Lecture 2 Module 1

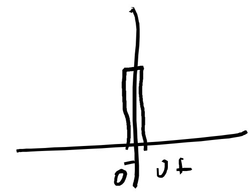
- Inverse Laplace Transforms.
- Response types depending on pole location.

An improper transfer function corresponds to an acausal system.

$\dot{x}(t) + 2x(t) = \delta(t)$, $\quad t > 0^-$ $\quad x(0^-) = 1$

classical method

$t > 0^+$



$$\dot{x}(t) + 2x(t) = 0$$

$$\Rightarrow x(t) = x(0^+)e^{-2t}$$

$$= 2e^{-2t}$$

$$x(0^+) - x(0^-) = \int_{0^-}^{0^+} \delta(t)\,dt$$

$$= 1$$

$$x(0^+) = 2$$

$$\int_{0^+}^{\infty} f(t) e^{-st}\, dt = \mathcal{L}_+ [f(t)]$$

$$\dot{x} + 2x = \delta(t)$$

$$s X_+(s) - x(0^+) + 2X(s) = 0$$

$$X_+(s) = \frac{x(0^+) \Rightarrow ?}{s+2}$$

$$\int_{0^-}^{\infty} f(t) e^{-st}\, dt$$

$$s X(s) - x(0^-) + 2X(s) = 1$$

$$\Rightarrow X(s) = \frac{1}{s+2} + \frac{x(0^-)}{s+2}$$

$$= \frac{2}{s+2} = 2e^{-2t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Suppose $x(t) = e^{\alpha t}$ $\alpha \in \mathbb{C}$.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{-\alpha\tau} \cdot e^{\alpha t}\, d\tau = e^{\alpha t} \underline{\int_{-\infty}^{\infty} h(\tau) e^{-\alpha\tau}\, d\tau}$$

$$= e^{\alpha t}\, \underline{\hat{H}(\alpha)}$$

$|\hat{H}(\alpha)| = M$

$\phi = \angle \hat{H}(\alpha)$

$M \angle \phi$

For causality,

$h(t) = 0 \quad t < 0$

$$y(t) = e^{\alpha t} \int_{0^-}^{\infty} h(\tau)\, e^{-\alpha\tau}\, d\tau = e^{\alpha t} H(\alpha)$$

$$\dot{x} + 2x = e^{\alpha t}$$

Let $x(t) = H(\alpha)\, e^{\alpha t}$

$$\alpha H(\alpha) e^{\alpha t} + 2H(\alpha) e^{\alpha t} = e^{\alpha t}$$

$$\Rightarrow H(\alpha) = \frac{1}{\alpha + 2}$$

$$H(s) = \frac{1}{s+2}$$

$$h(t) = \mathcal{L}^{-1}[H(s)] = e^{-2t} \cdot 1(t)$$

- $F(s) = \int_{0^-}^{\infty} f(t) e^{-st} dt = \mathcal{L}[f(t)]$

$\bar{F}(s) = \int_{0^-}^{\infty} e^{at} f(t) . e^{-st} dt = F(s-a)$ Ex

$\Rightarrow \mathcal{L}[e^{at} f(t)] = F(s-a)$. — (i)

- $I_k = \int_{0^-}^{\infty} t^k e^{-st} dt = \left. \frac{t^k e^{-st}}{-s} \right|_{t=+0^-}^{t\to\infty} - \int_{0^-}^{\infty} \frac{k t^{k-1} e^{-st}}{-s} dt$, $k \in \mathbb{Z}_+$

$= \frac{k}{s} I_{k-1}$

$I_0 = \frac{1}{s}$

$I_1 = \frac{1}{s^2}$, $I_2 = \frac{2}{s^3}$. . $I_k = \frac{\lfloor k}{s^{k+1}}$ — (ii)

$$G(s) = \frac{N(s)}{\prod_{i=1}^{k} (s - p_i)^{r_{p_i}}}$$

p_i are k distinct complex nos.

$$\deg(D(s)) = \sum_{i=1}^{k} r_{p_i}.$$

$$G(s) = \sum_{i=1}^{k} \left(\sum_{d_i=1}^{r_{p_i}} \frac{A_{i,d_i}}{(s - p_i)^{d_i}} \right)$$

$A_{i,d_i} \in \mathbb{C}$

$$\Downarrow$$

$$\frac{\underline{A_{i,d_i}}\, t^{d_i - 1}}{\lfloor d_i} e^{\underline{p_i} t}$$

$$G(s) = \frac{s}{s^2+3s+2} = \frac{s}{(s+1)(s+2)}$$

$$= \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

$$\frac{s}{(s+1)(s+2)} = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

$$s = k_1(s+2) + k_2(s+1) \quad \forall s$$

$s = -1, \ s = -2$

$k_1 + k_2 = 1 \quad ; \quad 2k_1 + k_2 = 0$

$k_1 = -1$

$k_2 = 2.$

$Ax = b$

$A = [\quad]$

$A = [\quad]$

$$\lim_{s \to -1} (s+1) G(s) = k_1 \qquad \lim_{s \to -2} (s+2) G(s) = k_2$$

$$\ddot{x} + 2\dot{x} + x = 0 \qquad \dot{x}(0^-) = b \qquad x(0^-) = a.$$

$$s^2 X(s) - s\,x(0) - \dot{x}(0) + 2sX(s) \quad 2\,x(0) + X(s) = 0$$

$$\Rightarrow \quad X(s) = \frac{as + (2a+b)}{(s+1)^2}$$

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2}$$

$$as + (2a+b) = A_2 + A_1(s+1)$$

$$A_2 + A_1 = 2a+b$$

$$A_1 = a$$

$$A_2 = a+b$$

$$X(s) = \frac{a}{s+1} + \frac{a+b}{(s+1)^2}$$

$$x(t) = a e^{-t} + (a+b) t e^{-t}$$

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2}$$

$$\lim_{s \to -1} (s+1)^2 X(s) = \lim_{s \to -1} A_1 \cancel{(s+1)}^{0} + A_2$$

$$\frac{A_1}{(s+1)}(s+1)^2 + \frac{A_2}{(s+1)^2}(s+1)^2 = as + (2a+b)$$

$$A_2 + A_1(s+1) = as + (2a+b) \Rightarrow A_1 = a$$

Complex poles

$a, b > 0$

$$\frac{5s+2}{(s+a)^2+b^2} = \frac{A+j\bar{A}}{s+a+jb} + \frac{B+j\bar{B}}{s+a-jb}$$

