Assignment 3 Solutions

Q1 Let $G(j\omega) = x + jy$

Closed loop transfer function in unity feedback:

$$T(s) = \frac{G(s)}{1+G(s)}$$ For standard -ve unity feedback

We want locus of x, y s.t.,

$$\left|\frac{G(j\omega)}{1+G(j\omega)}\right| = c \rightarrow \text{Constant}$$

$$\Rightarrow \left|\frac{x+jy}{1+x+jy}\right| = \frac{|x+jy|}{|1+x+jy|} = c$$

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$$\Rightarrow \frac{x^2+y^2}{(1+x)^2+y^2} = c^2$$

$$x^2+y^2 = c^2x^2 + 2c^2x + c^2 + c^2y^2$$

$$(c^2-1)x^2 + 2c^2x + c^2 + (c^2-1)y^2 = 0$$

If $c \neq 1$: $(c^2-1)\left[x^2 + \frac{2c^2x}{(c^2-1)} + \left(\frac{c^2}{c^2-1}\right)^2\right] - \frac{c^4}{c^2-1} + c^2$

$$+ (c^2-1)y^2 = 0$$

$$\therefore (c^2-1)\left(x+\frac{c^2}{c^2-1}\right)^2 + (c^2-1)y^2 = \frac{+c^2}{c^2-1}$$

$$\left(x+\frac{c^2}{c^2-1}\right)^2 + y^2 = \left(\left|\frac{c}{c^2-1}\right|\right)^2$$

Which is a circle with radius $\left|\frac{c}{c^2-1}\right|$

and centre at $\left(-\frac{c^2}{c^2-1}, 0\right)$

If $c=1$, $2c^2x + c^2 = 0$

$$\Rightarrow x = -\frac{1}{2} \rightarrow \text{straight line}$$

If positive unity feedback is assumed,

Radius $\left|\frac{c}{c^2-1}\right|$

Centre $\left(+\frac{c^2}{c^2-1}, 0\right)$

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Now for constant phase, ϕ:

$$\angle T(j\omega) = \phi$$

$$\Rightarrow \angle x+jy - \angle 1+x+jy = \phi$$

$$\Rightarrow \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{y}{1+x} = \phi$$

$$\tan\phi = \frac{\frac{y}{x} - \frac{y}{1+x}}{1 + \left(\frac{y}{x}\right)\left(\frac{y}{1+x}\right)}$$

$$\tan\phi = \frac{y}{x^2 + x + y^2}$$

If $\phi \neq 0$: $\therefore \; x^2 + x + y^2 - \frac{y}{\tan\phi} = 0$

$$\therefore \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2\tan\phi}\right)^2 = \left(\frac{1}{4} + \frac{1}{4\tan^2\phi}\right)$$

$\therefore$ Circle with

Radius $= \sqrt{\frac{\tan^2\phi + 1}{4\tan^2\phi}} = \sqrt{\frac{1}{4}\operatorname{cosec}^2\phi}$

Centre $= \left(\frac{-1}{2}, \frac{+1}{2\tan\phi}\right)$

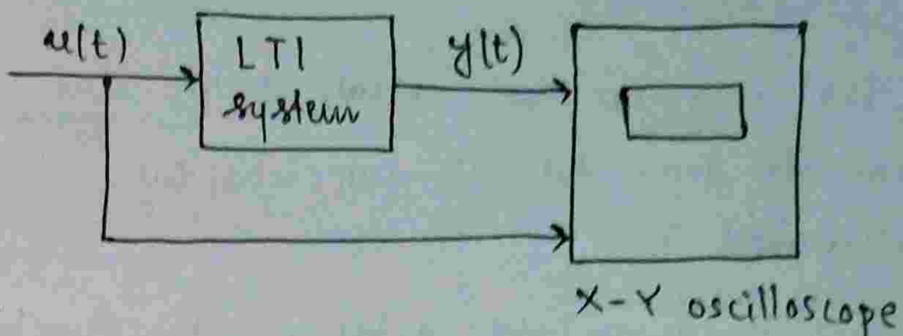
If positive feedback is assumed,

Radius is same,

Centre $= \left(+\frac{1}{2}, -\frac{1}{2\tan\phi}\right)$

If $\phi = 0$: $\Rightarrow y = 0$ which is a straight line.

Q.2 set up:



- $u(t) = A \sin \omega t$
- LTI system's steady state behavior can be characterized by its magnitude & phase functions $[M(\omega), \phi(\omega)]$

so LTI system can be represented by $G(j\omega) = M(\omega) e^{j\phi(\omega)}$

$\therefore$ $\boxed{y(t) = A M(\omega) \sin[\omega t + \phi(\omega)]}$ (show how)
- use Euler's formula
- take $M(-\omega) = M(\omega)$
  $\phi(-\omega) = -\phi(\omega)$

- For the display in the x-y oscilloscope, x-channel gets the input i.e. $x = A \sin \omega t$

similarly 'y' gets the output

$\longrightarrow$ represent y in terms of x ; (x, y are f^n of time and the display on the oscilloscope is constant wrt time for a given 'w' value)

$y(t) = A M(\omega) [\sin \omega t \cos \phi + \cos \omega t \sin \phi]$

$= M(\omega) [x(t) \cos \phi(\omega) + \sqrt{A^2 - x^2(t)} \sin \phi(\omega)]$

Hence, $y = M(\omega)\, x \cos \phi + M(\omega) \sqrt{A^2 - x^2} \sin \phi$ —— (*)

(where ϕ is f^n of w and x is f^n of time)

Also, $M(\omega), \phi(\omega)$ are fixed for a given 'w'

* curves to be expected on the display of XY scope : $\boxed{\text{Ellipse}}$ (how??)
  - elaborate using eq^n (*)
  - mention the special cases (it can be circle or straight line)

⇝ For a given 'w', $M(w)$, $\phi(w)$ can't be directly read off from the XY slope display. It needs to be calculated

