Assignment 4 : Hints

Qn 1 Either use Laplace inverse of $(sI-A)^{-1}$ to determine e^{At} and replace $t=1$., or use the Cayley Hamilton based method discussed during live interactions to infer that options (a) & (b) are the correct answers.

Qn 2 Obtain $C(sI-A)^{-1}B$ to see that the transfer fn has one pole. (indicating pole-zero cancellation) This rules out options (a) & (b). It is straightforward to see that (c) & (d) are true.

Qn 3 It is a 2×2 system. Recall that $\det(A) =$ product of eigenvalues $= \lambda_1\lambda_2$. Similarly, trace $(A) =$ sum of eigenvalues $= \lambda_1+\lambda_2$ [Even if you do not recall this you should be still able to derive this: Take

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix};\ |(\lambda I - A)| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = (\lambda-a)(\lambda-d) - bc$$
$$= \lambda^2 - (a+d)\lambda + ad - bc$$
$$= \lambda^2 - tr(A).\lambda + \det(A) \quad (*)$$

However, this result is true, in general, for any $n\times n$ matrix, i.e. $tr(A) = \sum$ eigenvalues & $\det(A) = \prod$ eigenvalues]

Clearly, $\lambda_1\lambda_2 < 0$, implies λ_1, λ_2 are real. since for complex eigenvalues, they appear in conj. pairs so $\lambda_1 = p+jq$ & $\lambda_2 = p-jq$. $\Rightarrow \lambda_1\lambda_2 = p^2+q^2 > 0$.

Also $sgn(\lambda_1) = -sgn(\lambda_2)$. Hence, (a) is true. (b) is ruled out similarly.

Recall from (*) that $\lambda = \dfrac{+\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$

$\delta > 0 \Rightarrow \tau^2 - 4\delta < \tau^2$

$\Rightarrow |\tau| > \sqrt{\tau^2-4\delta}$

& Moreover, $\tau < 0 \Rightarrow \lambda = \dfrac{\tau \pm \sqrt{\tau^2-4\delta}}{2} < 0 \Rightarrow$ stable.

Further, for $\tau > 0$, $\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2} > 0$.

$\Rightarrow$ instability

So (c) is true.

If $\delta > 0$ & $\tau = 0$, $\lambda = \pm j\sqrt{\delta}$

$\Rightarrow$ imaginary eigenvalues

$\Rightarrow$ marginally stable system

$\Rightarrow$ oscillatory response to non-zero initial conditions

[Recall: $x(t) = e^{At}x_0$ & e^{At} will contain pure undamped sinusoids/cosinusoids].

Hence, (d) is also true.

Qn 4) Write down the o/p of every integrator as a state. Readily follows that the parallel realization corresponds to the diagonal A matrix.

Others can be seen similarly too.

It follows that (i) & (v) are the correct matches

Qn 5). Asymptotic stability can be readily checked by obtaining eigenvalues and ascertaining if they are all in the open left-half plane (excludes imaginary axis) or not.

Turns out the (iii) is true.

For systems (A) & (B), $|(\lambda I - A)| = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1$

$\Rightarrow$ eigenvalues $= 1, -1$.

distinct real eigenvalues! Hence diagonalizable.

For (C) it follows that eigenvalues are imaginary. So cannot be diagonalised over the real field.

For (D), again eigenvalues are a complex conjugate pair. So, not diagonalizable over the reals.

Thus, (iv) is true. & (v) is not

~~(v) is~~ (vi) is not true since e^{At} is not the same for (C) & (D) as D is asympt stable while (C) is not.

Check ~~B~~ $\mathcal{C}(A,B) = [B \quad AB]$ for (A)

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

This is not invertible, so both poles cannot be placed. None of the original poles is at $\{-2$ or -3. So cannot be placed (vii) is not true.

By the same token, $\mathcal{C}(A,B)$ is invertible for (B), (C) & (D). So poles can be arbitrarily placed. Thus, (viii), (ix) & (x) are true.

(A) & (B) have the same eigenvalues and are both diagonalizable (since (iv) is true above). Thus we have

for (A) $T_1^{-1} A_1 T_1 = D \rightarrow$ diag matrix.

