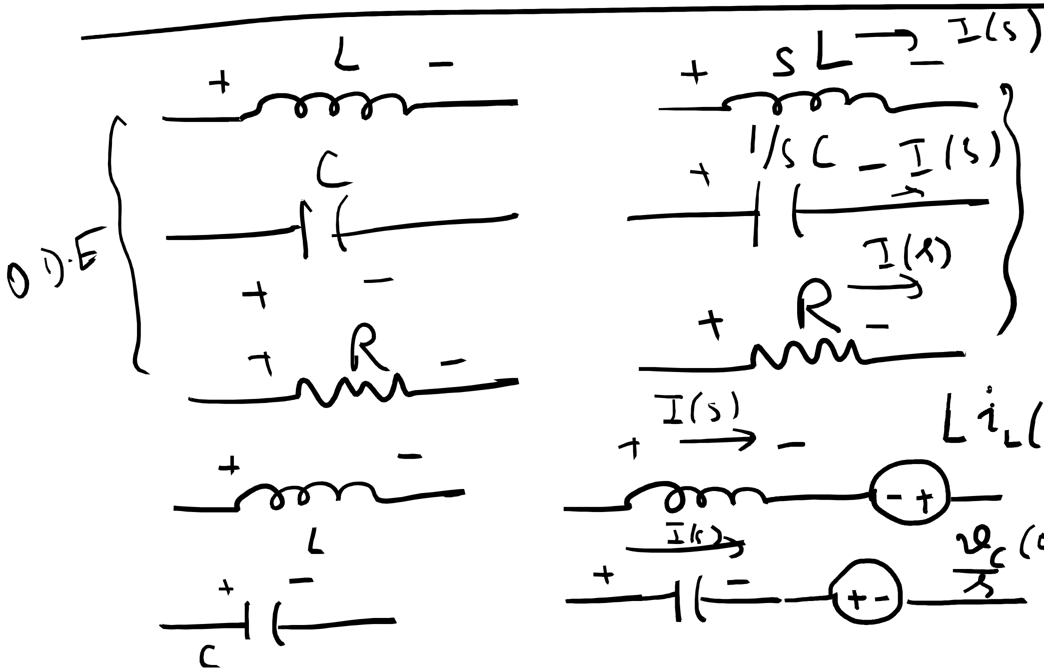
Lecture 3 Module 1

Modelling of mechanical systems
- translational dynamics
- rotational dynamics



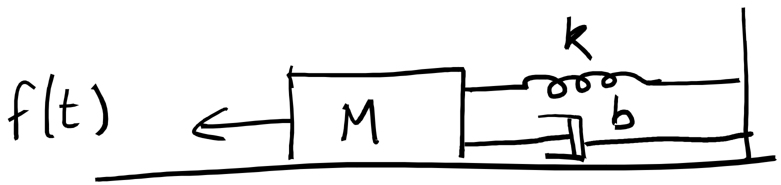
Algebraic eqns

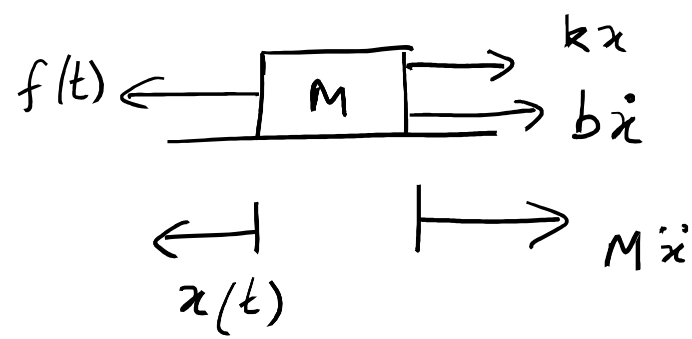
$$v_L(t) = L\frac{di_L}{dt}$$

$$\Rightarrow V_L(s) = sL\,I_L(s) - Li_L(0^-)$$

$$i_c = C\frac{dv_c}{dt}$$

$$\Rightarrow I_c(s) = CsV_c(s) - Cv_c(0^-)$$





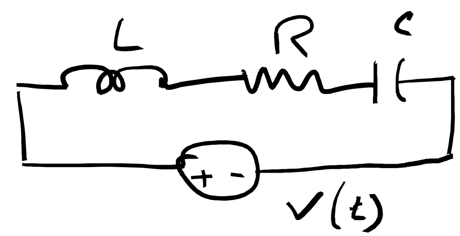
$$M\ddot{x} + b\dot{x} + kx = f(t)$$

$$\Rightarrow (Ms^2 + bs + k)X(s) = F(s).$$

$$\Downarrow$$

$$\Rightarrow \left(Ms + b + \frac{k}{s}\right) \underbrace{sX(s)}_{\mathcal{L}T.\text{ of velocity}} = F(s)$$