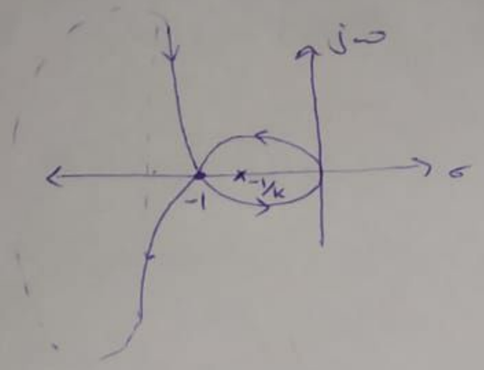
- Once the curve (plot) settle on the display (i.e. attain steady state), y-intercept (y_0) and x-intercept (x_0) is calculated by putting $x=0$, $y=0$ respectively (in the eq^n obtained for calculation)

- x_0, y_0, A are used to obtain magnitude $M(\omega)$ and phase $\phi(\omega)$. [give proper sign, observing the slope]

  (elaborate your arguments in detail)

Q.3 Using Nyquist Criteria

$$G(s) = \frac{K(s+2)}{s\left(\frac{s}{20}-1\right)}$$



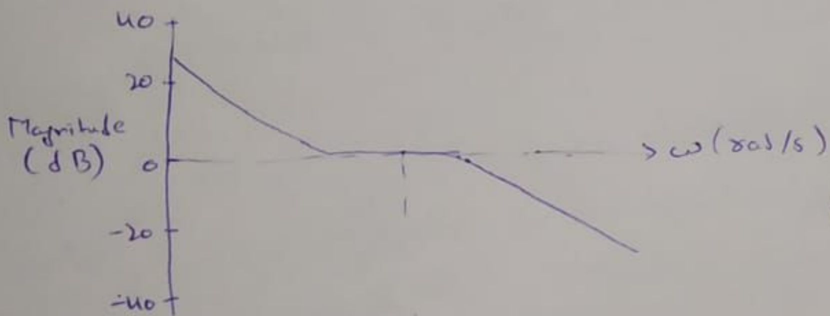
as $P = 1$ → no. of poles in RHP

$N = 1$ → counter clockwise rotation

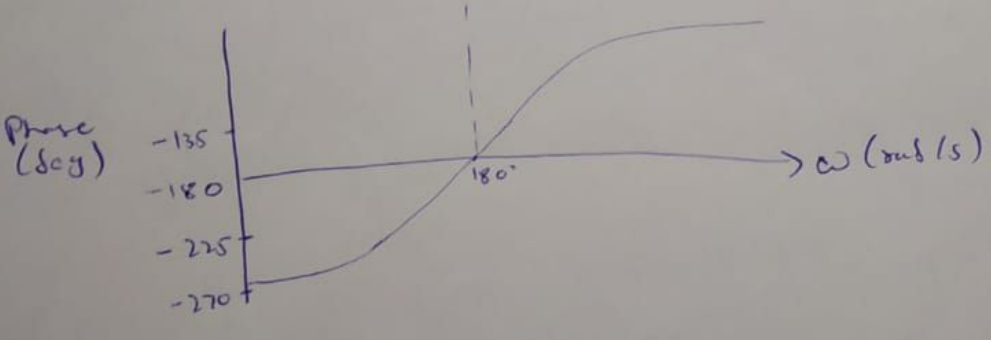
$Z = P - N = 0$

hence for ~~K~~ $K > 1$ → closed loop stability

Using Bode Plot

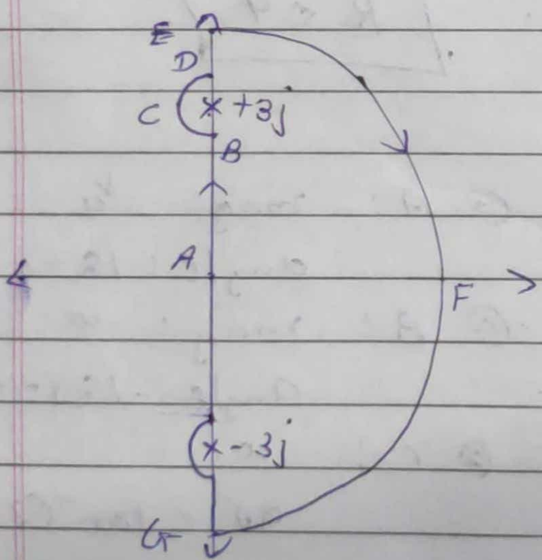


hence Gain Margin here is 0 dB. To keep it high (-ve) for stability → $K > 1$



Q:4 $G(s) = \frac{s-1}{s^2+9}$ $G(j\omega) = \frac{j\omega-1}{9-\omega^2}$

(a)



@ pt. A : mag : $1/9$

angle : $180°$

@ pt B : $+\infty$,

$180° - \tan^{-1}(3)$

@ pt. C : ∞

$270° - \tan^{-1}(3)$

@ pt. D : ∞

$360° - \tan^{-1}(3)$

@ pt E : 0

$-90°$

@ pt. F : 0 , $0°$

@ pt. G : 0 , $90°$

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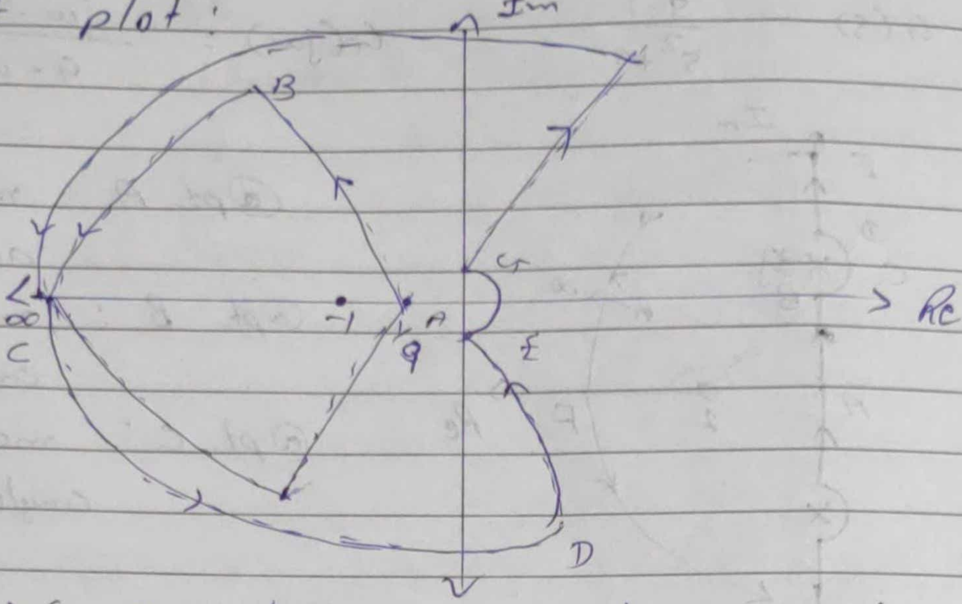
Nyquist plot:



→ 2 CCW encirclements around $-1+0j$

$P = 2$

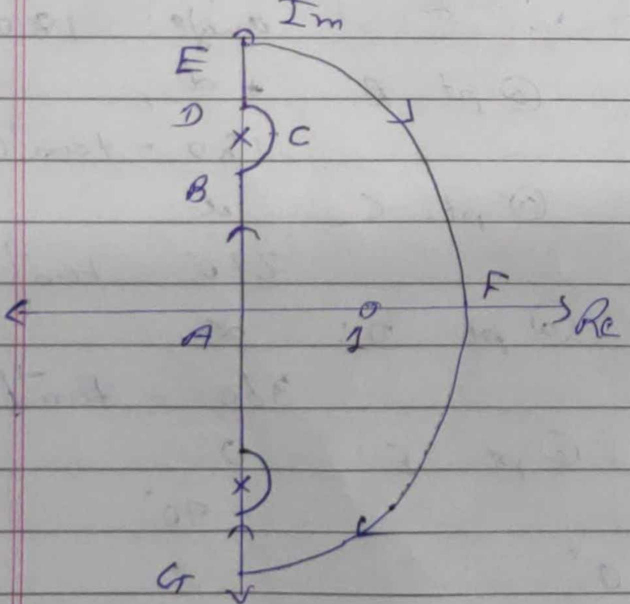
if $K > 9$: $Z = 2 - 1$

$= 1$ ∴ atleast 1 closedloop pole in RHP

⇒ unstable

so, stability condition: $\boxed{K < 9}$

(b)



@ A: mag: $1/9$

angle: $180°$

@ B: mag: ∞

angle: $180° - \tan^{-1}(3)$

@ C: ∞

$90° - \tan^{-1}(3)$

@ D: ∞

$-\tan^{-1}(3)$

@ E: 0, $-90°$

@ F: 0, $0°$

@ G: 0, $+90°$

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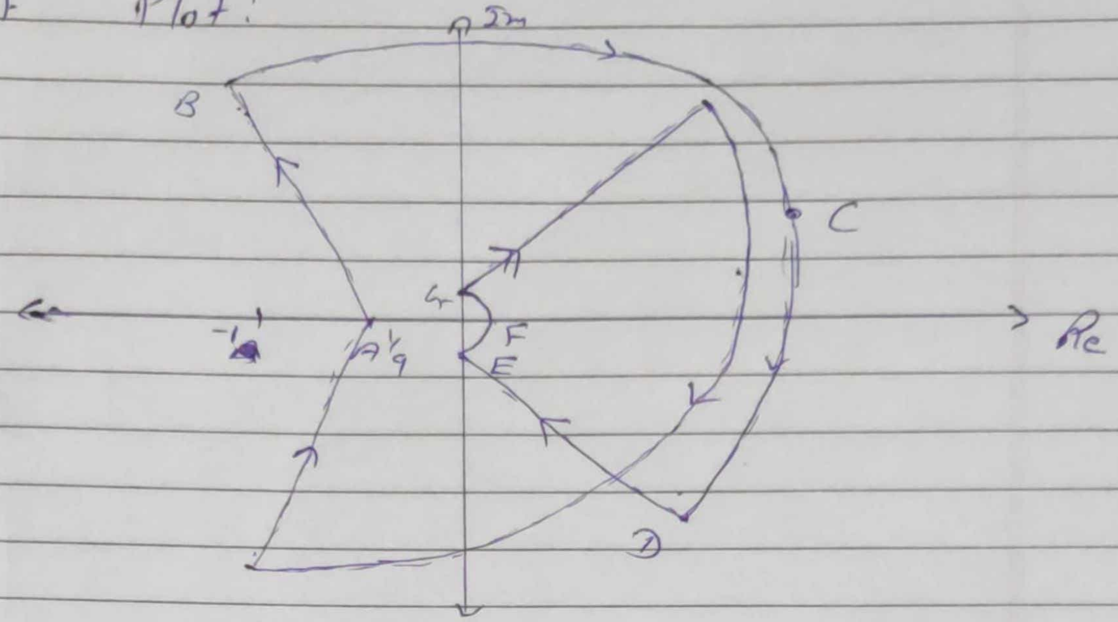
Nyquist Plot:



@ K = 1 : Z = 0 stable

for K > 9 - left part of -1+0j

L Z = -1

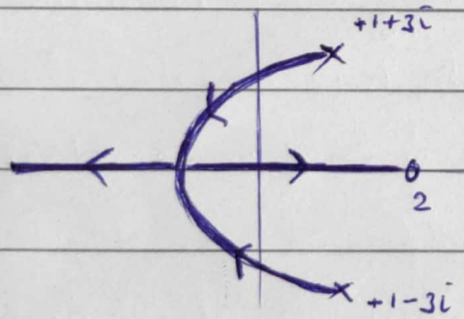
stability condition: K < 9

5. $G(s) = \dfrac{s-2}{s^2-2s+10}$ CLTF $\Rightarrow \dfrac{kG}{1+kG} = \dfrac{s-2}{s^2+s(k-2)+10-2k}$

Routh Table for denominator.

s^2	1	$10-2k$
s^1	$k-2$	0
s^0	$10-2k$	0

To make entire row zero we get $k=2$ & $k=5$.