$A, \bar{A}, B, \bar{B} \in \mathbb{R}$

Ex

$$(A+j\bar{A})^* = (B+j\bar{B})$$

$p, \xi \in \mathbb{R}$

$$(A+j\bar{A})(s+a-jb) + (B+j\bar{B})(s+a-jb) = ps+\xi$$

$$\left.\begin{aligned} \bar{A} &= -\bar{B} \\ A &= B \end{aligned}\right| \quad \text{EX}$$

$$z_1 e^{p_1 t} + z_2 e^{p_2 t} \qquad z_1, z_2, p_1, p_2 \in \mathbb{C}$$

$$e^{\alpha t}\cos\beta t \qquad e^{\alpha t}\sin\beta t$$

$$\mathcal{L}^{-1}\left[\frac{z_1}{s-p_1}\right] = z_1 e^{p_1 t} \qquad \mathcal{L}^{-1}\left[\frac{z_1^*}{s-p_1^*}\right] = z_1^* e^{p_1^* t}$$

$$z_1 e^{\alpha t} e^{j\beta t} + z_1^* e^{\alpha t} \cdot e^{-j\beta t}$$

$$= \left[(a+jb)e^{j\beta t} + (a-jb)e^{-j\beta t}\right]e^{\alpha t}$$

$$= \left[2a\left[\frac{e^{j\beta t}+e^{-j\beta t}}{2}\right] \cancel{+}\, 2b\left[\frac{e^{j\beta t}-e^{-j\beta t}}{2j}\right]\right]e^{\alpha t}$$

$$\mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin\omega t \qquad \mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos\omega t$$

$$\mathcal{L}^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = e^{-at}\sin\omega t \qquad e^{-at}\cos\omega t$$

$$G(s) = \frac{3}{s(s^2+2s+5)} = \frac{k_1}{s} + \frac{k_2 s + k_3}{(s^2+2s+5)}$$

$$\lim_{s\to 0} s\,G(s) = k_1 = 3/5, \quad 3 = \left(k_2 + 3/5\right)s^2 + \left(k_3 + \frac{6}{5}\right)s + 3.$$

$$G(s) = \frac{N(s)}{D(s)} e^{-s\tau}$$

$$\int_{0^-}^{\infty} f(t-\tau) e^{-st} dt = e^{-s\tau} F(s)$$

$$F(s) = \mathcal{L}[f(t)]$$

$$G(s) = \frac{2s e^{-2s} + 3s e^{-5s}}{(s+2)(s+3)(s+4)}$$

Ex

$$= e^{-2s} \left[\frac{A}{(s+2)} + \frac{B}{s+3} + \frac{C}{s+4} \right]$$

$$+ e^{-5s} \left[\frac{D}{s+2} + \frac{E}{s+3} + \frac{F}{s+4} \right]$$

Ex

Take a transfer fn with repeated complex poles and see which method is easier.

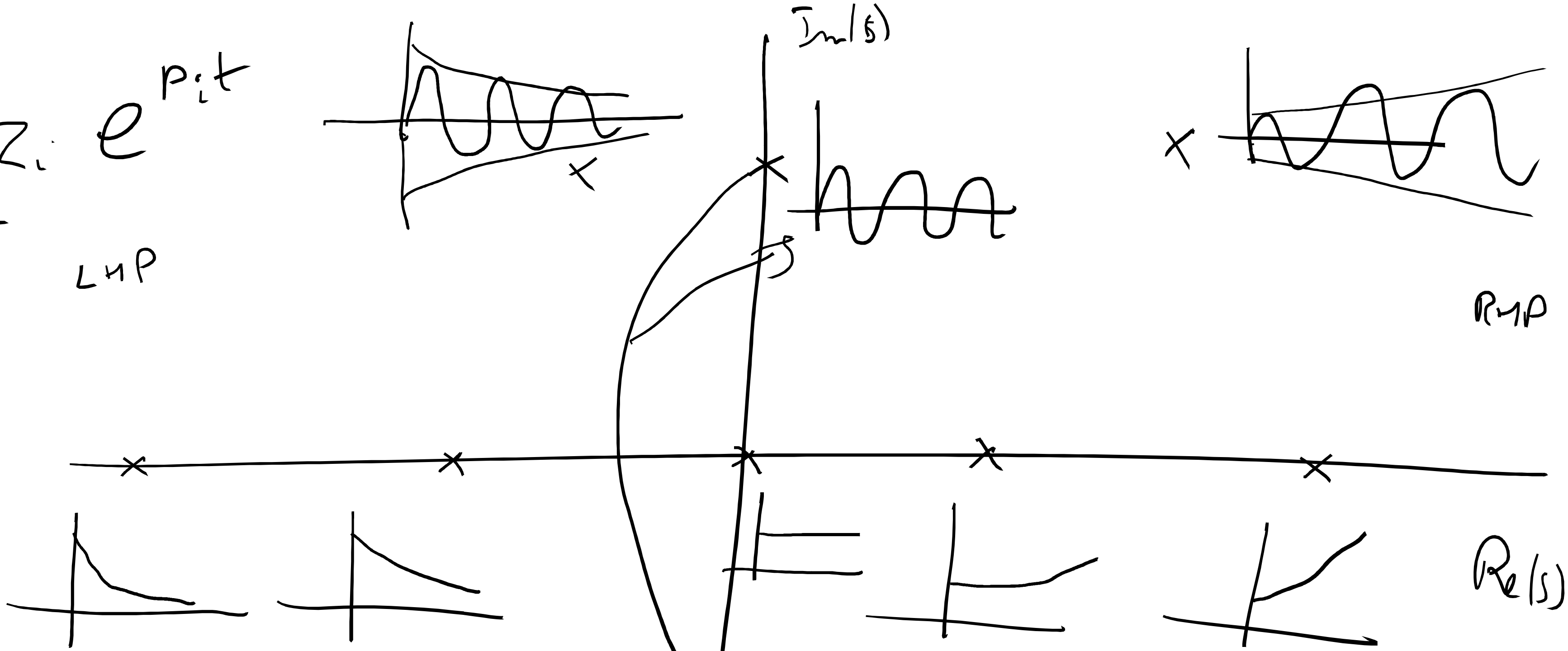
$$G(s) = \frac{7s+8}{[s^2+5s+20]^2}$$

$\boxed{Aa = b}$

$$\frac{7s+8}{(s+2)^2(s+1)} = \frac{A}{(s+2)^2} + \frac{B}{(s+1)} + \boxed{\frac{C}{s+2}}$$

LC

$$\sum z_i \, e^{p_i t}$$



stable

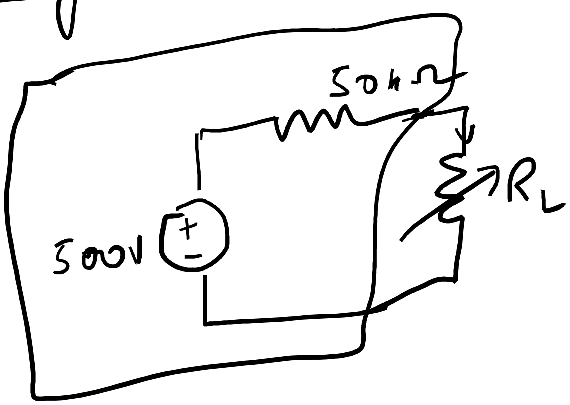
unstable

$\boxed{t^k e^{-\alpha t}} \downarrow \quad \alpha > 0$

Ex Additional practice problems (Inverse Laplace Transforms).