$$\left(Ls + R + \frac{1}{sC}\right) I(s) = V(s)$$

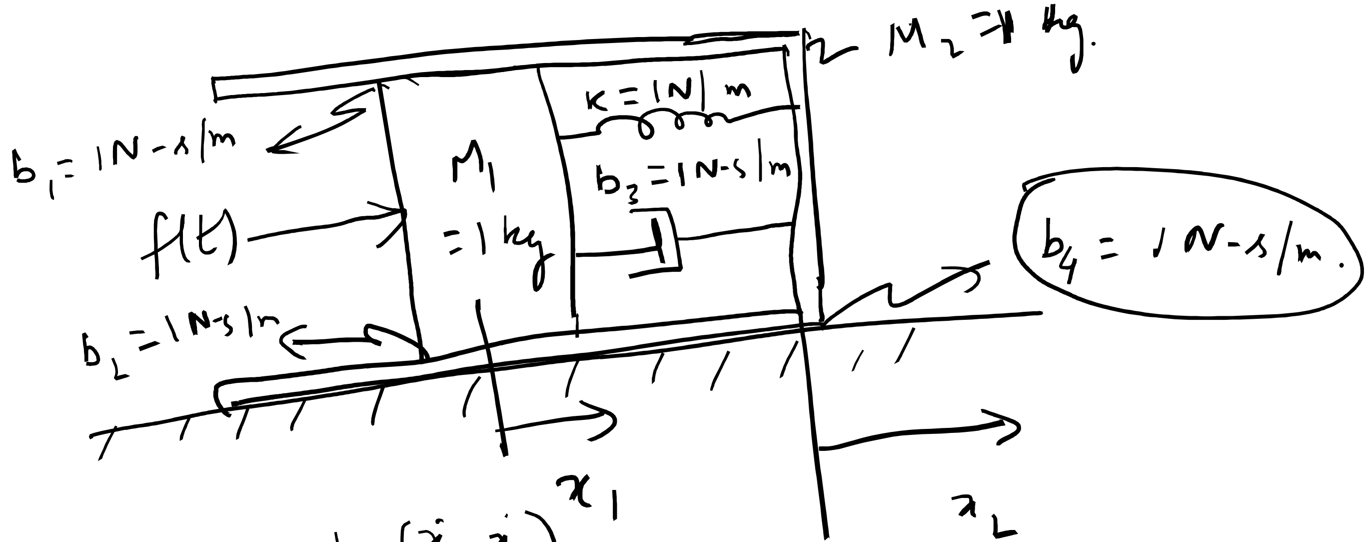


Series Analogy

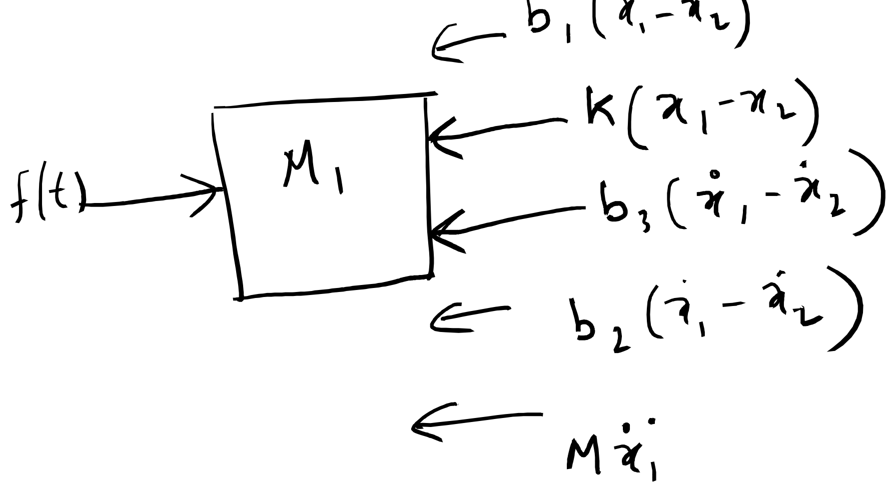
$K \rightarrow \frac{1}{C}$

$R \rightarrow b$

$L \rightarrow M.$



Goal: obtain $G(s) = \dfrac{X_2(s)}{F(s)}$



$$F(s) = s^2 X_1(s) + 3s X_1(s) + X_1(s) - 3s X_2(s) - X_2(s) \quad (i)$$

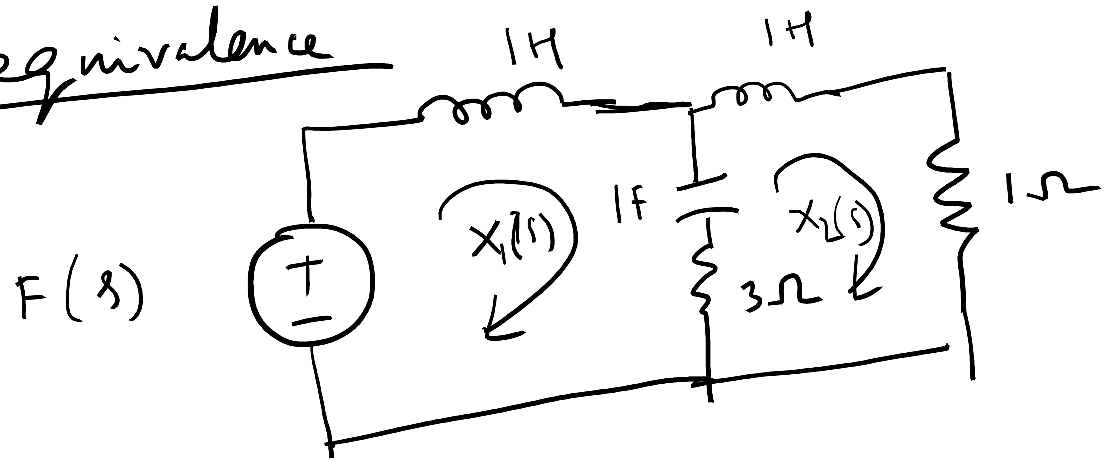
$$0 = -(3s+1) X_1(s) + (s^2 + 3s + 1) X_2(s) + s X_2(s) \quad (ii)$$

Ex/

$$\frac{3s+1}{s^4 + 7s^3 + 5s^2 + s} = G(s)$$

Equivalent ckt

Series equivalence

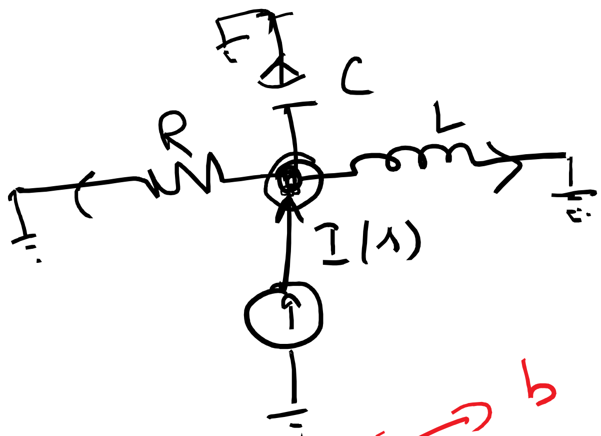


Parallel equivalence

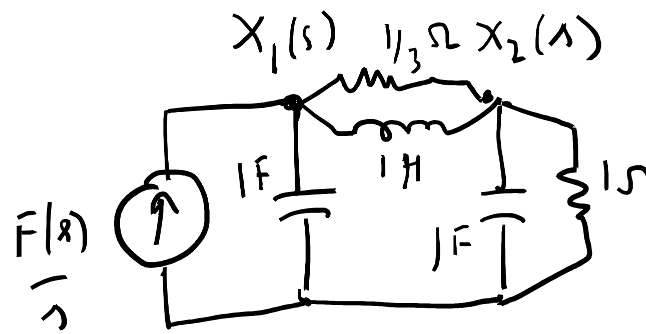
"through" variables : current, force, t…

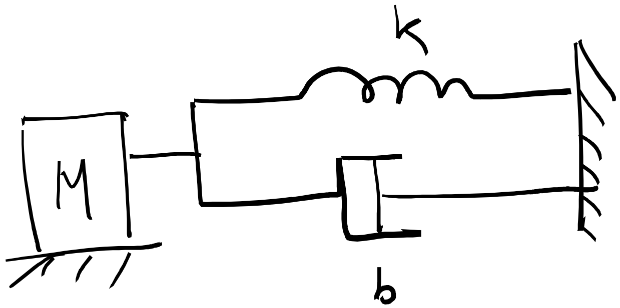
"across" variables : volt / displ / angular displ.

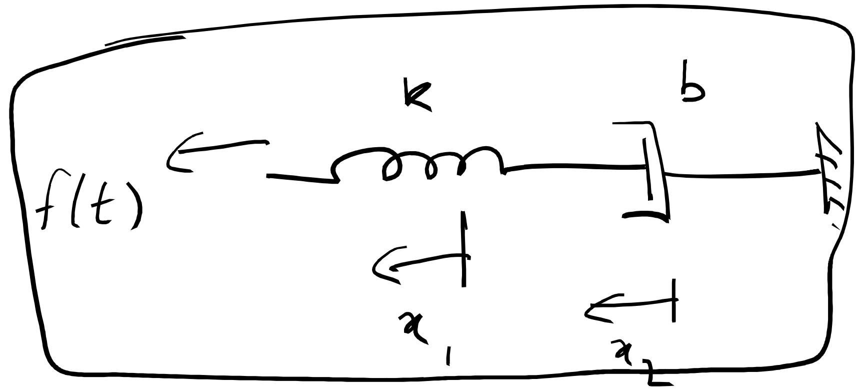
$$I(s) = V(s)\cdot\frac{1}{sL} + V(s)\,Cs + \frac{V(s)}{R}$$



$$I(s) = V(s)\left[\underbrace{\frac{1}{sL}}_{K} + \underbrace{Cs}_{M} + \overbrace{\frac{1}{R}}^{b}\right]$$







$$F(s) = k X_1(s)$$
$$= s b X_2(s)$$

$$f(t) = k x_1 \quad \text{—(i)}$$
$$f(t) = b \dot{x}_2 \quad \text{(ii)}$$

Tot disp

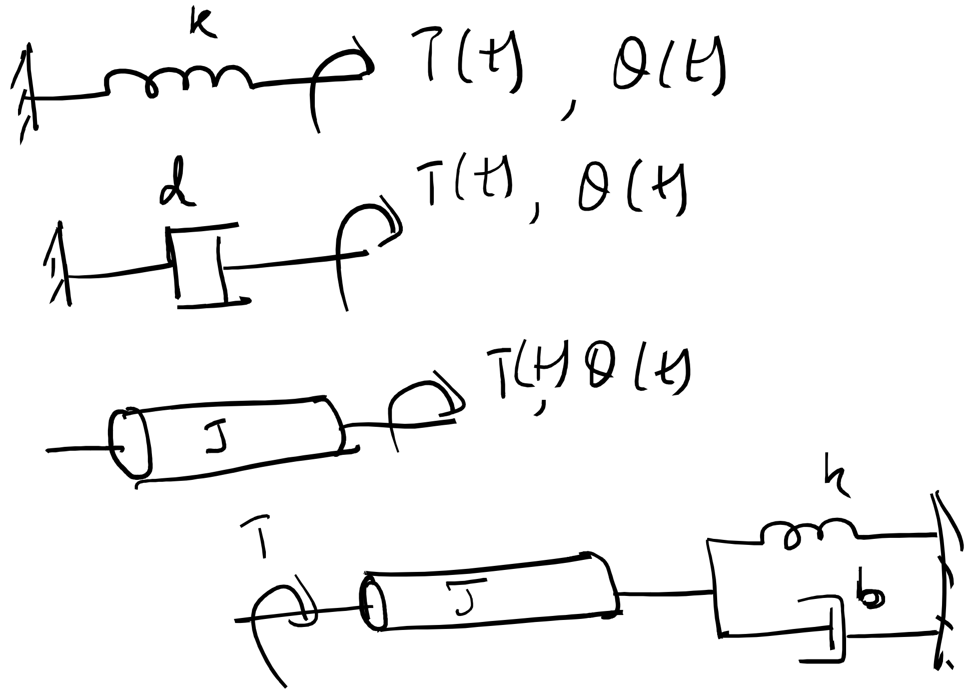
$$X(s) = x_1(s) + x_2(s) \Rightarrow \frac{F(s)}{k} + \frac{F(s)}{bs}$$