Looking at the RL we can see the two $j\omega$ crossings one corresponding to $k=2$ (the symmetric case) & the other to $k=5$.

So for $k \in (2,5)$ the RL stays in left half plane & hence contributes to system stability.

(Q3)

OLTF:- $G(s) = \dfrac{k}{(s+1)(s+10)(s+100)}$

CLTF $G(s) = \dfrac{k}{(s+1)(s+10)(s+100)+k}$

Ch eq^n = $s^3 + 111s^2 + 1110s + 1000 + k$.

R.H Tab

s^3	1	1110
s^2	111	1000+k
s^1	$\dfrac{122210-k}{111}$	
s^0	1000+k	

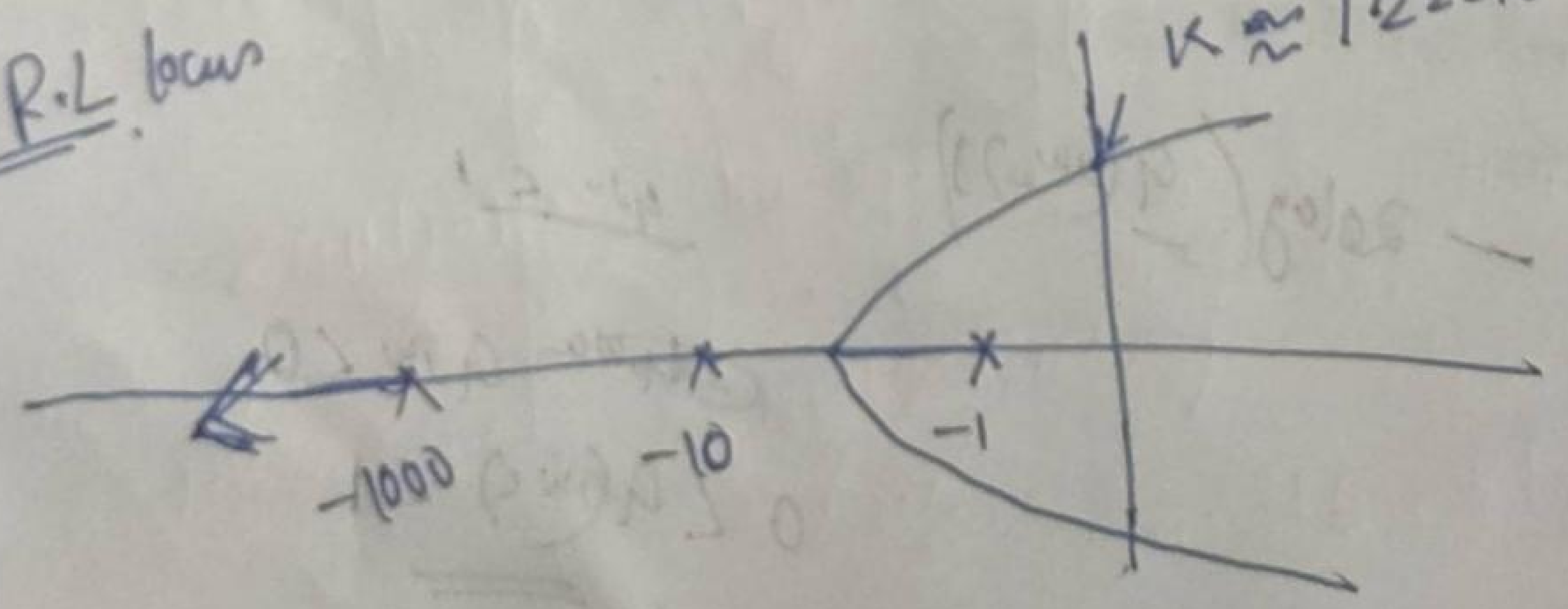
for stable $\dfrac{122210-k}{111} > 0^+ \Rightarrow k < 122210$

$1000+k > 0 \Rightarrow k > -1000$.

we can take gain value $k > 0$ to $k < 122210$.

$k \approx 122210$ (MATLAB).

R.L locus



Bode plot:-

$$G(j\omega) = \frac{k}{(j\omega+1)(j\omega+10)(j\omega+100)}$$

$$\tilde{k} = \frac{k}{1000.}$$

$$= \frac{\tilde{k}}{(1+j\omega)(1+j\omega/10)(1+j\omega/100)}$$

$$|G(j\omega)| = \frac{\tilde{k}}{\sqrt{1+\omega^2}\ \sqrt{1+\omega^2/10^2}\ \sqrt{1+(\omega/100)^2}}$$

$$|G(j\omega)|_{\text{in dB}} = 20\log\hat{k} - 20\log\sqrt{1+\omega^2} - 20\sqrt{1+\omega^2/10^2} - 20\log\sqrt{1+(\omega/100)^2}$$

$$\angle GH(j\omega) = -\tan^{-1}\omega - \tan^{-1}\omega/10 \ \tan^{-1}\omega/100$$

1) phase cross freq. = $\angle GH(j\omega_c) = -180$

$\omega_c = 33.316$ rad/sec.