- $\dfrac{2s+3}{s^2+3s+2}$
- $\dfrac{s+2}{(s+1)^2}$
- $\dfrac{1}{s^2+2s+10}$
- $\dfrac{2s^2+3s+2}{(s+1)^3}$
- $\dfrac{s+2}{(s+1)^2(s+3)}$
- $\dfrac{s(1+e^{-2s}+e^{-7s})+e^{-2s}}{s(s+6)}$

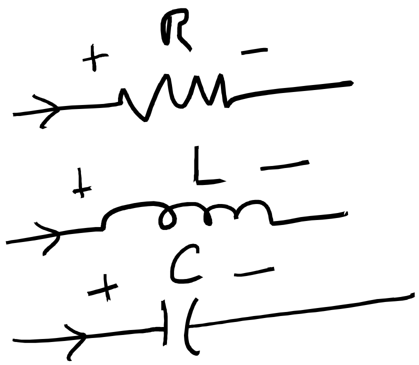
- Analysis of electrical ckts. Module 2



Current Source.

$10\Omega < R_L < 50\Omega$

KCL, KVL



$v = Ri \longrightarrow V(s) = R\,I(s)$

$v = L\dfrac{di}{dt} \longrightarrow V(s) = sL\,I(s)$

$v = \dfrac{1}{C}\displaystyle\int_0^t i\,dt \longrightarrow V(s) = \dfrac{1}{sC}\,I(s)$

Ex

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \int_{0^-}^{\infty} \underbrace{\left(\int_0^t f(t)\,dt\right)}_{u} \underbrace{e^{-st}\,dt}_{dv} = \frac{1}{s}F(s)$$

$F(s) = \mathcal{L}[f(t)]$

$\int u\,dv = uv - \int v\,du$

$V(s) = \boxed{R}\,I(s)$

$V(s) = \boxed{sL}\,I(s)$

$V(s) = \boxed{\dfrac{1}{sC}}\,I(s)$

$\boxed{\text{KVL}}$

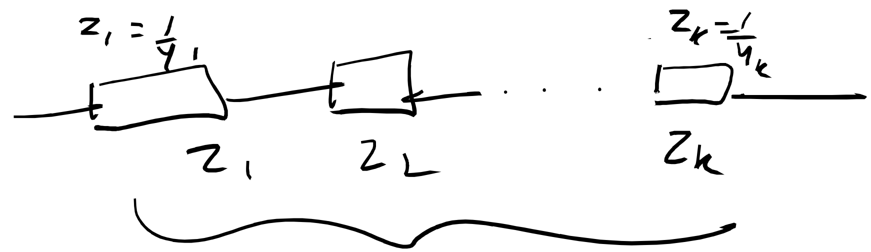
$\boxed{\text{KCL}}$

$I(s) = \dfrac{V(s)}{R}$ $\qquad \dfrac{1}{R}$

$I(s) = \dfrac{V(s)}{sL}$ $\qquad \dfrac{1}{sL}$

$I(s) = sC\,V(s)$ $\qquad \dfrac{1}{sC}$

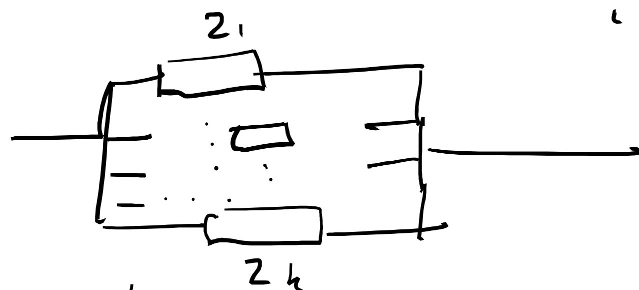
series ckts.



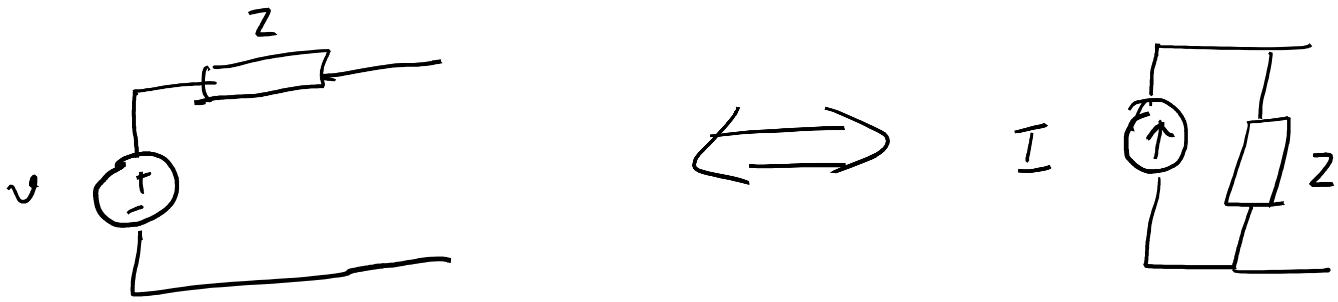
$$Z_{eq} = \sum_{i=1}^{k} Z_i$$

$$Y_{eq} = \frac{1}{\sum_{i=1}^{k} \frac{1}{Y_i}}$$

|| ckts

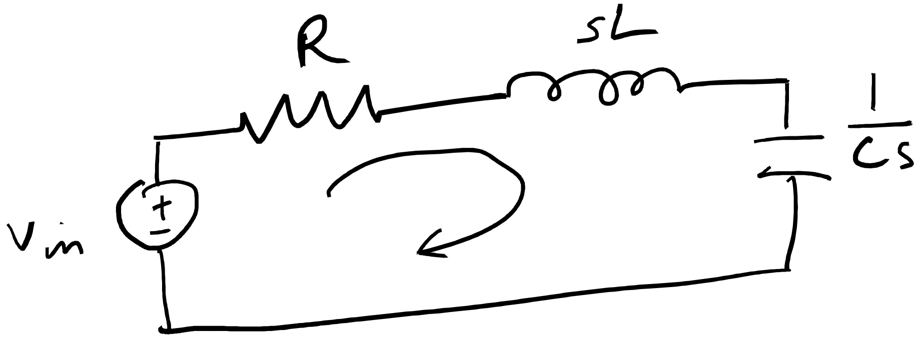


$$Y_{eq} = \sum_{i=1}^{k} Y_i \qquad Z_{eq} = \frac{1}{\sum_{i=1}^{k} \frac{1}{Z_i}}$$