$$\Rightarrow \frac{X(s)}{F(s)} = \frac{1}{k} + \frac{1}{bs}$$

$$T(t) = k\theta(t)$$

$$T(t) = d\dot{\theta}(t)$$

$$T(t) = J\ddot{\theta}(t)$$

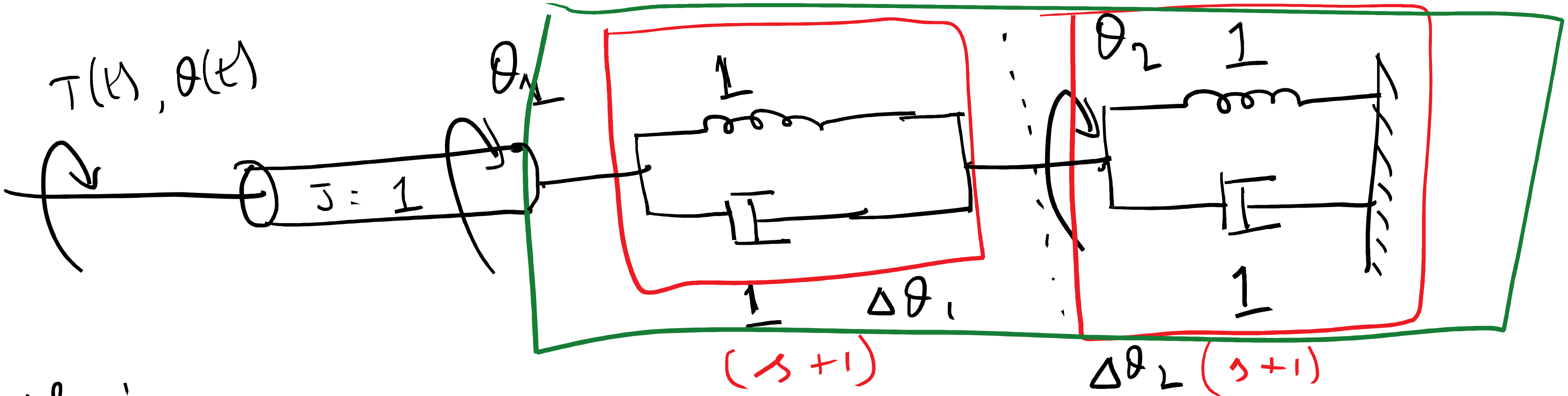


$$T(t) = J\ddot{\theta} + d\dot{\theta} + k\theta$$

$$\Rightarrow T(s) = (Js^2 + ds + k)\,\theta$$

$L \quad R \quad \frac{1}{C} \longrightarrow$ Series Analog.

$C \quad \frac{1}{R} \quad \frac{1}{L} \longrightarrow$ Parallel analog.



obtain

$\theta_2(s) / T(s)$

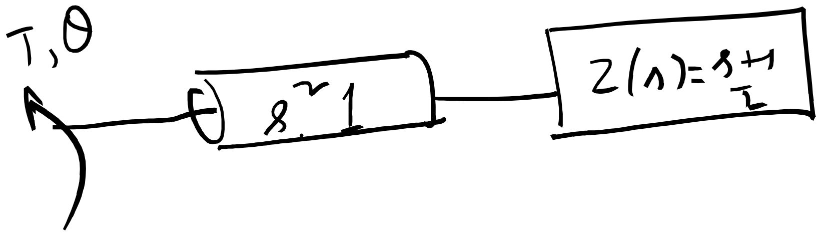
$$T = (s^2 + s + 1)\,\theta_1(s) + (s+1)\,\theta_2(s) \qquad (1)$$

$$0 = (\theta_2 - \theta_1) + (\dot\theta_2 - \dot\theta_1) + \theta_2 + \dot\theta_2$$

$$\Rightarrow\ 0 = -(s+1)\,\theta_1(s) + 2(s+1)\,\theta_2(s) \qquad (2)$$

$$\boxed{G(s) = \frac{1}{2s^2 + s + 1}}$$

$$\boxed{\frac{(s+1)(s+1)}{(s+1) + (s+1)}} = \frac{s+1}{2}$$



$$T(s) = \left(s^2 + \frac{s+1}{2}\right)\theta(s)$$

$$\Rightarrow \frac{\theta(s)}{T(s)} = \frac{2}{2s^2+s+1}$$

$$\frac{T}{s+1} = \Delta\theta_1$$

$$\frac{T}{s+1} = \Delta\theta_2$$

$$\Rightarrow \Delta\theta_1 = \Delta\theta_2$$

Now,

$$\theta = \Delta\theta_1 + \Delta\theta_2$$

$$= 2\Delta\theta_2 = \theta_1$$

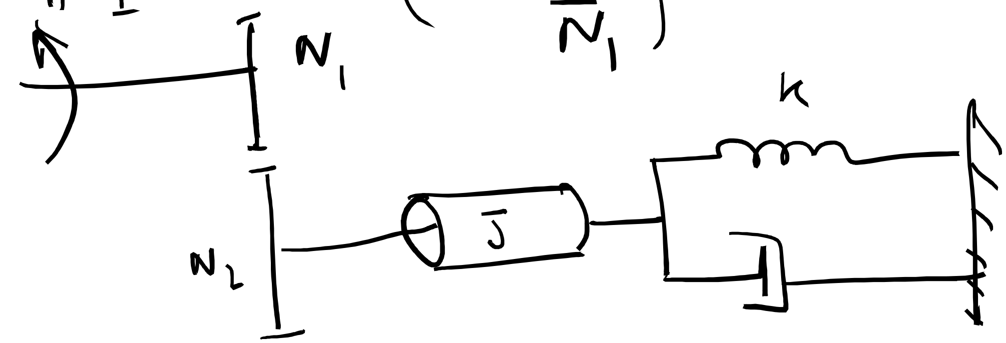
$$\Delta\theta_2 = \theta_2$$

$$\Delta\theta_1 = \theta_1 - \theta_2$$

$$\boxed{\frac{\theta_2}{T} = \frac{1}{2s^2+s+1}}$$

Gears

$$\left(T_1 \frac{N_2}{N_1}\right)$$



$$\frac{r_1}{r_2} = \frac{N_1}{N_2} \quad \text{— (1)}$$

$$r_1\theta_1 = r_2\theta_2 \quad \text{— (2)}$$

$$N_1\theta_1 = N_2\theta_2 \quad \text{— (3)}$$

$$T_1\theta_1 = T_2\theta_2 \quad \text{— (4)}$$

Combining (3) & (4) $\rightarrow$

$$\boxed{\frac{\theta_1}{\theta_2} = \frac{N_2}{N_1} = \frac{T_2}{T_1}}$$

$$T_1 \left(\frac{N_2}{N_1}\right) = (Js^2 + ds + k)\,\theta_2 = (Js^2 + ds + k)\,\theta_1 \left(\frac{N_1}{N_2}\right)$$

$$\Rightarrow \quad \frac{\theta_1}{T_1} = \frac{N_2/N_1}{\left(\frac{N_1}{N_2}\right)(Js^2 + ds + k)} = \frac{1}{J_{eq}s^2 + d_{eq}s + k_{eq}}$$

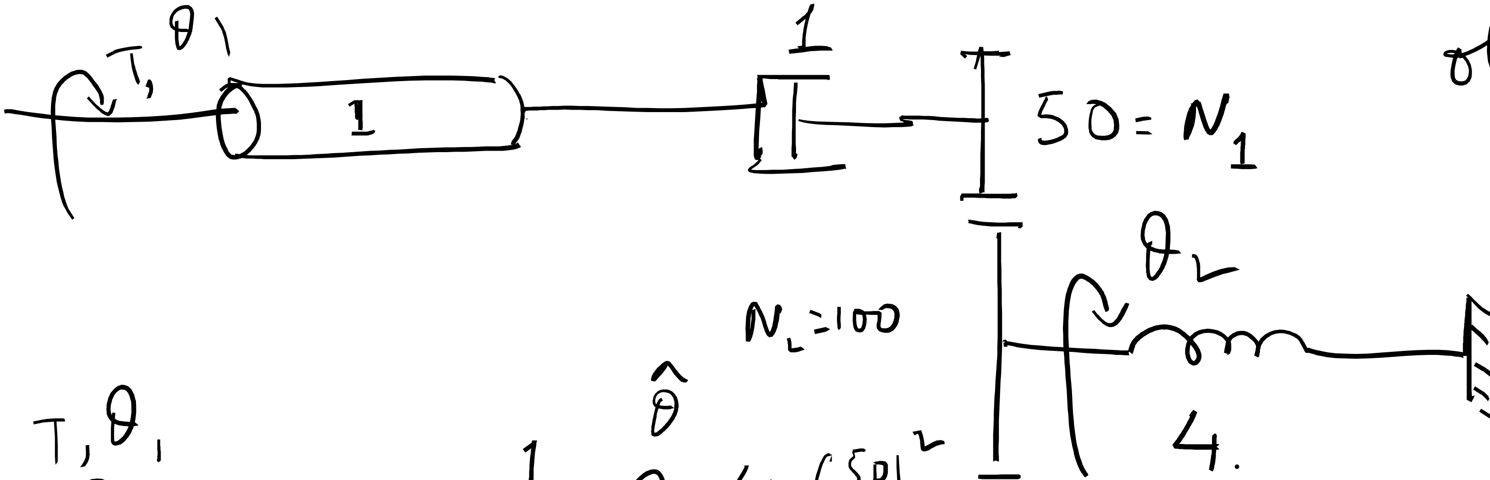
$$J_{eq} = J \cdot \left(\frac{N_1}{N_2}\right)^2$$