gain margin. $= \dfrac{1}{\left| \dfrac{K}{\sqrt{\omega_{pc}^2+1}\,\sqrt{\omega_{pc}^2+100}\,\sqrt{\omega_{pc}^2+(100)^2}} \right|}$

system to be stable :-

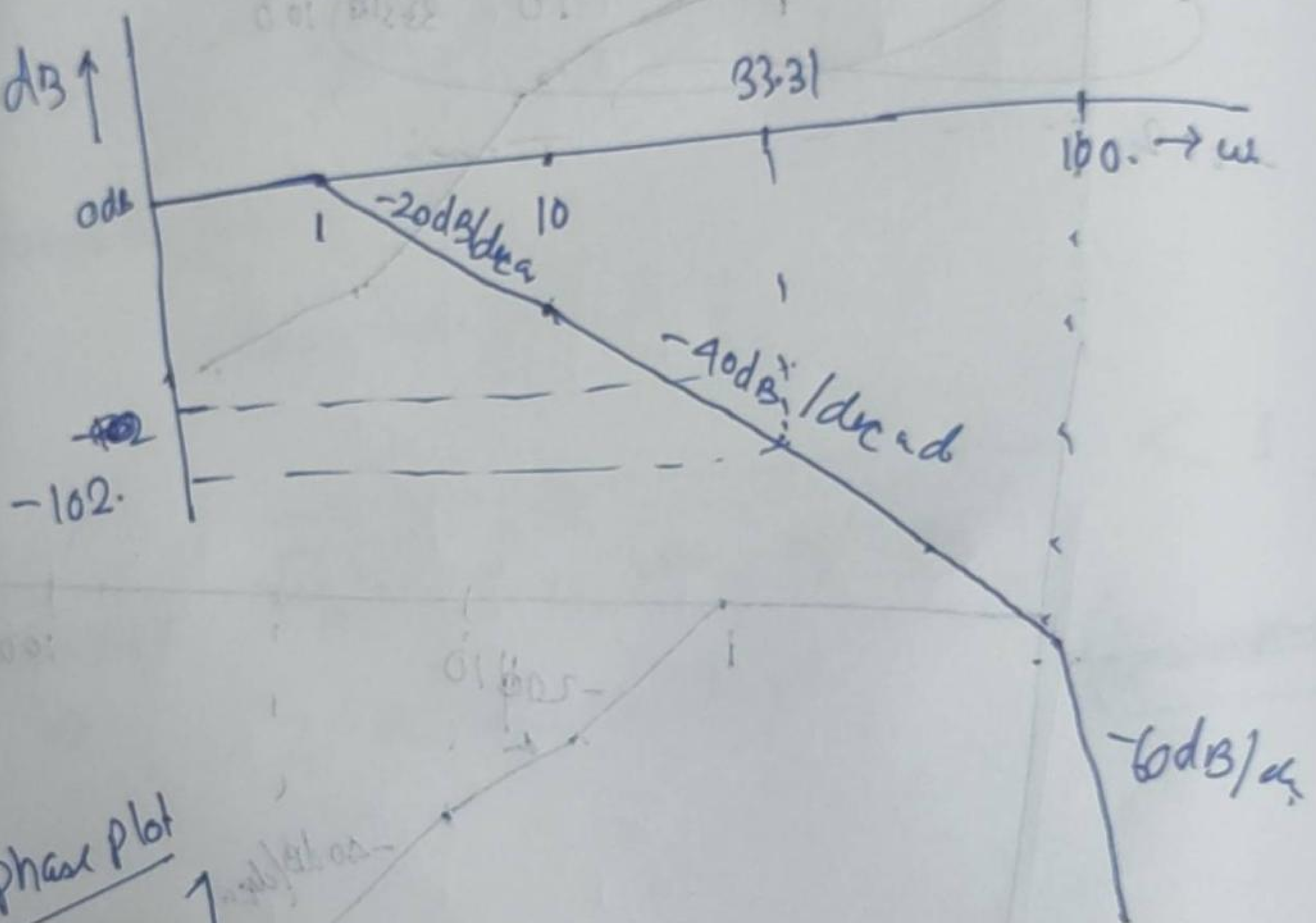
Gain margin > 1

$$\Rightarrow \frac{K}{\sqrt{\omega_{pc}^2+1}\,\sqrt{\omega_{pc}^2+100}\,\sqrt{\omega_{pc}^2+(100)^2}} < 1.$$