KVL $\Longleftrightarrow$ KCL

$I = v/Z$



$$I(s)\left[R + \frac{1}{sC} + sL\right] = V_{in}(s)$$

$$\Rightarrow \frac{Cs}{LCs^2+RCs+1} = \frac{I(s)}{V_{in}(s)}$$

$$\Rightarrow \frac{I(s)}{V_{in}(s)} = \frac{s/L}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Suppose R is negligibly small.

Additionally consider $V_{in}(s) = \frac{1}{s}$

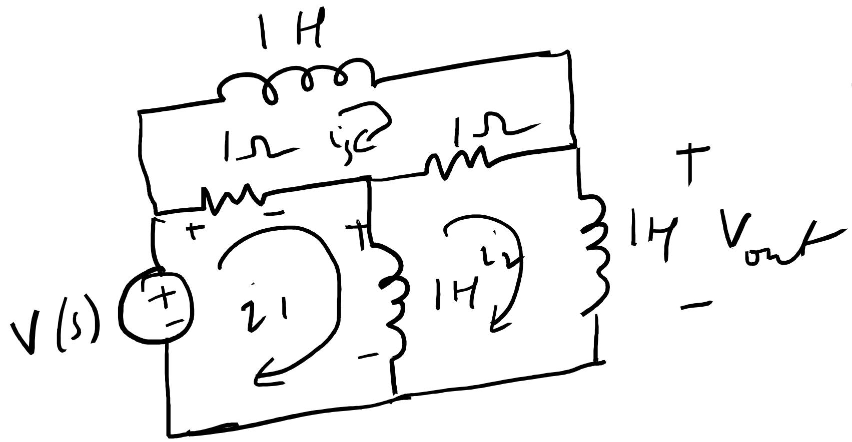
$$I(s) = \frac{1/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$i(t) = \mathcal{L}^{-1}\left[I(s) \right] = \mathcal{L}^{-1}\left[\frac{1/L}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \right]$$

$$i(t) = \mathcal{L}^{-1}\left[\frac{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}{\left(s + \frac{R}{2L}\right)^2 + \left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)^2} \right] \frac{1/L}{\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

$$= \frac{1}{\sqrt{\frac{L}{C} - R^2}} \cdot e^{-\frac{Rt}{2L}} \sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\, t \right)$$

study RL & RC ckts w.r.t. their response to a const. voltage.



obtain $\dfrac{V_{out}(s)}{V(s)}$

$$V(s) = [I_1(s) - I_2(s)]s + [I_1(s) - I_3(s)]$$
$$= (s+1)I_1 - sI_2 - I_3 \quad (i)$$

$1H \to s.$

$1\Omega \to 1$

$$0 = (I_2 - I_3) + s(I_2) + s[I_2 - I_1]$$

$$\left|\begin{matrix} s+1 & -s & -1 \\ -s & 2s+1 & -1 \\ -1 & -1 & s+2 \end{matrix}\right| \quad \begin{matrix} V(s) \\ 0 \\ 0 \end{matrix}$$

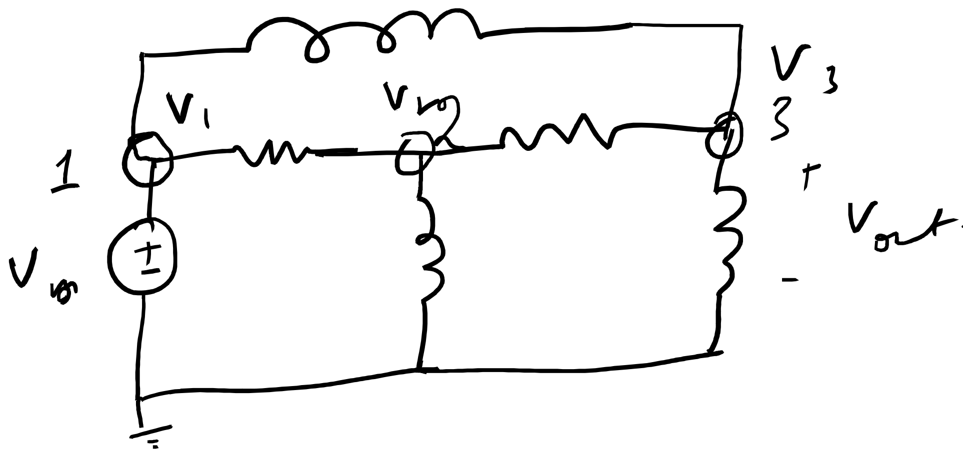
$$= s(s^2+5s+2) \quad \Delta$$

$$\Rightarrow -sI_1 + (2s+1)I_2 - I_3 = 0 \quad (ii)$$

$$0 = -I_1 - I_2 + (s+2)I_3 \quad (iii)$$

$V(s)(s^2+2s+1)$

$$\frac{I_2(s)}{V(s)} = \frac{s^2+2s+1}{s(s^2+5s+2)} \Rightarrow V_{out}/V_{in} = \frac{s^2+2s+1}{s^2+5s+2}.$$



$V_3 = V_{out}$

(Ex) Verify the tf using KCL

For application of KVL you can merge together elements in series.