$$k_{eq} = k \cdot \left(\frac{N_1}{N_2}\right)^2$$

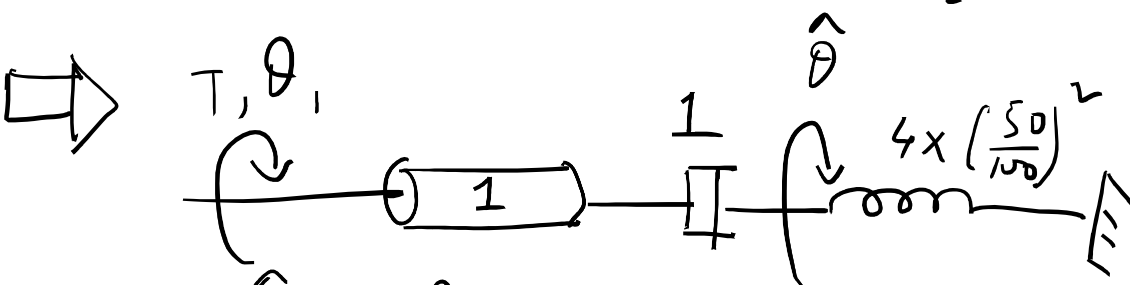
$$d_{eq} = d \left(\frac{N_1}{N_2}\right)^2$$

Ex

obtain θ_2/T



⇒



Note: $\hat{\theta} = 2\theta_2$

$\Rightarrow T = s^2\theta_1 + s(\theta_1 - \hat{\theta})$ (i)

& $0 = s(\hat{\theta} - \theta_1) + \hat{\theta}$ (ii)

$\Rightarrow \theta_1 = \frac{(s+1)}{s}\hat{\theta}$

$\Rightarrow T = s(s+1)\hat{\theta} + (s+1)\hat{\theta} - s\hat{\theta}$

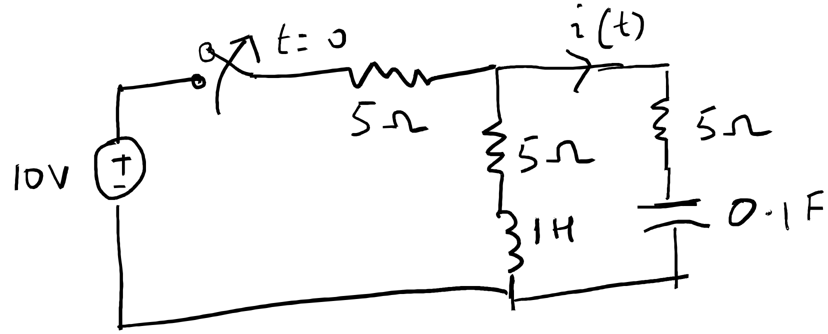
$= (s^2 + s + 1)\hat{\theta}$

$\Rightarrow \hat{\theta}/T = \frac{1}{s^2+s+1}$

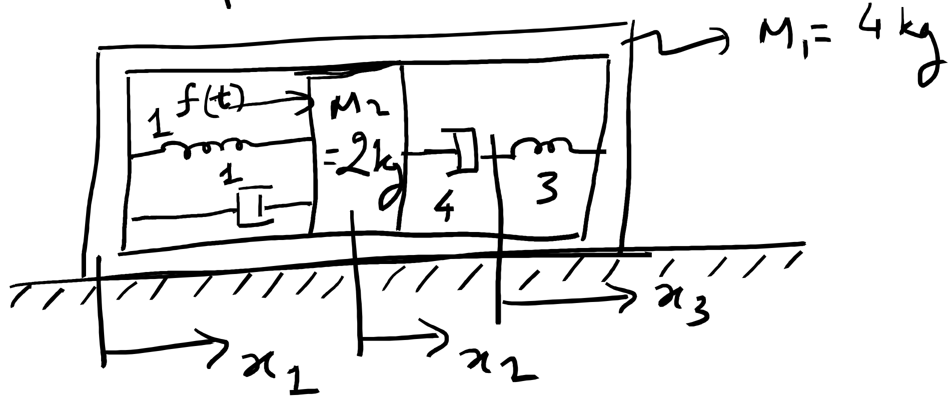
$\frac{\theta_2}{T} = \frac{1/2}{s^2+s+1}$

Exercises

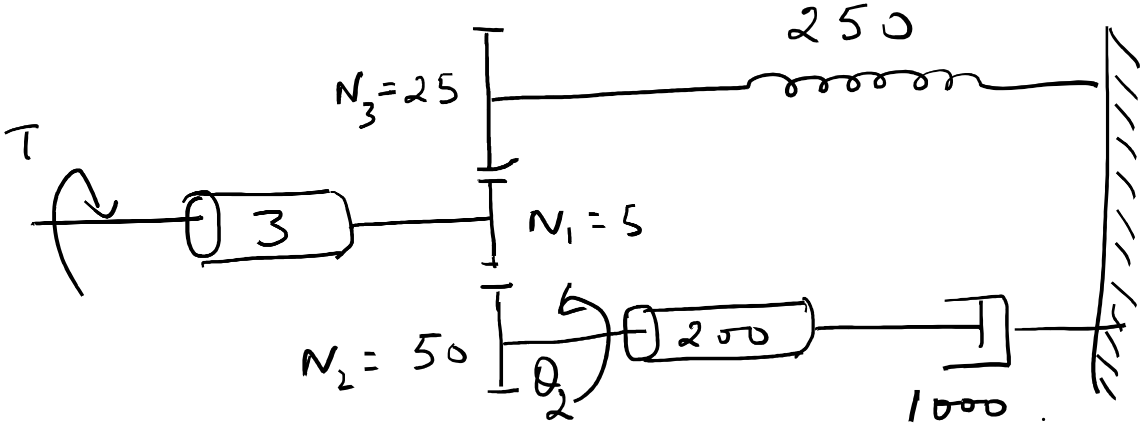
- Solve for $i(t)$, $t > 0$, using Laplace transform.



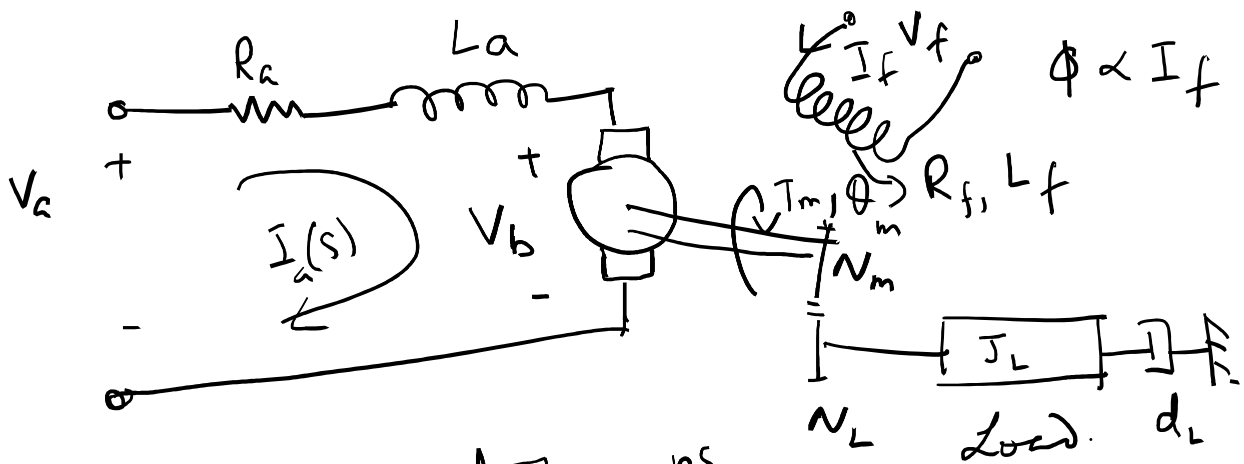
- Obtain $X_3(s)/F(s)$ for the following system. (units are consistent)
Assume all surfaces to be frictionless.