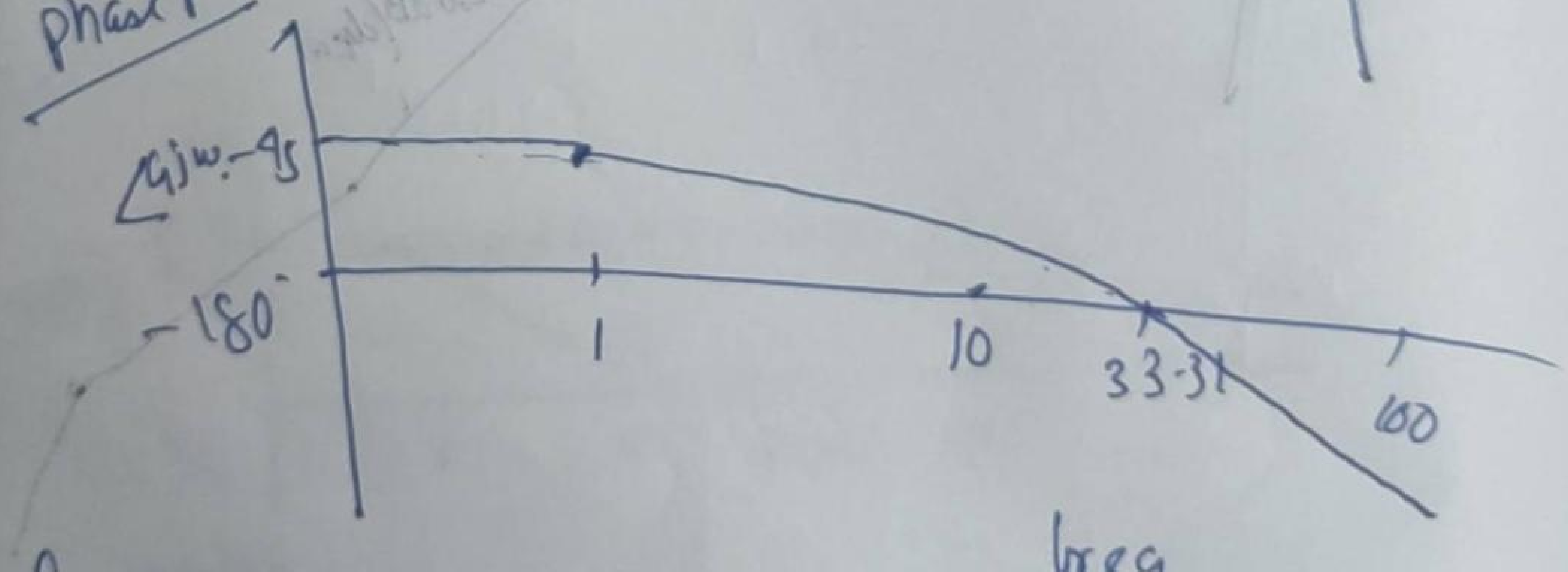
$\omega_{pc} = 33.3167$

$$\Rightarrow K < 122210$$

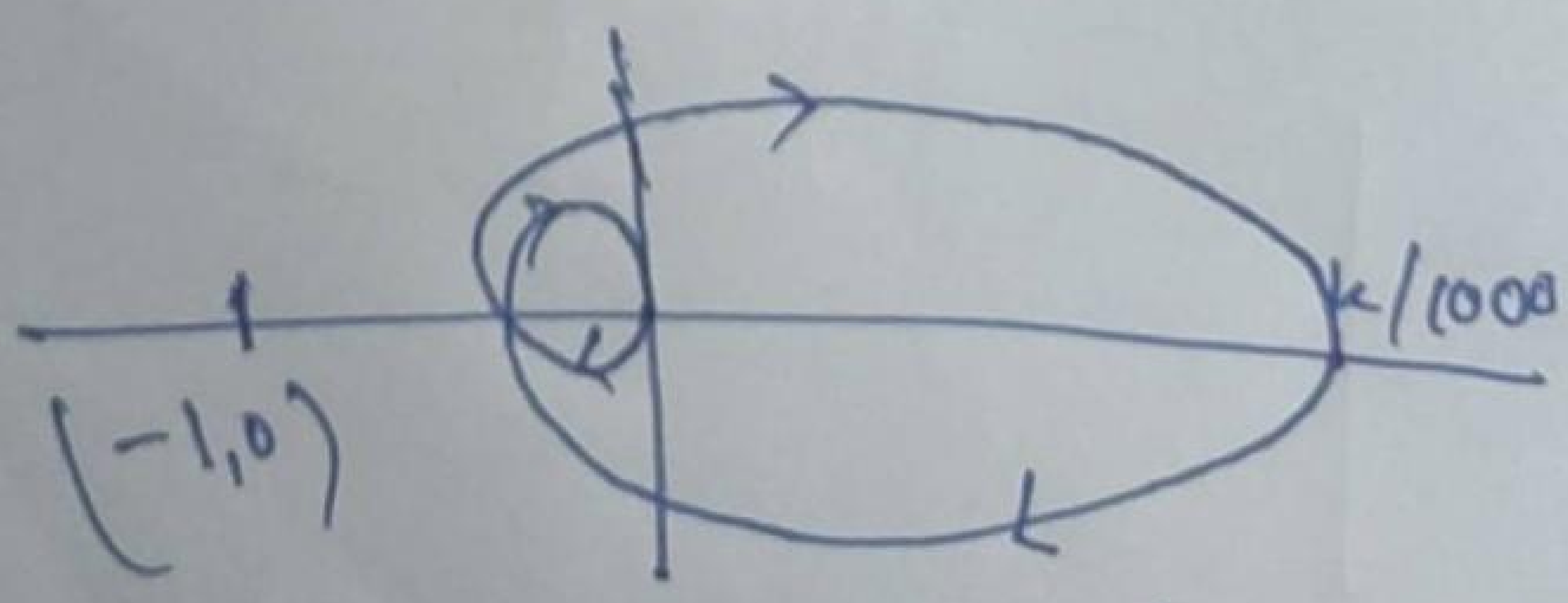
Bode Plot



phase plot



Nyquist plot



$$\angle G(j\omega) = -\tan^{-1}\omega - \tan^{-1}\omega/10 - \tan^{-1}\omega/100$$

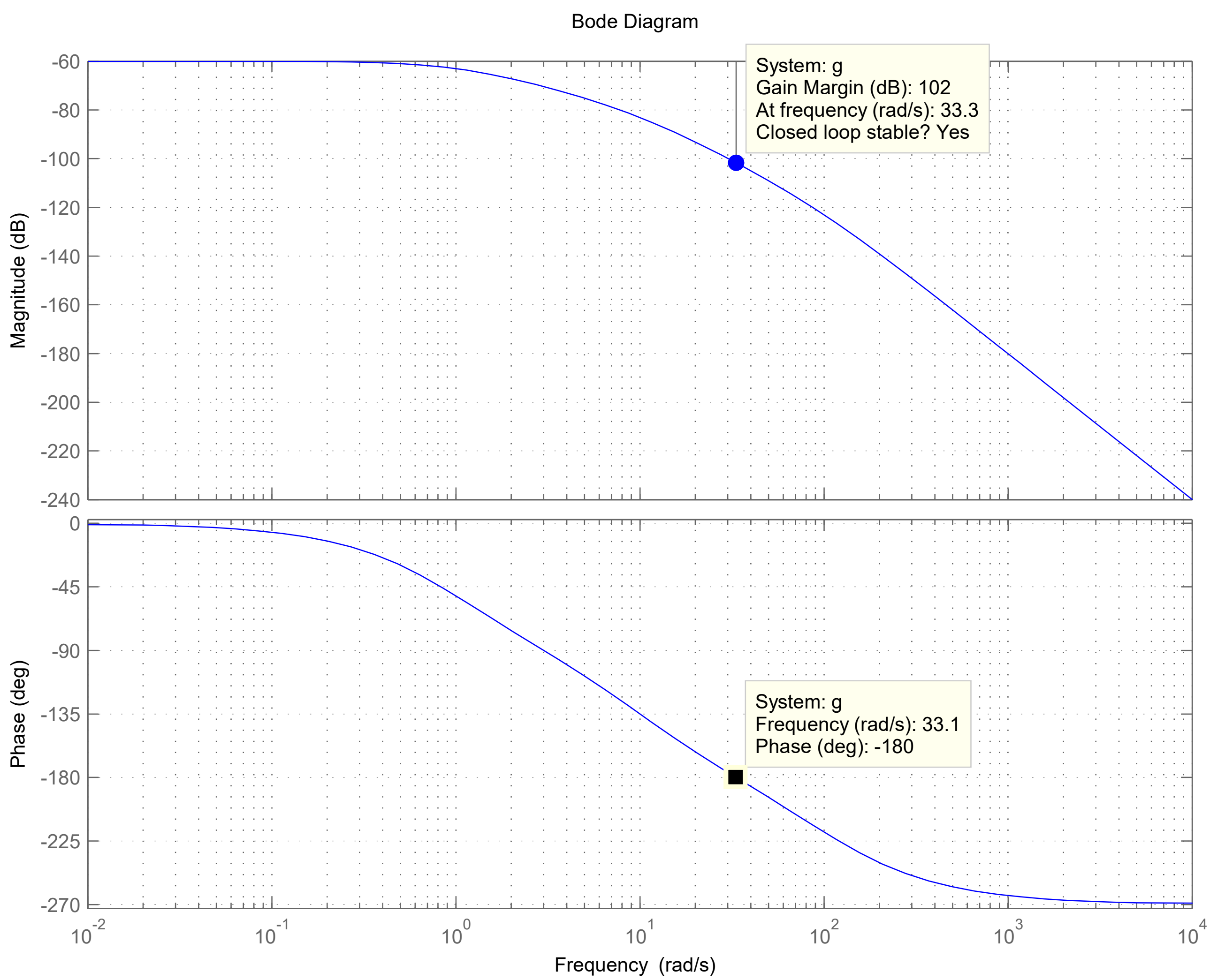
Phase crossover frequency. $\omega_{pc} = 33.3167$ rad/sec.