For application of KCL you can merge together elements in parallel.

$$V_1 = V_{in} \longrightarrow (1)$$

$$\frac{(V_2 - V_1)}{1} + \frac{V_2 - V_3}{1} + \frac{V_2}{s} = 0 \quad -(2)$$

$$\frac{V_3}{s} + \frac{V_3 - V_2}{1} + \frac{V_3 - V_1}{s} = 0 \quad (3)$$

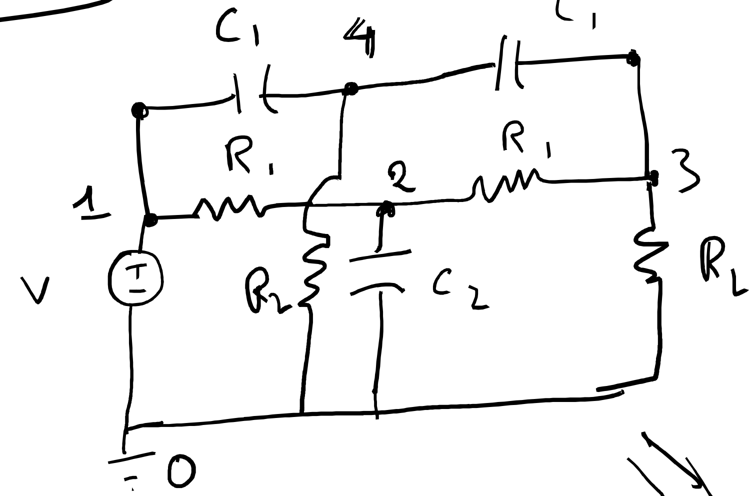
Ex

$$V_1 = V_{in}$$

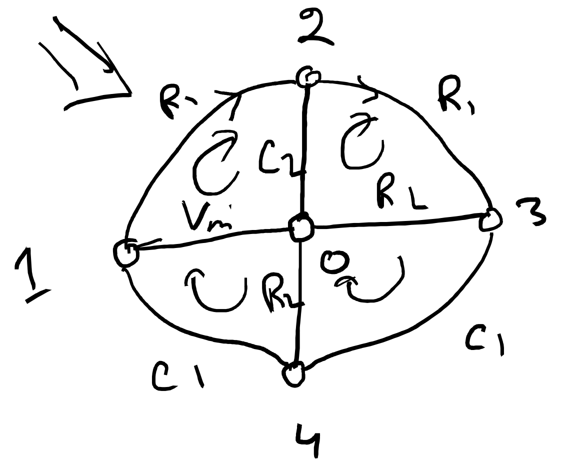
$$V_2\left(\frac{1}{1} + \frac{1}{1} + \frac{1}{s}\right) - V_1\left(\frac{1}{1}\right) - V_3\left(\frac{1}{1}\right) = 0$$

$$V_3\left(\frac{1}{s} + \frac{1}{s} + \frac{1}{1}\right) - \frac{1}{1}V_2 - \frac{1}{s}V_1 = 0$$

Twin T ckt

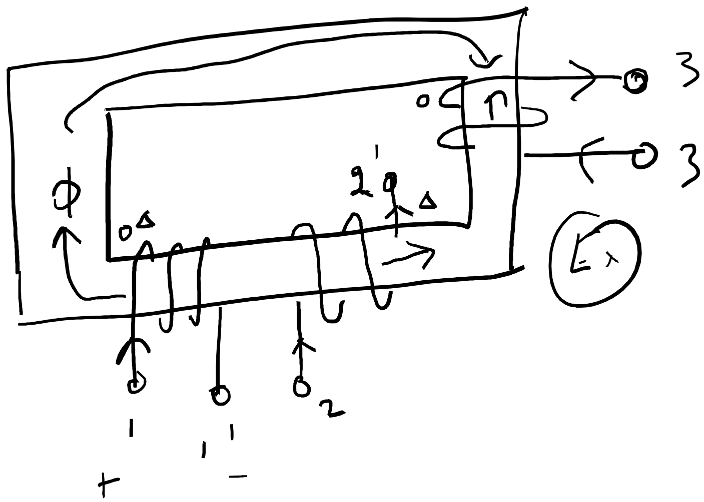


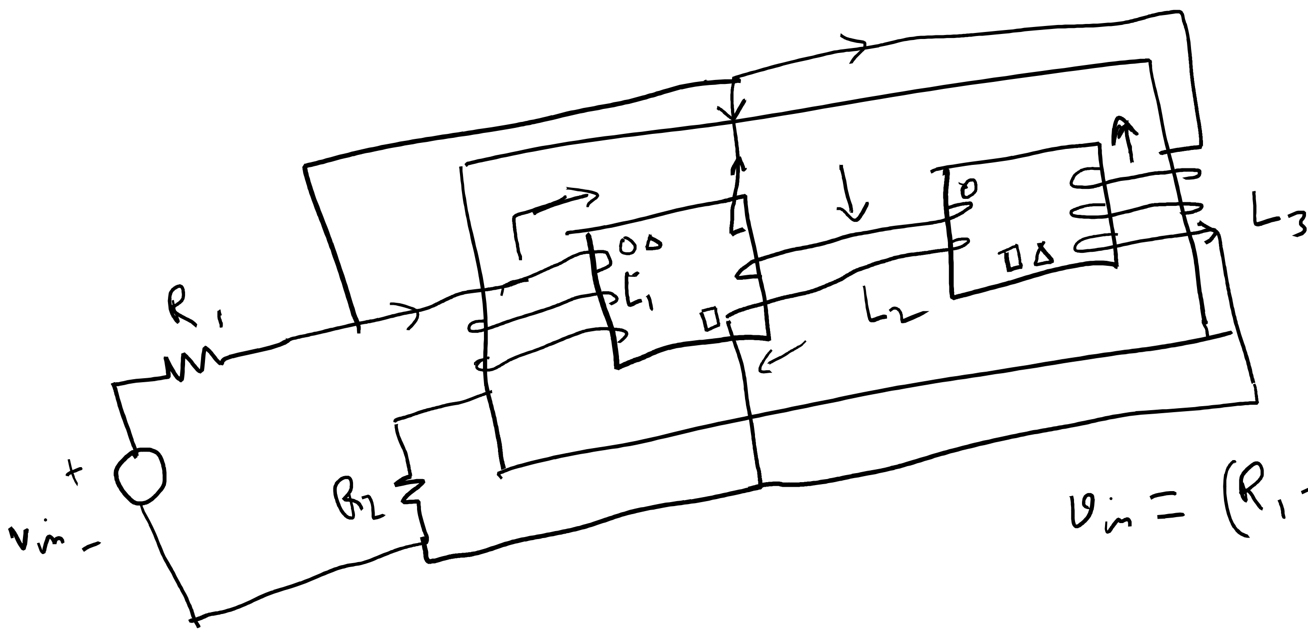
Graph theoretic method

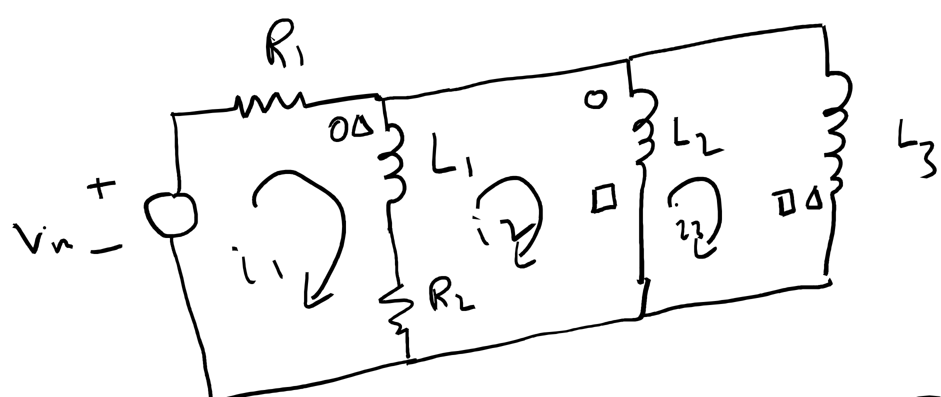


Module 3

- Ckts with Mutual Inductance
- OPAMPS & their transfer fns.