- Obtain $\dfrac{\theta_2(s)}{T(s)}$ for the following system. (All units are consistent)



Module 2 : Modelling electromechanical systems



$\phi \propto I_f$

Armature Control

Obtain $\dfrac{\theta_m(s)}{V_a(s)}$

$$V_b = \frac{p\phi \boxed{Z} n}{a} \quad n \to rps$$

lap/wave

$P/2$ $\quad$ 2

Elect.

$$V_b(s) = K_b \cdot s\,\theta_m(s) \quad — (1)$$

$v_b = K_b \cdot \omega_m$, $\omega_m = \dfrac{d\theta_m}{dt}$.

$$V_a(s) - V_b(s) = (R_a + sL_a)\,I_a(s) \quad — (2)$$

Mech.

$$T_m(s) = \left(\Big[\underbrace{J_m + J_L\left(\frac{N_m}{N_L}\right)^2}_{J_e}\Big] s^2 + \Big[\underbrace{d_m + d_L\left(\frac{N_m}{N_L}\right)^2}_{d_e}\Big] s\right)\theta_m(s)$$

$T \propto \phi,$ $\boxed{T = k_m I_c(s)}$ Electromechanical part.

$T \propto I_c$

- Obtain $$\frac{\theta_m(s)}{T(s)} = \frac{1}{J_e s^2 + d_e s} \longrightarrow \text{I}$$

- $$V_c(s) = k_b s \theta_m(s) + (R_c + sL_c) I_c(s).$$
$$= k_b s \theta_m(s) + (R_c + sL_c) \frac{T(s)}{k_m}$$
$$= \left[k_b \cancel{s} \frac{1}{J_e s^{\cancel{2}} + d_e \cancel{s}} + \frac{(R_c + sL_c)}{k_m} \right] T(s)$$

$$\Rightarrow \frac{T(s)}{V_c(s)} = \frac{1}{\frac{R_c + sL_c}{k_m} + \frac{k_b}{J_e s + d_e}} \longrightarrow \text{II}$$

Ex

$$\frac{\theta_m(s)}{V_a(s)} = \frac{k_m}{s\left[k_b k_m + (R_a + sL_a)(J_e s + d_e)\right]}$$

$L_a << R_a$

$$= \frac{k_m}{s\left[k_b k_m + R_a\left(1 + \frac{sL_a}{R_a}\right)\left[J_e s + d_e\right]\right]}$$

$$= \frac{k_m}{s\left[J_e R_a s + R_a d_e + k_b k_m\right]}$$

$$= \frac{k_m / J_e R_a}{s\left[s + \underbrace{\frac{d_e}{J_e} + \frac{k_b k_m}{J_e R_a}}_{\alpha}\right]} = \boxed{\frac{k}{s(s+\alpha)}}$$

$\dfrac{k_m^{*}}{J_e R_a^{*}}$, k_b

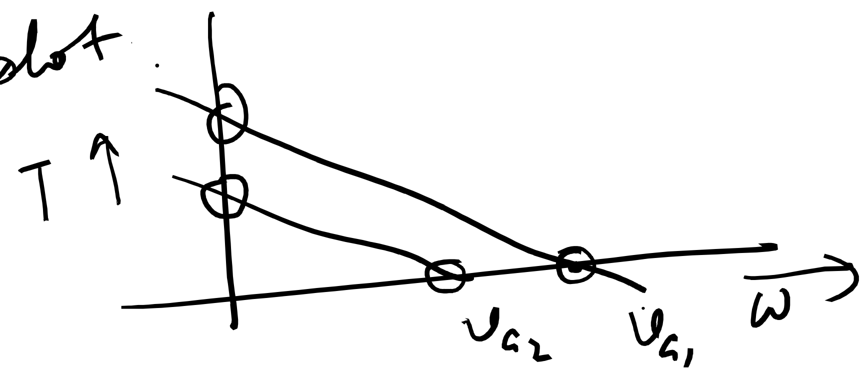
$$\Omega(s) = s\,\theta(s)$$

$$V_a(s) = k_b\,\Omega(s) + \left(R_a + s\cancel{L_a}\right)\frac{T(s)}{k_m}$$

$$V_a(s) = k_b\,\Omega(s) + \frac{R_a}{k_m}\,T(s)$$

$$\boxed{v_a(t) = k_b\,\omega(t) + \frac{R}{k_m}\,T(t)}$$

For a given $v_a(t)$, we have a straight line plotted in the T-ω plot.



$$\omega_{no\text{-}load} = \frac{v_a^{*}}{k_b}$$

$$T_{stall} = e_a \cdot \frac{k_m}{R_a}$$

ex Derive the t.f. betn $\theta_m(s)$ and $V_f(s)$ (field control)

Module 3 : Transient response

$$G(s) = \frac{s^3+s^2+3s+4}{s^4+5s^3+7s^2+8s+5}$$

$u = e^{\alpha t}$

$U(s) = \frac{1}{s-\alpha}$

$Y(s) = G(s)\boxed{U(s)}$

$y(t) = \mathcal{L}^{-1}[G(s)U(s)]$

$$y^{(4)} + 5y^{(3)} + 7y'' + 8y' + 5y = u^{(3)} + u'' + 3u' + 4u.$$

$$y = y_f + y_n$$

$+\boxed{e^{\alpha t}}$

$\boxed{\sum c_i e^{p_i t}}$ $i = 1, 2, 3, 4.$

$s^4+5s^3+7s^2+8s+5=0$

p_1, p_2, p_3, p_4

$$Y(s) = \sum_{r_i=1}^{d_i} \frac{a_i}{(s+p_i)^{r_i}}$$

$$\frac{a}{s+a} \qquad \frac{as+b}{cs^2+ds+b}$$

<u>First order systems</u>

$$G(s) = \frac{a}{s+a}$$

$$Y(s) = \frac{1}{s} \cdot \frac{a}{s+a}$$

$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{a}{s+a} = 1 = \lim_{t\to\infty} y(t)$$

<u>d.c gain</u> $= 1$

Ex $y(t) = [1 - e^{-at}]\, \underbrace{1(t)}$

- time constant : the time at which the o/p reaches 63% of its s.s. value

$$1 - e^{-a\frac{1}{a}} = 1 - e^{-1} = 0.63$$

$$\frac{1}{a} = \tau$$

$$\frac{a}{s+a} = \frac{1}{1 + s\tau_{\to\, tc}}$$

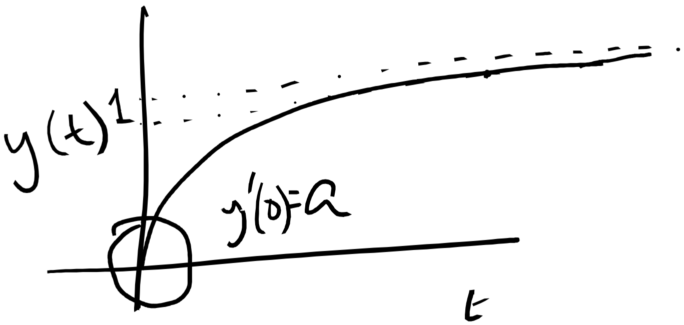
- rise time : time taken by y to go from 10% to 90% of its s.s. value.

$$1 - e^{-at_1} = 0.1 \Rightarrow e^{-at_1} = 0.9 \Rightarrow$$
$$1 - e^{-at_2} = 0.9 \Rightarrow e^{-at_2} = 0.1 \Rightarrow$$

$$t_2 - t_1 = \frac{1}{a}\ln 9 \approx \frac{2.2}{a}$$

Ex

__settling time__ : time taken by $y(t)$ to reach within 2% of its s.s value.



$$1 - e^{-a t_s} = 0.98$$

$$t_s = \frac{1}{a}\boxed{\ln 50} = \frac{3.9}{a} \approx \frac{4}{a}$$

Step response of second order systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

damping ratio

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$$

• $\zeta > 1 \longrightarrow$ Overdamped.

$$G(s) = \frac{p_1 p_2}{(s+p_1)(s+p_2)}$$

p_1, p_2 are distinct

$$Y(s) = \frac{1}{s} \cdot G(s)$$

Ex

A $y(t) = \left[1 - \frac{1}{p_1 - p_2}\left[p_1 e^{p_2 t} - p_2 e^{p_1 t}\right]\right]$

$p_1, p_2 < 0.$

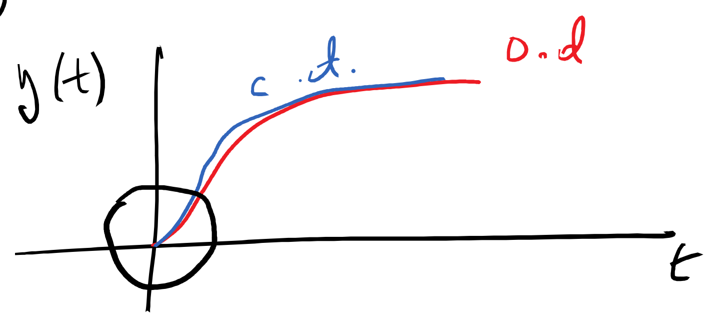
- $\zeta = 1 \longrightarrow$ critically damped

$$G(s) = \frac{p^2}{(s+p)^2} \qquad p \in \mathbb{R}_-$$

$$Y(s) = \frac{1}{s} \frac{p^2}{(s+p)^2}$$

Ex

$$y(t) = 1 - e^{-pt} - pte^{-pt}$$



$$y'(0) = 0$$

- Suppose you look at the step response for a 1st order system & a second order system. Is there a way to distinguish between the two if ζ (for 2nd order system) is ≥ 1?