& for (B) $T_2^{-1} A_2 T_2 = D \rightarrow$ same diag. matrix.

$$= T_1^{-1} A_1 T_1$$

$$\Rightarrow A_2 = T_2 T_1^{-1} A_1 T_1 T_2^{-1}$$

Let $T = T_1 T_2^{-1}$

$$\therefore \quad A_2 = T^{-1} A_1 T.$$

But for a similarity transf, we must also have.

the same transfer fn. Check to see that
$C(sI-A)^{-1}B$ does not match in the two systems
Hence (xi) is not true
(xii) is also not true since the eigenvalues of the system matrices do not match.

Qn 6

$[B \quad AB]$ is not invertible, (check!) so both poles cannot be placed arbitrarily. Original poles are not at $-2, -3$. So (a) is not true
Due to same reason, (b) is true
check to see that

$$A+BF = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} [f_1 \quad f_2]$$

$$= \begin{bmatrix} 2+5f_1 & 5f_2+1 \\ 0 & 3 \end{bmatrix}$$

Since $A+BF$ is Δar matrix, closed loop poles are at $2+5f_1$ & 3.
Choosing $f_1 = -4/5$ and f_2 = any real no. we can place poles at $(-2, 3)$.
Hence, soln is non-unique. Option (c) is true
By same reasoning, there exists a non-unique feedback, $F = [-1 \quad f_2]$ (f_2 is any real no.) to place poles at $(-3, 3)$. Hence (d) is not true.

Qn 7 Clearly (a) is false. & (b) is true.
Take $C(sI-A)^{-1}B$ with $C = [c_1 \quad c_2]$

$$(sI-A)^{-1}B = \begin{pmatrix} s-2 & -1 \\ 0 & s-3 \end{pmatrix}^{-1}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{(s-2)(s-3)}\begin{bmatrix} s-3 & 1 \\ 0 & s-2 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{s-2} \end{bmatrix}$$

For arbitrary c_1, c_2, $G(s) = \frac{5c_1}{s^2}$

Thus the pole at 3 has been cancelled.

So (c) is true.

Q. 8 Since both poles of the t.f. are at 0, both eigenvalues of the SSR will be at 0.

Thus. A matrix can either be of rank 1 or rank 0. (zero matrix).

Suppose it is the zero matrix.

Then $(sI-A)^{-1} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{bmatrix}.$

There is no way that this can lead to a t.f of $\frac{1}{s^2}$ for any $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and $C = [c_1\ c_2]$

$\therefore$ A must have rank 1, but not rank 2 (since its eigenvalues are at 0, so it is singular).

Hence, option (b) is true.

If A is diagonalizable, we will again land up with the zero matrix & the transfer fn for the diagonal system cannot be $\frac{1}{s^2}$. But for diagonalizable systems (diagonalized through similarity transforms), ~~eigenvalue~~ transfer fns are invariant. Hence, (c) is true.

Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\therefore \mathcal{C}(A,B) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\cancel{\therefore (sI-A)^{-1} = \frac{1}{s^2}\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}}$

not invertible!

So (d) is not true.

Q.9 (a) Take the case when $a_i = 0$ for $i = 0, 1, 2, \dots, n-1$.

A has all its eigenvalues at 0. (Check!).

Suppose it is diagonalizable. Then $T^{-1}AT = O_{n\times n}$ (zero matrix)

But this would lead to $A = T\, O_{n\times n}\, T^{-1}$

$= 0$ (contradiction).

Hence (a) is not true.

(b) Check to see that $[B\ \ AB\ \ A^2B \cdots A^{n-1}B]$

$$= \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ & & 1 & & \\ & 1 & -a_{n-1} & & \\ 1, & -a_{n-1} & (-a_{n-2}+a_{n-1}^2) & & \end{bmatrix}$$

which clearly has full rank (why?) Hint: consider rows and see that they are always linearly independent. Thus, (b) is true.

(c) straightforward calculation of $|(sI-A)|$ yields that (c) is true.

(d) $A + BF = A + \begin{bmatrix} 0 & & & \\ f_0 & f_1 & \cdots & f_{n-1} \end{bmatrix}$ where $F = [f_0\ f_1 \cdots f_{n-1}]$

$$\therefore\ A+BF = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ (-a_0+f_0) & (-a_1+f_1) & \cdots & & (-a_{n-1}+f_{n-1}) \end{bmatrix}$$

clearly, choosing f_i such that $f_i - a_i = -d_i$

$\Rightarrow f_i = a_i - d_i$ yields a closed loop char. eqn of $s^n + d_{n-1}s^{n-1} + \dots + d_1 s + d_0$

Thus, (d) is true.