$$v_{in} = (R_1 + R_2 + sL_1) I_1 - (sL_1 + R_2) I_2 + sM_{12}(I_2 - I_3) - sM_{13} I_3 \quad \text{— (1)}$$

$$-sM_{12}(I_2 - I_3) - sM_{21}(I_2 - I_1) \quad \text{(2)}$$

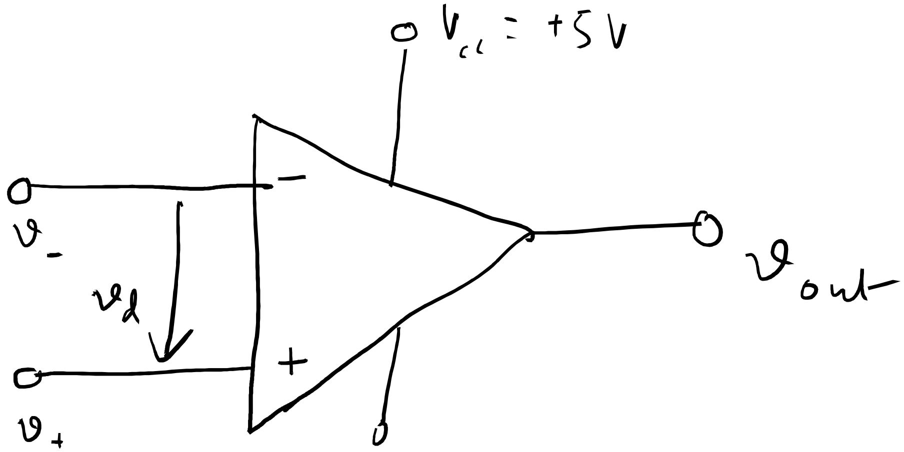
$$sM_{13} \quad sM_{23}$$

— (3)

(EX)

OP Amps

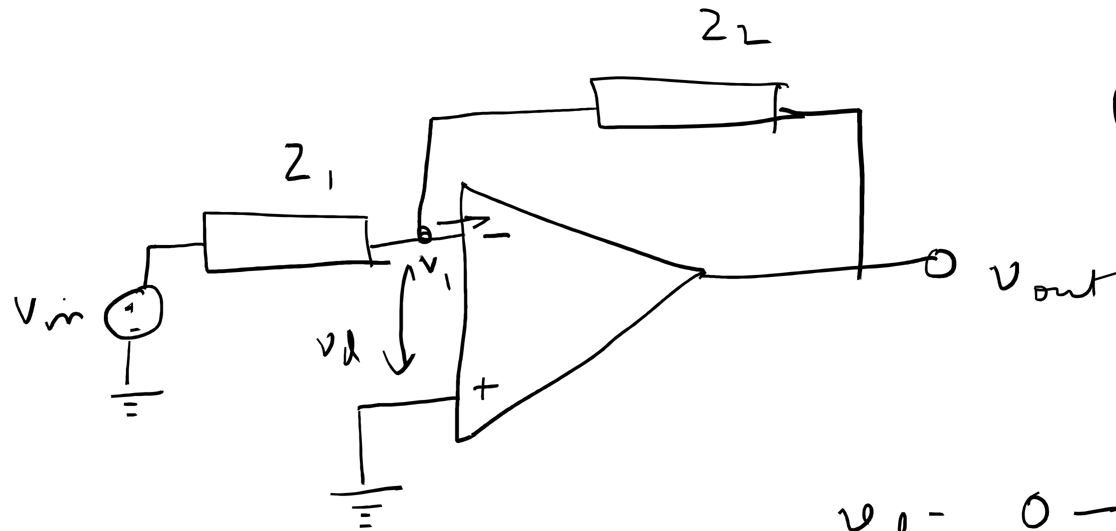
Differential Amplifiers



$$v_{out} = A\, v_d \quad (ii)$$

$$v_d = v_+ - v_- \quad (i)$$

- Infinite i/p impedance
- Zero o/p impedance.
- $A \rightarrow \infty$



$R_{in} \to \infty$

$v_d = 0 - v_1$ — (1)

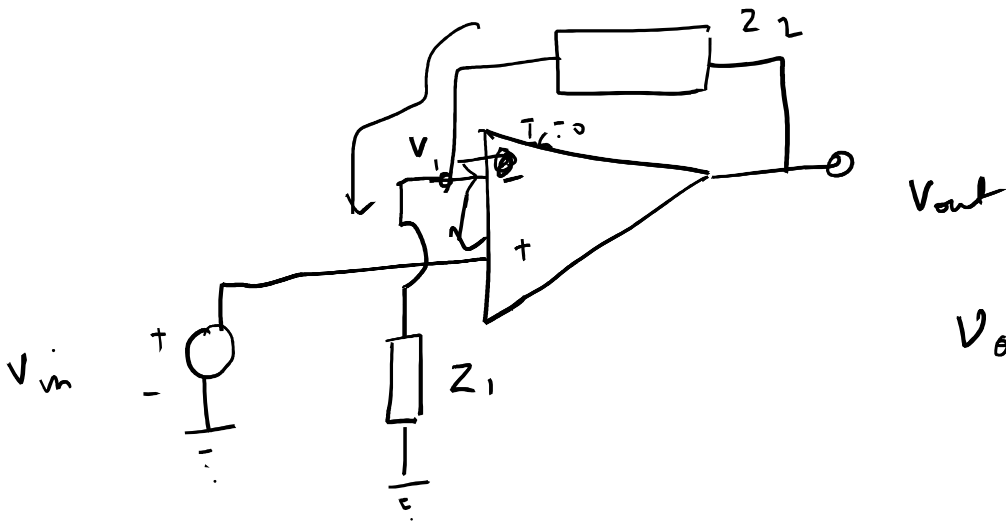
$v_{out} = A v_d$ —— (2)

$= -A v_1$

$v_1 = -\frac{v_{out}}{A} \to 0$ Virtual short.