Ans: Look for $y'(0)$. EX

Hint: Apply IVT to $y'(t)$

- Undamped system : $\zeta = 0$

$$G(s) = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$U(s) = 1/s$

Ex $y(t) = 1 - \cos\omega_n t.$

- Underdamped system $(0 < \zeta < 1)$

$\omega_n \to$ actual freq. $\zeta \to$ damping ratio.

$U(s) = 1/s$

$$Y(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Ex $\quad A = 1 \;;\; B = -1 \;;\; C = -2\zeta\omega_n$

$$Y(s) = \frac{1}{s} - \frac{(s + \zeta\omega_n)}{(s+\zeta\omega_n)^2 + (\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d})^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + (\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d})^2}$$

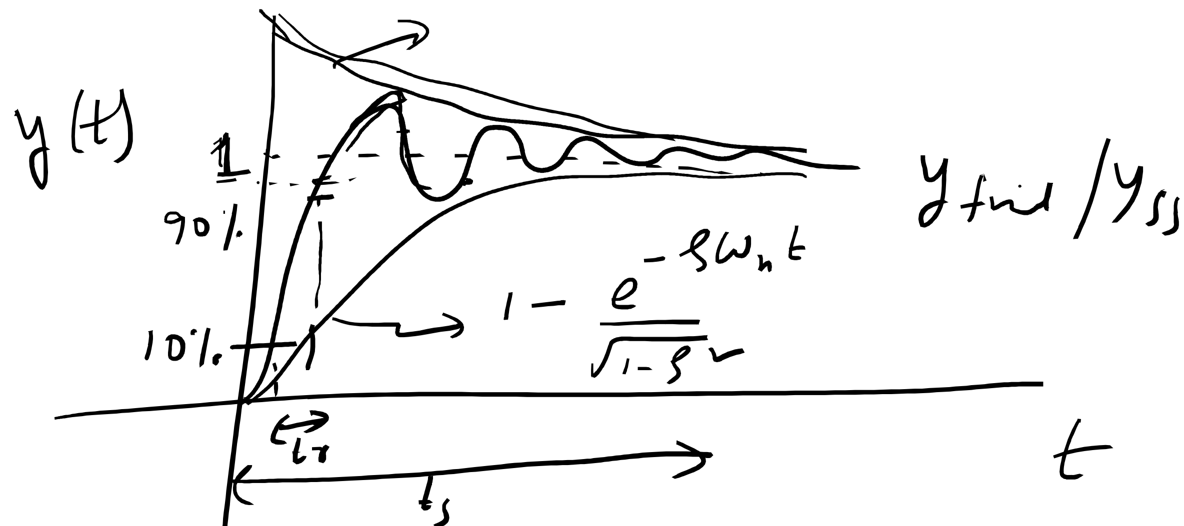
$$y(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}\mathcal{L}^{-1}\left[\frac{\overbrace{\omega_n\sqrt{1-\zeta^2}}^{\omega_d}}{(s+\zeta\omega_n)^2 + \omega_d^2}\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right]$$

suppose $\tan\phi = \frac{\zeta}{\sqrt{1-\zeta^2}}$.

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cdot \cos(\omega_d t - \phi)$$

- t_s
- t_r
- % O.S.
- t_p



- it enters the 2% band of s.s value and "continues to remain within the band thereafter" $\forall t > t_s$
- $\frac{y_{max} - y_{ss}}{y_{ss}} \times 100 = \%\ OS$
- time at which y_{max} is attained by $y(t)$.

- $t_r \approx \dfrac{1.8}{\omega_n}$
- $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{\omega_d}$
- $\% OS$

$$y(t_p) = 1 - \frac{e^{-\frac{\zeta\omega_n \pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\cos\left(\omega_n\sqrt{1-\zeta^2}\,\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} - \phi\right)$$

$$= 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\% OS = \frac{y(t_p) - \overset{=1}{y_{ss}}}{y_{ss}} = \boxed{e^{-\pi\zeta/\sqrt{1-\zeta^2}}}$$

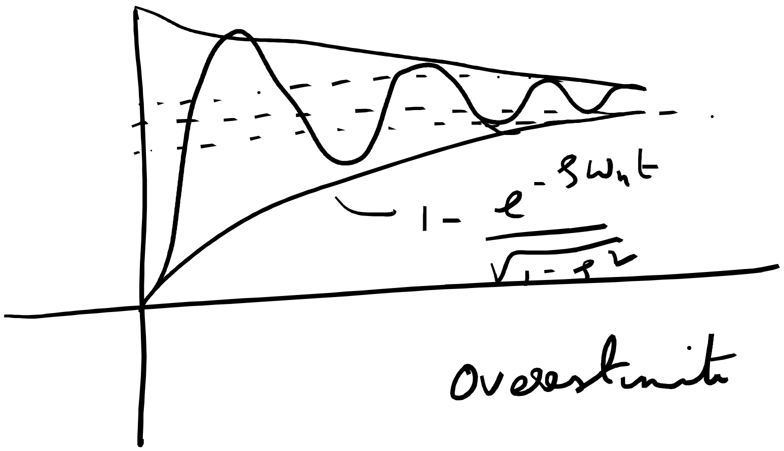
$$y'(t) = \mathcal{L}^{-1}\left[\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\mathcal{L}^{-1}\left[\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2}\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\sin\omega_d t = 0$$

$$\Rightarrow \omega_d t = n\pi \qquad n = 0, 1, 2, \ldots$$

$n = 1 \rightarrow$ maxima.



$$1 - \frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 0.98$$

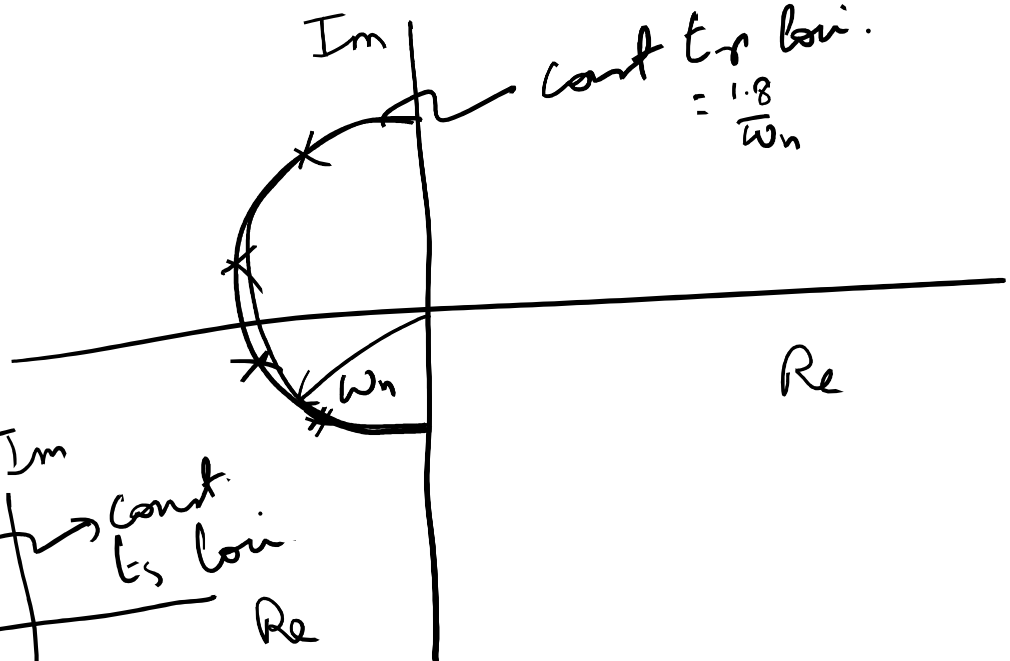
$$\Rightarrow t_s = -\frac{1}{\zeta\omega_n} \boxed{\ln\left(0.02\sqrt{1-\zeta^2}\right)}$$