for stability it should not encircle the point $(-1, 0)$

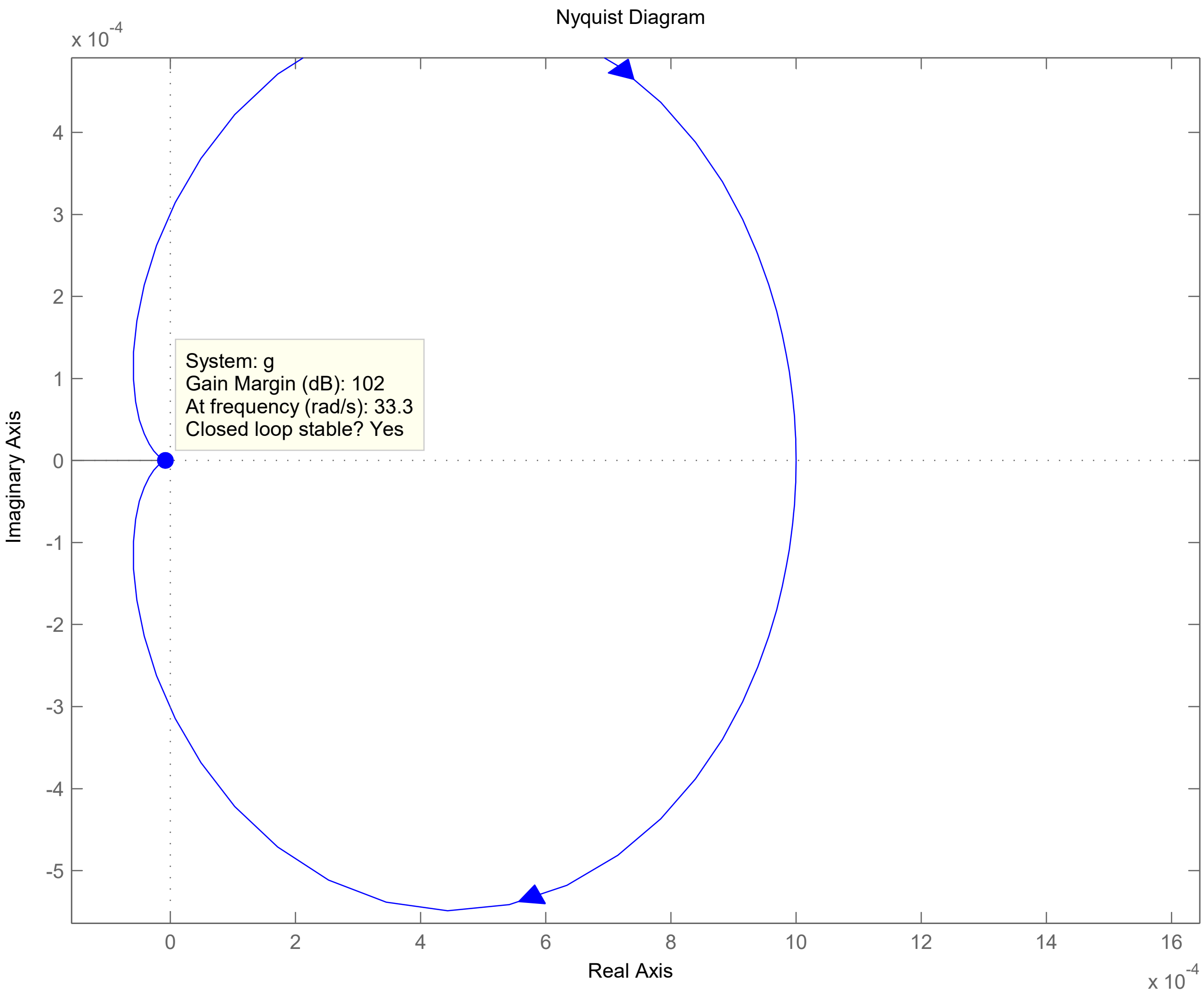
so,

$$\left| \frac{k}{\sqrt{1+\omega_{pc}^2}\sqrt{10^2+\omega_{pc}^2}\sqrt{100^2+\omega_{pc}^2}} \right| < 1$$

$\Rightarrow$ $K \leq 122210$ (MATLAB)



Bode Diagram
System: g
Gain Margin (dB): 102
At frequency (rad/s): 33.3
Closed loop stable? Yes
System: g
Frequency (rad/s): 33.1
Phase (deg): -180
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
-60
-80
-100
-120
-140
-160
-180
-200
-220
-240
0
-45
-90
-135
-180
-225
-270
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}


Nyquist Diagram
x 10^-4
System: g
Gain Margin (dB): 102
At frequency (rad/s): 33.3
Closed loop stable? Yes
Imaginary Axis
Real Axis
x 10^-4
4
3
2
1
0
-1
-2
-3
-4
-5
0
2
4
6
8
10
12
14
16

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Q7 (a) Given Settling time (2%) = 2 sec.

