'(t)] = \underbrace{\int_{0^-}^{\infty} e^{-st} \frac{d}{dt} f(t)\,dt}_{\displaystyle \int_{0^-}^{0^+} (\quad)\,dt + \int_{0^+}^{\infty} (\quad)\,dt} = sF(s) - f(0^-) \quad \text{— (1)}$$

$$\Downarrow$$

$$= f(0^+) - f(0^-) + \int_{0^+}^{\infty} f'(t) e^{-st}\,dt \quad \text{— (2)}$$

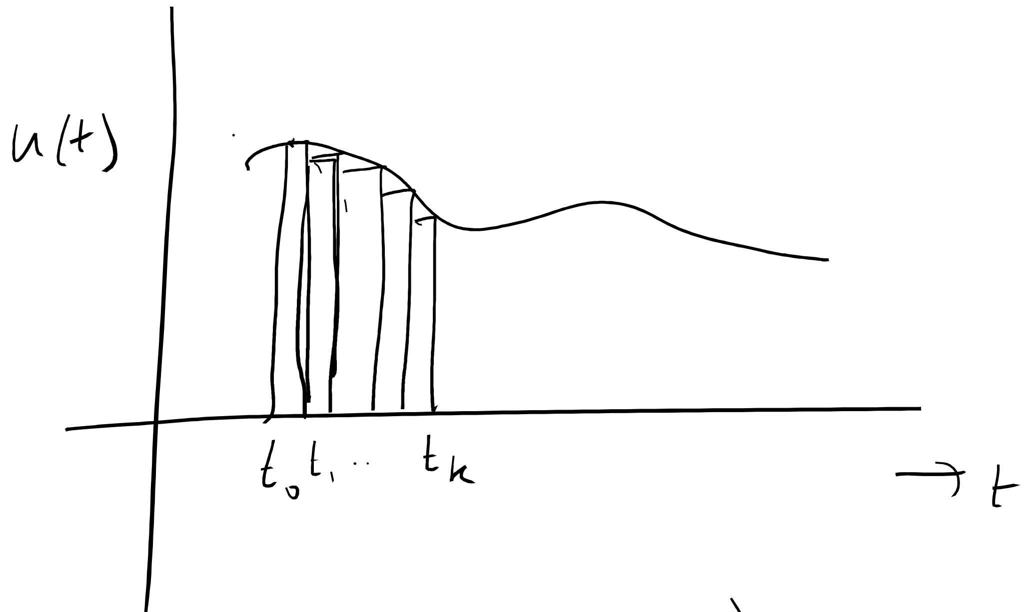
From (1) & (2),

$$f(0^+) - \cancel{f(0^-)} + \int_{0^+}^{\infty} (\quad)\,dt = sF(s) - \cancel{f(0^-)}$$

$$\Rightarrow \boxed{\int_{0^+}^{\infty} f' e^{-st}\,dt} = sF(s) - f(0^+) \quad \Rightarrow sF(s) - f(0^+) \to 0$$

$\to$ tends to 0 as $s \to \infty$ as $s \to \infty$

$$\boxed{\lim_{s \to \infty} sF(s) = f(0^+).}$$



$$\Delta = t_{k+1} - t_k$$

$$u_\Delta(t_i) = u(t_i) \cdot \delta_\Delta(t - t_i) \cdot \Delta$$

$\delta_\Delta(t - t_i) \cdot \Delta$ is a pulse of unit amplitude.

$$\bar{u}(t) = \sum_{i=0}^{\infty} u(t_i)\, \delta_\Delta(t - t_i)\, \Delta$$

$$u(t) = \mathop{\mathrm{Lt}}_{\Delta \to 0} \sum_{i=0}^{\infty} u(t_i)\,\delta_\Delta(t-t_i)\,\Delta = \int_0^\infty u(\tau)\,\delta(t-\tau)\,d\tau$$

$$u(t) \longrightarrow \boxed{\;G\;} \longrightarrow y(t)$$

G : linear, t.i., initially relaxed.

$$y(t) = G\left[\sum_{i=0}^{\infty} u(t_i)\,\delta_\Delta(t-t_i)\,\Delta\right]$$

$$= \left[\sum_{i=0}^{\infty} G\big(u(t_i)\,\delta_\Delta(t-t_i)\,\Delta\big)\right]$$

$$= \left[\sum_{i=0}^{\infty} G\big(\delta_\Delta(t-t_i)\,\Delta\big)\cdot u(t_i)\right]$$

$G(\delta_\Delta(t-t_i))$: response or o/p of the system at time t due to the application of the signal δ_Δ at time t_i.

$G\left[\delta_\Delta(t-t_i)\right]$

$t - t_i$

Now let $\Delta \to 0$. Summation is replaced by integration.

$$y(t) = \int_0^\infty g(t-\tau)\, r(\tau)\, d\tau \qquad t > \tau$$

$g(t-\tau)$: <u>Impulse response</u> of system at t due to $\delta(t-\tau)$ applied at τ.