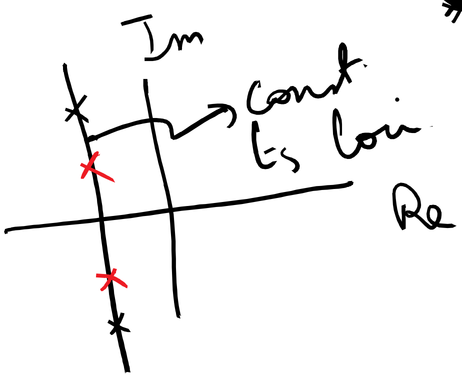
$0 < \zeta < 0.9$

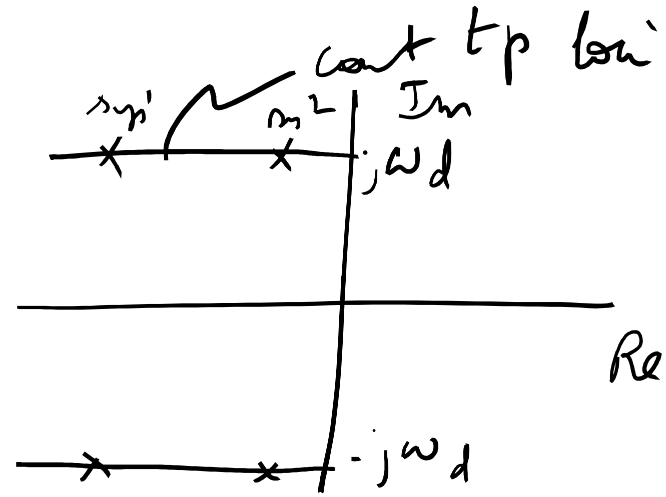
$3.91 - 4.7$

$$\approx \frac{4}{\zeta\omega_n}$$

$$t_s = \frac{4}{\zeta\omega_n}$$

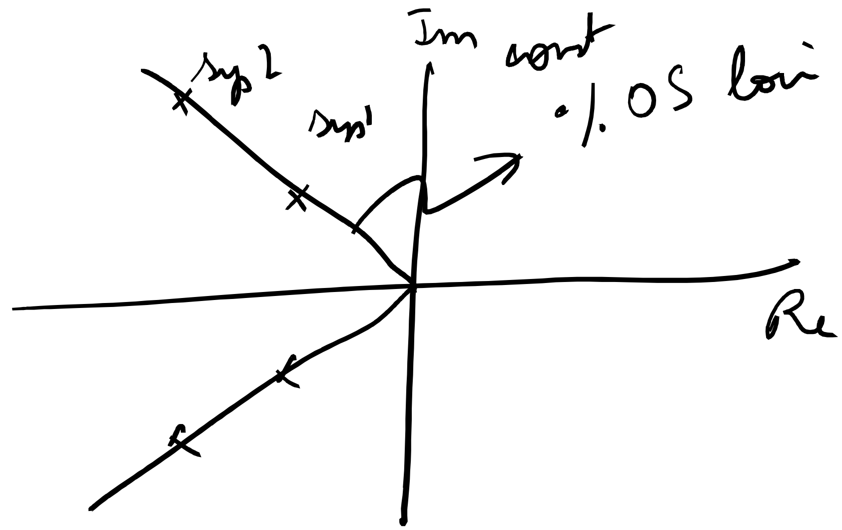


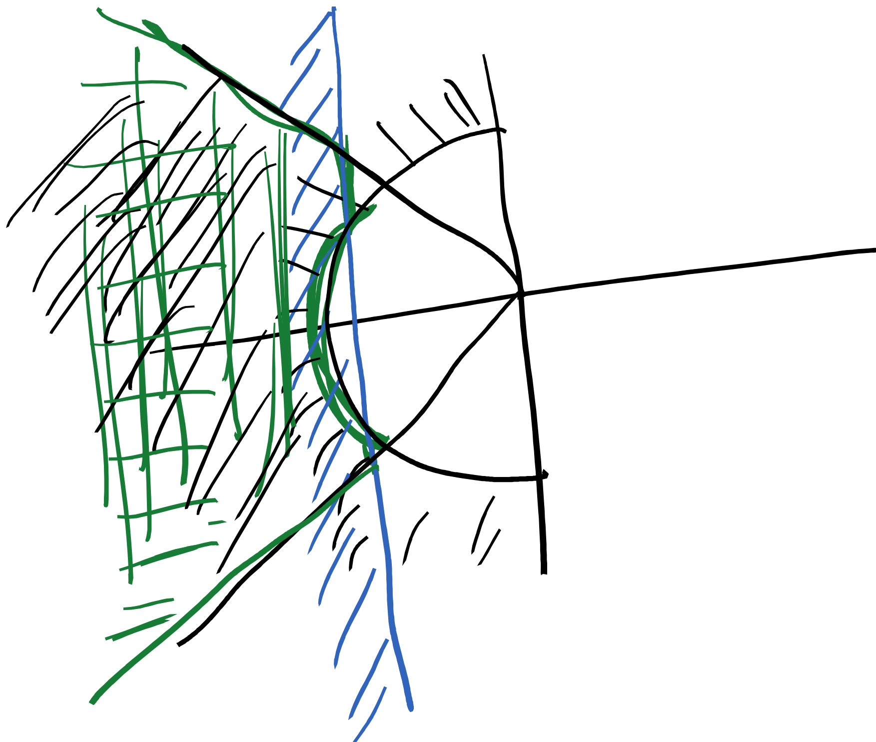
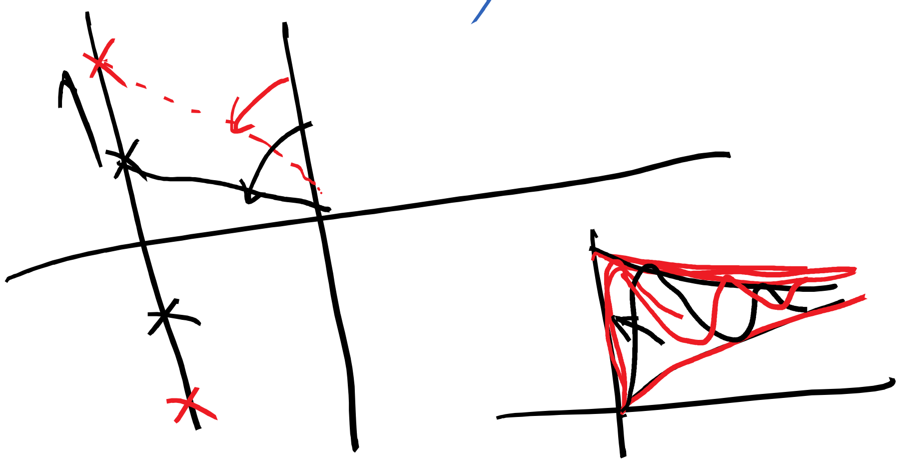
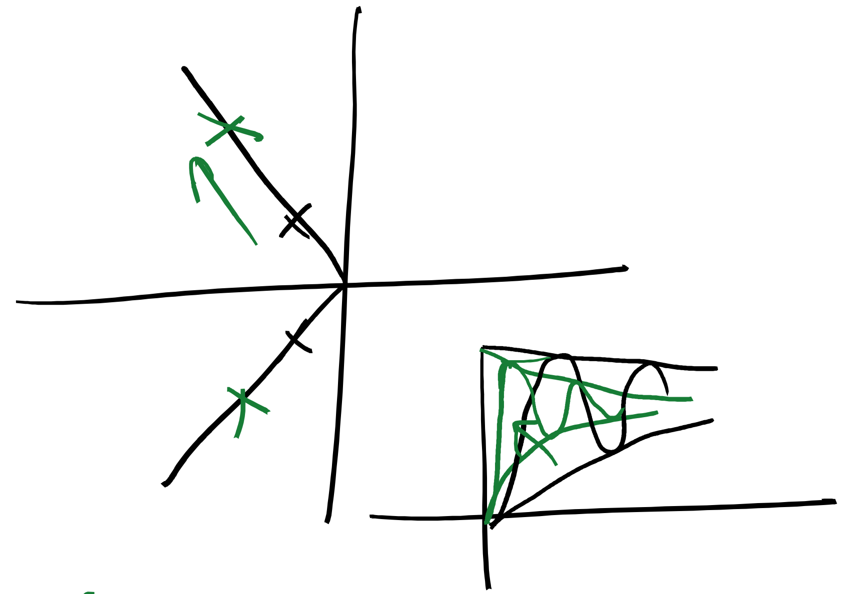
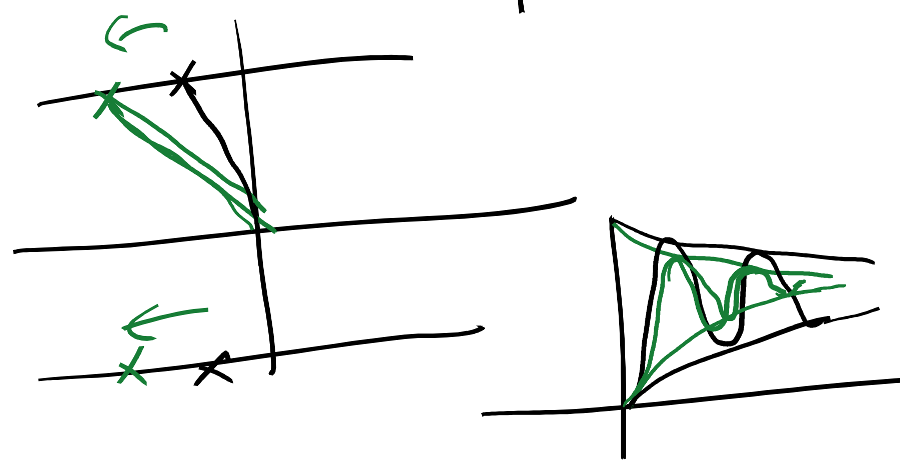