$$\frac{v_{in} - 0}{Z_1} + \frac{v_{out} - 0}{Z_2} = 0$$

$$\frac{v_{out}}{v_{in}} = -\frac{Z_2}{Z_1}$$



$$v_{out} = A v_d$$
$$= A(v_{in} - v_1) \quad (1)$$

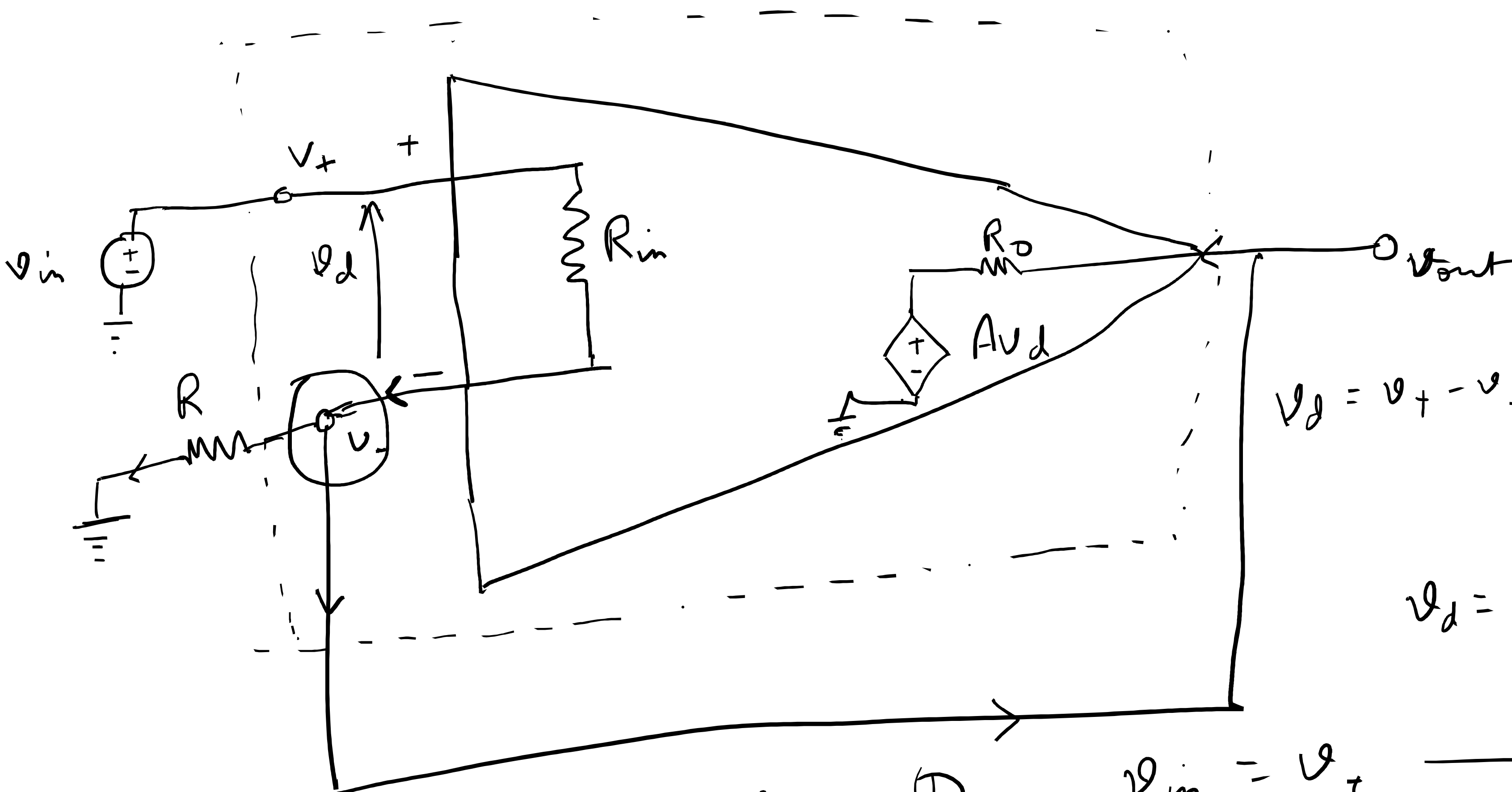
$$\frac{v_{out} Z_1}{Z_1 + Z_2} = V_1$$

$$v_{in} - v_1 = \frac{v_{out}}{A} \rightarrow 0$$

$$v_{in} = V_1$$

$$v_{out} = \left(\frac{Z_1 + Z_2}{Z_1}\right) V_1$$

$$= \left(\frac{Z_1 + Z_2}{Z_1}\right) V_{in} \Rightarrow \frac{V_{out}}{V_{in}} = \frac{Z_1 + Z_2}{Z_1} = 1 + \frac{Z_2}{Z_1}$$



$$v_d = v_+ - v_-$$

$$v_d = v_+ - v_-$$

$$v_{in} = v_+ \quad \text{—— } 2$$

$$v_{out} = v_- \quad \text{— } 1$$

$$\frac{v_{out} - A(v_+ - v_-)}{R_o}$$

$$\frac{v_+ - v_-}{R_{in}} = \frac{v_-}{R} + \frac{v_{out} - A(v_+ - v_-)}{R_o}$$

$$v_0\left[\frac{1}{R}+\frac{1}{R_0}+\frac{A}{R_0}+\frac{1}{R_{in}}\right]=v_{in}\left[\frac{1}{R_{in}}+\frac{A}{R_0}\right]$$

$$\frac{v_0}{v_{in}}=\frac{\frac{1}{R_{in}}+\frac{A}{R_0}}{\frac{1}{R}+\frac{1}{R_0}+\frac{A}{R_0}+\frac{1}{R_{in}}}$$