$\Rightarrow \frac{4}{\xi\omega_n} = 2 \Rightarrow \boxed{\xi\omega_n = 2}$

Overshoot $= 0.05 = e^{-\pi\xi/\sqrt{1-\xi^2}}$

$\Rightarrow \frac{\pi\xi}{\sqrt{1-\xi^2}} = 2.9957$

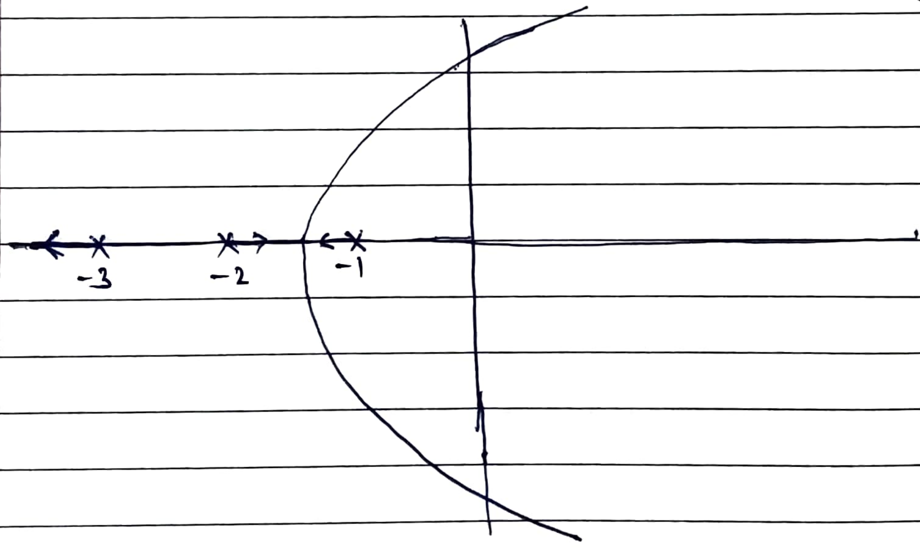
$\Rightarrow \boxed{\xi = 0.6901}$

$\Rightarrow \boxed{\omega_n = 2.8981}$

$\therefore$ Required pole $= -\xi\omega_n \pm j\sqrt{1-\xi^2}\,\omega_n$

$= -2 \pm j\,2.0974$

Root Locus Without P.D Controller $G(s) = \frac{1}{(s+1)(s+2)(s+3)}$



Let PD Controller be $K_c(s+Z_c)$

It should cancel pole at $Z=2$ so that our T.F is second order

$\therefore$ O.L.T.F. $= \frac{K_c(s+2)}{(s+1)(s+2)(s+3)} = \frac{K_c}{(s+1)(s+3)}$

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Now C.L.T.F has roots given by eqⁿ

$$s^2+4s+3+K_c = 0$$

$$\Rightarrow K_c = -(s^2+4s+3)$$

$$\Rightarrow \boxed{K_c = 5.3991}$$

∴ Required PD Controller $= 5.3991(s+2)$

Modified transfer function $\hat{G}(s) = \frac{5.3991}{(s+1)(s+3)}$

Ans 7 (b) ∵ It is type-0 System Steady State Error

$$e_{ss} = \lim_{s\to 0} \frac{1}{1+\lim_{s\to 0}\hat{G}(s)}$$

$$= \frac{1}{1+\frac{5.3991}{3}} = \frac{1}{1+1.7997}$$

$$= 0.3572$$

Desired Steady State Error $= \frac{1}{10} \times 0.3572$

$$= 0.03572$$

Let Lag compensator be $\frac{(s+z_L)}{(s+P_L)}$

$$\Rightarrow 0.03572 = \frac{1}{1+\lim_{s\to 0}\hat{G}(s)\frac{(s+z_L)}{(s+P_L)}} = \frac{1}{1+(1.7997)\left(\frac{z_L}{P_L}\right)}$$

$$\Rightarrow 1+(1.7997)\left(\frac{z_L}{P_L}\right) = \frac{1}{0.03572}$$

$$\Rightarrow \boxed{\frac{z_L}{P_L} = 15}$$

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Lets take the pole near to the origin at
$s = -0.005$

$\Rightarrow$ $P_L = +0.005$
$Z_L = +0.075$

$\therefore$ Lag Compensator $= \dfrac{s+0.075}{s+0.005}$

& Overall transfer function with desired properties

is $\tilde{G}(s) = \dfrac{5.3991\,(s+0.075)}{(s+1)(s+3)(s+0.005)}$

Note: Although it meets the steady state error criteria it does not meet the desired time specification
$\therefore$ It does not lead to a second order approximation of the transfer function

Now if we choose $P_L = 0.0667$
$Z_L = 1$

$\Rightarrow$ $\tilde{G}(s) = \dfrac{5.3991}{(s+3)(s+0.0667)}$

Note: ~~This meets the~~ It seems at first that since the ~~However~~ third pole is a problem as it cannot be cancelled out, it may be a good idea to try and cancel one of the plant poles using a lag compensator. But simulations, unsurprisingly, reveals that this is not such a good idea because it ends up distorting the original root locus so much that closed loop poles from part (a) no longer satisfy the angle criteria.

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In fact, the closed loop poles are now at $-1.5 \pm j1.8020$, leading to following performance metrics. Settling time: 2.5384 & % Overshoot - 6.9052%. & Steady state Error = 0.03572.

Conclusion: For this system, it seems to be very hard, if not impossible, to meet all design specs using the PD & lag compensator combination.