$$s = -\zeta\omega_n \pm j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}.$$

$$|s| = \omega_n$$



Effects of additional poles & zeros.

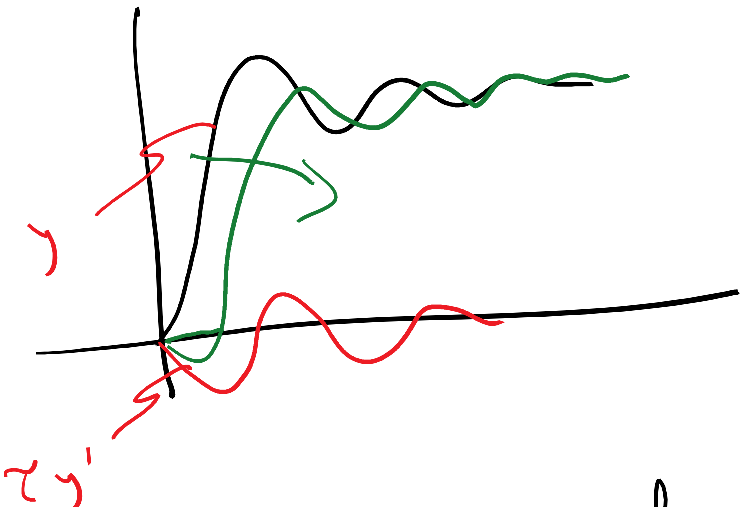
$$G_1(s) = \frac{\omega_n^2 (1+s\tau)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \tau > 0$$

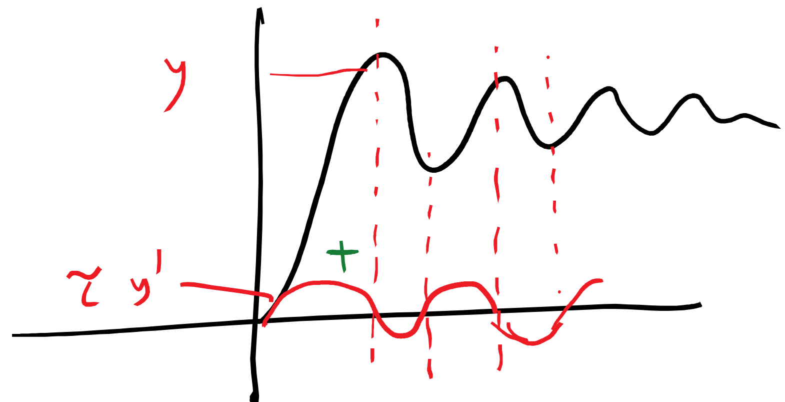
$$G_2(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)(1+s\tau)} \qquad \tau > 0$$

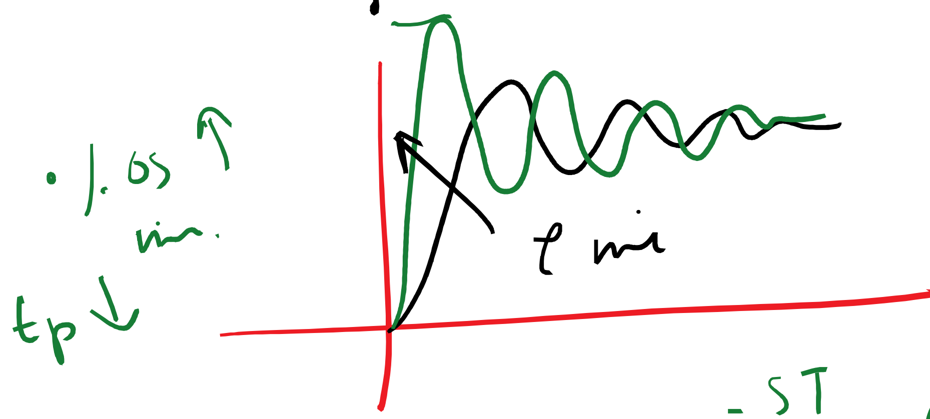
$$\boxed{-1/\tau \approx -\zeta\omega_n} \longrightarrow$$

All the analysis we have carried out does not help!

Suppose $0 < \tau << \zeta\omega_n$







Non-minimum phase systems ($\tau < 0$)

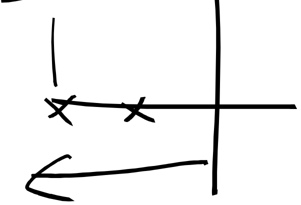
Pade approx

$$e^{-sT} \approx \frac{1 - sT/2}{1 + sT/2}$$

Effect of additional poles

$$G_2(s) = \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)(1+s\tau)}$$

dom pole



$$Re(p_1) = -\zeta\omega_n$$

Dominant pole

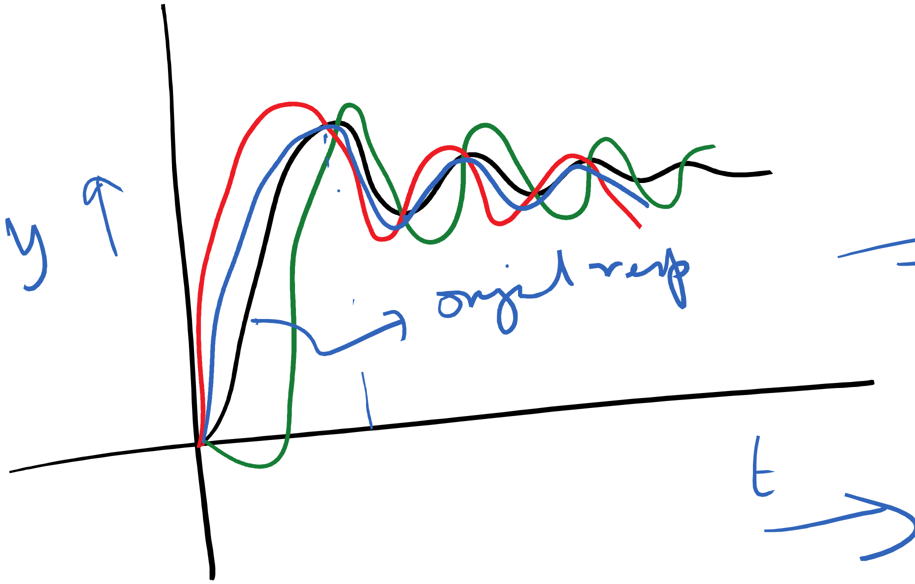
$$\left|\frac{1}{\tau}\right| > 5\,|Re(\text{dom pole})|$$

$\frac{1}{5}$ $G_2(s) \longrightarrow$

$$G_3 = \frac{\omega_n^2(1+s\tau_1)}{(s^2+2\zeta\omega_n s+\omega_n^2)(1+s\tau_2)} \quad \tau_1 \to \tau_2$$

Let $\tau \to 0$.

$$G > G_2 > G_3 \quad \boxed{C}\, e^{-t/\tau}$$



$\not\Rightarrow \quad \boxed{\tau y'} + \boxed{y}$