(i) $A\to\infty$

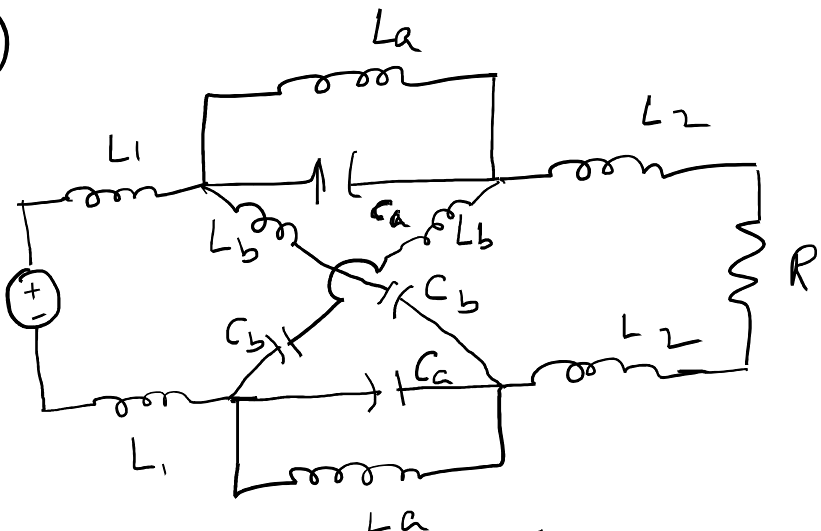
(ii) $R_0\to 0$

(iii) $R_{in}\to\infty$

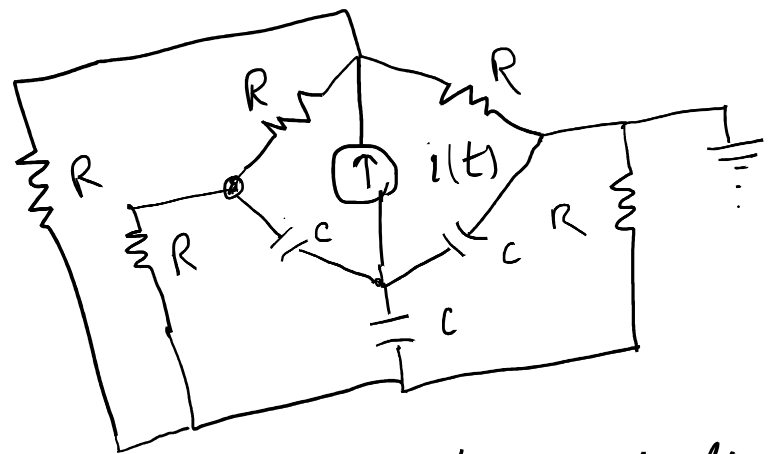
Ex

<u>Goal</u> $\frac{v_0}{v_{in}}=1$

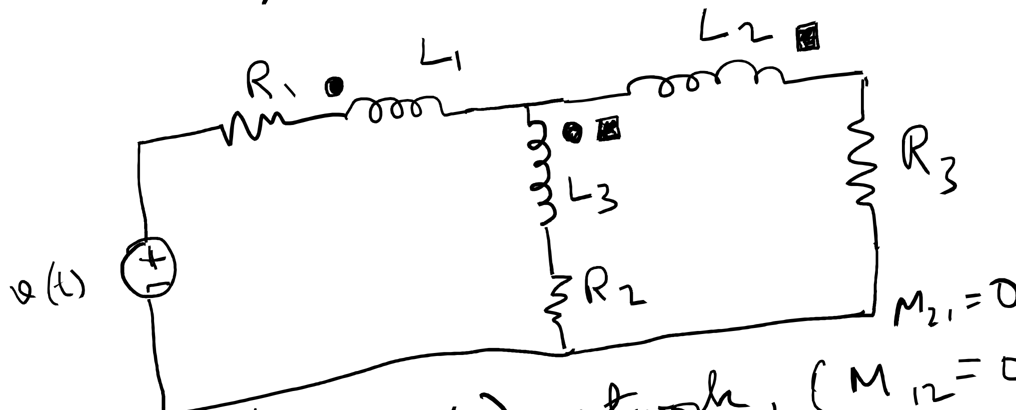
Ex



Apply KVL to write the eqns



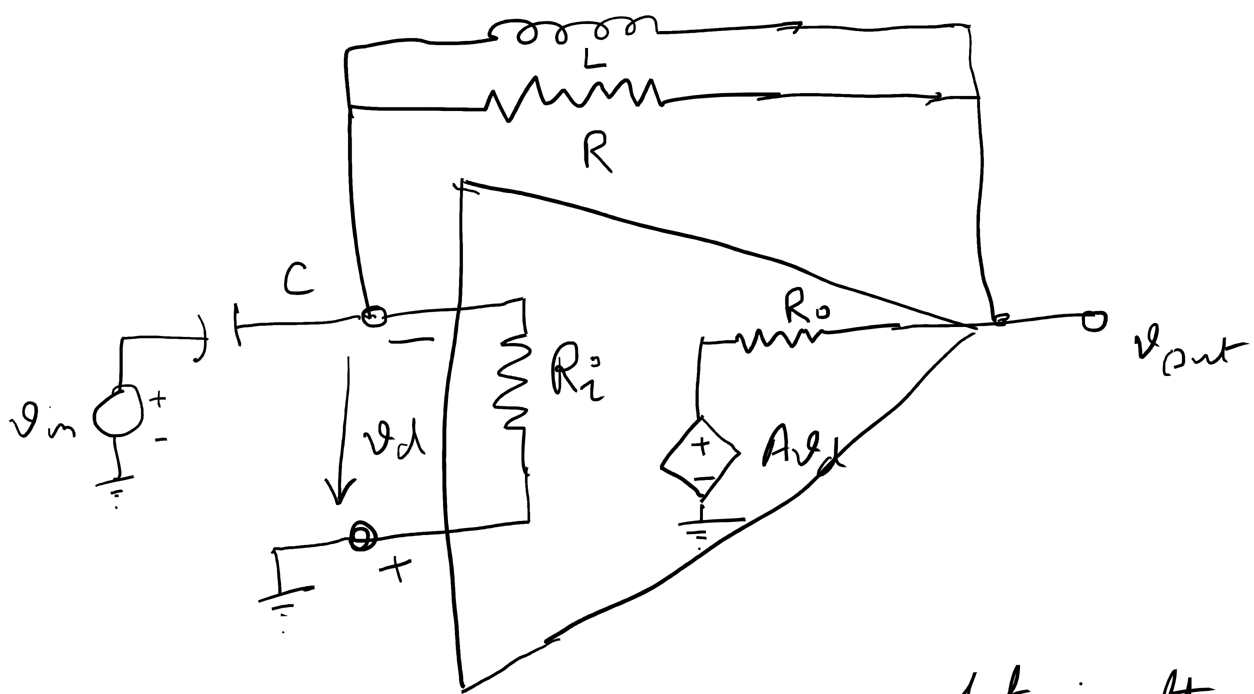
Apply KCL to write the eqns.



$M_{21}=0$

For the magnetically coupled network, ($M_{12}=0$), formulate loop eqns using KVL.

Ex contd



For the non-ideal OPAmp, obtain the t.f. $\dfrac{V_{out}(s)}{V_{in}(s)}$.

Test the effects of $A \to \infty$, $R_o \to 0$ & $R_i \to \infty$ individually, and together.