Transfer function : L.T. of the impulse response.

$$y(t) = \int_0^\infty g(t-\tau)\, u(\tau)\, d\tau$$

suppose sys. is causal.

$$= \int_0^t g(t-\tau)\, u(\tau)\, d\tau$$

Convolution integral.

Ex

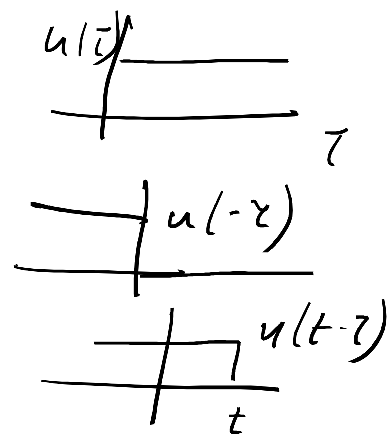
$$\int_0^t g(t-\tau)\, u(\tau)\, d\tau = \int_0^t g(\tau)\, u(t-\tau)\, d\tau$$

Laplace Transform of the Output

$$\mathcal{L}[y(t)] = \mathcal{L}\left[\int_0^t g(t-\tau)\,u(\tau)\,d\tau\right]$$

$$= \mathcal{L}\left[\int_0^t g(\tau)\,u(t-\tau)\,d\tau\right]$$

$$= \mathcal{L}\left[\int_0^\infty g(\tau)\,u(t-\tau)\,d\tau\right]$$



$\because\ \underline{u(t-\tau) = 0\ \forall\ \tau > t}$

$$= \int_0^\infty \left\{ \int_0^\infty g(\tau)\,u(t-\tau)\,d\tau \right\} e^{-st}\,dt$$

$$\mathcal{L}[y(t)] = \int_0^\infty \int_0^\infty g(\tau)\, u(t-\tau)\, e^{-st}\, dt\, d\tau$$

$$= \int_0^\infty \int_0^\infty g(\tau)\, u(t-\tau)\, e^{-s\tau} \cdot e^{-s(t-\tau)}\, dt\, d\tau$$

$$= \int_0^\infty g(\tau) e^{-s\tau} \left\{ \overbrace{\int_0^\infty u(t-\tau)\, e^{-s(t-\tau)}\, dt} \right\} d\tau$$

$$= \int g(\tau) e^{-s\tau} \left\{ \overbrace{\int_0^\tau (\quad)}^{0} + \int_\tau^\infty (\quad) \right\} d\tau$$

$$u(t-\tau) = 0 \;\; \forall \; t < \tau.$$

$$\mathcal{L}[y(t)] = \int_0^\infty g(\tau)e^{-s\tau}\left\{\int_\tau^\infty u(\underline{\underline{t-\tau}})e^{-s(\underline{\underline{t-\tau}})}\,dt\right\}d\tau$$

$$= \int_0^\infty g(\tau)e^{-s\tau}\,d\tau \int_0^\infty u(\gamma)e^{-s\overset{\underline{t-\tau}}{\gamma}}\,d\gamma$$

$$= \underbrace{G(s)}_{\hookrightarrow\ \mathcal{L}T.\ \text{of}\ \underline{IR}} \cdot U(s)$$

$$Y(s) = G(s)\,U(s)$$

$G(s)$ ↳ Transfer function of the LTI system

$$a_0 y''' + a_1 y'' + a_2 y' + a_3 y = b_0 u'' + b_1 u' + b_2 u$$

Assume the initial condition to be zero.

Taking LT on both sides.

$$\Rightarrow \left[a_0 s^3 + a_1 s^2 + a_2 s + a_3\right] Y(s) = \left[b_0 s^2 + b_1 s + b_2\right] U(s).$$

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{a_0 s^3 + a_1 s^2 + a_2 s + a_3} = \boxed{G(s)}$$

$\mathcal{L}[y]/\mathcal{L}[u]$ with all initial condition set to zero.

Transfer fn for physical systems are at most

biproper

G(s) is a rational fn in s.

$$G(s) = \frac{N(s)}{D(s)}$$ $\Rightarrow$ polynomials in 'S'.

$$\deg(N) = m$$
$$\deg(D) = n$$

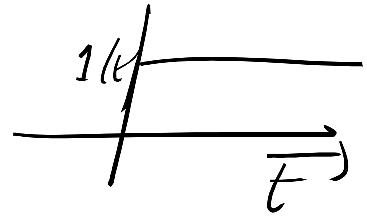
$n \geq m$

$n > m \rightarrow$ strictly proper

$n = m \rightarrow$ biproper

$G(s) = s$

$1(t)$



$$\mathcal{L}(1(t)) = \frac{1}{s}$$

$$y = \mathcal{L}^{-1}\left[s \cdot \frac{1}{s}\right] = \delta(t)$$

Roots of $D(s)$ are called <u>poles</u> of the system

Roots of $N(s)$ are the <u>zeros</u> of the system